# Approximation Algorithms and Hardness Results for the Joint Replenishment Problem with Constant Demands

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**Abstract.** In the Joint Replenishment Problem (JRP), the goal is to coordinate the replenishments of a collection of goods over time so that continuous demands are satisfied with minimum overall ordering and holding costs. We consider the case when demand rates are constant. Our main contribution is the first hardness result for any variant of JRP with constant demands. When replenishments per commodity are required to be periodic and the time horizon is infinite (which corresponds to the so-called general integer model with correction factor), we show that finding an optimal replenishment policy is at least as hard as integer factorization. This result provides the first theoretical evidence that the JRP with constant demands may have no polynomial-time algorithm and that relaxations and heuristics are called for. We then show that a simple modification of an algorithm by Wildeman et al. (1997) for the JRP gives a fully polynomial-time approximation scheme for the general integer model (without correction factor). We also extend their algorithm to the finite horizon case, achieving an approximation guarantee asymptotically equal to $\sqrt{9/8}$.

## 1 Introduction

In the deterministic Joint Replenishment Problem (JRP) with constant demands, we need to schedule the replenishment times of a collection of commodities in order to fulfill a constant demand rate per commodity. Each commodity incurs fixed ordering costs every time it is replenished and linear holding costs proportional to the amount of the commodity held in storage. Linking all commodities, a joint ordering cost is incurred whenever one or more commodities are ordered. The objective of the JRP is to minimize the sum of ordering and holding costs.

The JRP is a fundamental problem in inventory management. It is a natural extension of the classical economic lot-sizing model that considers the optimal trade-off between ordering costs and holding costs for a single commodity. With multiple commodities, the JRP adds the possibility of saving resources via coordinated replenishments, a common phenomenon in supply chain management.

For example, in manufacturing supply chains, a suitable replenishment schedule for raw materials can lead to significant reductions in operational costs. This reduction comes not only from a good trade-off between ordering and holding costs, but also from joint replenishment savings, such as those involving transportation and transactional costs.

Since an arbitrary replenishment schedule may be difficult to implement, it is natural to focus on restricted sets of schedules (often called policies in this context). The *general integer model with correction factor* (GICF) assumes an infinite horizon and constant inter-replenishment time per commodity. The joint ordering cost in the GICF model is a complicated function of the inter-replenishment times, so it is often assumed that joint orders are placed periodically, even if some joint orders are empty. This defines the *general integer model* (GI). In both cases, the time horizon is infinite.

The existence of a polynomial-time optimal algorithm for the JRP with constant demands remains open for all models, regardless of whether the time horizon is finite or not, whether the ordering points are periodic or not, or whether the incurred costs are modeled precisely or not. Given the significant amount of research in this area, it may be a bit surprising that only a few papers mention the lack of a hardness result for the JRP as an issue (the only recent paper we could find is [TB01]). Some papers addressing the JRP with constant demands (e.g. [LY03,MC06]) cite a result by Arkin et al. [AJR89], which proves that the JRP with variable demands is NP-hard. However, it is not clear how to adapt this result to the constant demand case. In fact, we believe that the two problems are completely unrelated from a complexity perspective as their input and output size are incomparable.

In this paper we present the first hardness result for the JRP with constant demands. We show that finding an optimal policy for the general integer model with correction factor is at least as hard as integer factorization, under polynomial-time reductions. Although integer factorization is unlikely to be NP-hard, it is widely believed to be outside P. In fact, this belief supports the hypothesis that RSA and other cryptographic systems are secure [RSA78].

We also give approximation results. An $\alpha$-approximation algorithm for a minimization problem is an algorithm that produces a feasible solution with cost at most $\alpha$ times the optimal cost. We show that a simple modification of an algorithm by Wildeman et al. [WFD97] gives polynomial-time approximation algorithms with ratios very close to or better than the current best approximations for the JRP with constant demands. We illustrate this in detail for dynamic policies in the finite horizon case and for GI. For the latter model, this yields a fully polynomial-time approximation scheme (FPTAS): for every $\epsilon > 0$, we provide a $(1 + \epsilon)$-approximation algorithm with running time polynomial in $n$ and $1/\epsilon$. Here, $n$ is the number of commodities. To the best of our knowledge, this is the first FPTAS for a model formally known as GI with *variable base*.

In the remainder of this section we formally describe the JRP models we consider in this paper, followed by a review of the literature and a summary of our results.

*Mathematical formulation.* For all JRP variants considered in this paper the input consists of a finite collection $\mathcal{I} = \{1, \ldots, n\}$ of commodities with constant demand rates $d_i \in \mathbb{Z}_+$, for $i \in \mathcal{I}$. The cost of an order is the sum of the *individual ordering costs* $K_i$ of the commodities involved in the order plus the *joint ordering cost* $K_0$. The acquired inventory is stored at a *holding cost rate* of $h_i$ per unit of commodity $i$ and per unit of time. The objective is to find an optimal ordering schedule in the time horizon $[0, T)$, where $T$ may be equal to $+\infty$ (the so-called *stationary case*).

If $T$ is finite, a *schedule* $S$ is a finite sequence of joint orders. (An order is called a joint order even if it consists of one commodity only.) If we place $N$ joint orders, the total joint ordering cost is $C_{\mathrm{ord}}^{\mathrm{joint}} \equiv NK_0$. If we replenish commodity $i$ at times $0 = t_1 < t_2 < \ldots < t_{n_i} < T$, its individual ordering cost is $C_{\mathrm{ord}}^{\mathrm{indiv}}(i) \equiv n_i K_i$ and its individual holding cost is $C_{\mathrm{hold}}(i) \equiv \frac{d_i h_i}{2} \sum_{j=1}^{n_i} (t_{j+1} - t_j)^2$, where $t_{n_i+1} = T$. The cost $C[S]$ of the schedule $S$ is the sum of the joint ordering cost $C_{\mathrm{ord}}^{\mathrm{joint}}$, the total individual ordering cost $C_{\mathrm{ord}}^{\mathrm{indiv}} \equiv \sum_{i\in\mathcal{I}} C_{\mathrm{ord}}^{\mathrm{indiv}}(i)$, and the total holding cost $C_{\mathrm{hold}} \equiv \sum_{i\in\mathcal{I}} C_{\mathrm{hold}}(i)$. The objective of the JRP is to minimize $C[S]$.

An arbitrary sequence of joint orders is called a *dynamic* schedule. The structure of an optimal dynamic schedule for the finite horizon case is not known. Potentially, it could be exponential in the size of the input. One can avoid this issue by adding more structure. In the JRP with *general integer policies*, joint orders can be placed only at multiples of a base period $p$ (to be determined), and each commodity $i \in \mathcal{I}$ is periodically replenished every $k_i p$ units of time, for some $k_i \in \mathbb{Z}_+$. The costs are just the time-average version of their counterparts in the finite horizon case. An accurate mathematical description of this scenario is the general integer model with correction factor:

$$\begin{aligned} \min \quad & \frac{K_0 \Delta(k_1, \ldots, k_{|\mathcal{I}|})}{p} + \sum_{i\in\mathcal{I}} \left( \frac{K_i}{q_i} + \frac{1}{2} h_i d_i q_i \right) \\ \text{s.t.} \quad & q_i = k_i p \\ & k_i \in \mathbb{Z}_+ \\ & p > 0, \end{aligned} \tag{GI-CF1}$$

where $\Delta(k_1, \ldots, k_{|\mathcal{I}|})/p$ is the average number of joint orders actually placed in a time interval of length $p$. With a simple counting argument, it is easy to see that

$$\Delta(k_1, \ldots, k_{|\mathcal{I}|}) = \sum_{i=1}^{|\mathcal{I}|} (-1)^{i+1} \sum_{I \subseteq \mathcal{I} : |I| = i} \mathrm{lcm}(k_i, i \in I)^{-1},$$

where $\mathrm{lcm}(\cdot)$ is the least common multiple of its arguments.

The GICF model is complicated to analyze because of the $\Delta$ term. Ignoring $\Delta$ (i.e. setting the joint ordering cost rate to be $K_0/p$ in GI-CF1) defines the *general integer* model (GI). Note that this change is equivalent to assuming that $K_0$ is paid at every multiple of the base period.

We defined the GI and GICF formulations in the *variable base model*. Both formulations have a variant where the base $p$ is restricted. The fixed base version of the GI formulation requires $p$ to be multiple of some constant $B$. The fixed base version of the GICF formulation requires $p$ to be fixed.

*Literature Review.* We only survey approximation algorithms (see Goyal and Satir [GS89] and Goyal and Khouja [KG08] for a review of other heuristics). For none of the models studied here any hardness results are known.

**General integer models.** A common approach in this case is to solve the problem for a sequence of values of $p$ and return the best solution found. For instance, Kaspi and Rosenblatt [KR91] approximately solve (GI) for several values of $p$ and pick the solution with minimum cost. They do not specify the number of values of $p$ to test, but they choose them to be equispaced in a range $[p_{\min}, p_{\max}]$ containing any optimal $p$. For example,

$$p_{\min} = \frac{K_0}{\sqrt{2(K_0 + \sum_{i \in \mathcal{I}} K_i)(\sum_{i \in \mathcal{I}} d_i h_i)}}, \quad p_{\max} = \sqrt{2 \frac{K_0 + \sum_{i \in \mathcal{I}} K_i}{\sum_{i \in \mathcal{I}} d_i h_i}} \tag{1}$$

are a lower and an upper bound for any optimal $p$. Wildeman et al. [WFD97] transform this idea into a heuristic that converges to an optimal solution. They exactly solve (GI) for certain values of $p$ determined using a Lipschitz optimization procedure. They do not establish a running time guarantee.

A completely different approach uses the rounding of a convex relaxation of the problem. This was introduced by Roundy [Rou85] for the One Warehouse Multi Retailer Problem (OWMR). For the JRP, Jackson et al. [JMM85] find a GI schedule with cost at most $\sqrt{9/8} \approx 1.06$ times the optimal cost for dynamic policies. This approximation is improved to $\frac{1}{\sqrt{2}\log 2} \approx 1.02$ when the base is variable [MR93]. The constants above have been slightly improved by considering better relaxations [TB01].

Using another method, Lu and Posner [Pos94] give an FPTAS for the GI model with fixed base. They note that the objective function is piecewise convex and the problem reduces to querying only a polynomial number of its break points.

**General integer policies with correction factor (GICF).** No progress has been reported in terms of approximation for this problem, other than the results inherited from the GI model. The incorporation of the correction factor leads to a completely different problem, at least in terms of exact solvability. For example, as the inter-replenishment period goes to 0 the joint ordering cost in the GI model diverges, in contrast to what happens in the GICF model. Porras and Dekker [PD08] show that the inclusion of the correction factor significantly changes the replenishment cycles $k_i$ and the joint inter-replenishment period with respect to those in an optimal GI solution. Moreover, they prove that there is always a solution under this model that outperforms ordering commodities independently, thereby neglecting possible savings from joint orders. This desirable property has not yet been proved for the GI model.

**Finite horizon.** Most of the heuristics for the finite horizon case assume variable demands and run in time $\Omega(T)$ [LRS04,Jon90]. Some of them can be extended to the constant demand case preserving polynomiality. To our knowledge, the only heuristic with a provable approximation guarantee in this setting is given by Joneja [Jon90]. Their algorithm is designed for variable demands, but for constant demands and $T = \infty$, it achieves an approximation ratio of 11/10.

*Summary of results.* In Sect. 2 we show that finding an optimal solution for the GICF model in the fixed base case is at least as hard as the integer factorization problem. This is the first hardness result for any of the variants of JRP with constant costs and demands. In Sect. 3 we present, based on [WFD97], a polynomial-time 9/8-approximation algorithm for the JRP with dynamic policies and finite horizon. As the time horizon $T$ increases, the ratio converges to $\gamma \equiv \sqrt{9/8}$. In Sect. 4, we observe that the previous algorithm, extended to the infinity horizon case, is an FPTAS for the class of GI policies (either variable or fixed base model). This result is new for the fixed base case.

## 2 A hardness result for GICF

In this section we prove a hardness result for the JRP in the *fixed base* GICF. In contrast to GI-CF1, this model has $p$ as a parameter:

$$\begin{aligned} \min \quad & \frac{K_0 \Delta(k_1, \ldots, k_{|I|})}{p} + \sum_{i \in \mathcal{I}} \frac{K_i}{q_i} + \frac{1}{2} h_i d_i q_i \\ \text{s.t} \quad & q_i = k_i p \\ & k_i \in \mathbb{Z}_+ \end{aligned} \tag{GI-CF2}$$

Essentially, we prove that if we are able to solve GI-CF2 in polynomial time, then we are able to solve the following problem in polynomial time:

INTEGER-FACTORIZATION: Given an integer $M$, find an integer $d$ with $1 < d < M$ such that $d$ divides $M$, or conclude that $M$ is prime.

*Reduction.* The reduction uses two commodities. The main idea is to set up the costs so that commodity 1 has a constant renewal interval of length $M$ in the optimal solution. Under this assumption, commodity 2 has some incentive to choose an inter-replenishment time $q_2$ not coprime with $M$, since this reduces the joint ordering cost with respect to the case when they are coprime. When this happens, and as long as $q_2 < M$, we can find a non-trivial divisor of $M$ by finding the maximum common divisor of $M$ and $q$, using Euclid's algorithm.

We initially fix $p = 1$, $\frac{1}{2}h_1 d_1 = H_1$, $K_2 = 0$ and $\frac{1}{2}h_2 d_2 = 1$ (here $H_1$ is a constant we will define later), and therefore GI-CF2 reduces to:

$$\begin{aligned} \min \quad & K_0 \left( \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{\text{lcm}(q_1, q_2)} \right) + \frac{K_1}{q_1} + H_1 q_1 + q_2 \\ \text{s.t} \quad & q_1, q_2 \in \mathbb{Z}_+ \end{aligned} \tag{2}$$

Note that, except for the term $K_0/\operatorname{lcm}(q_1, q_2)$, the objective function is the sum of two functions of the form $f(q) = A/q + Bq$. We will frequently use that the minimum of $f$ is equal to $2\sqrt{AB}$, and is attained at $q = \sqrt{A/B}$.

In order to force a replenishment interval $q_1 = M$ in any optimal solution of Program (2), we make commodity 1 "heavy". More precisely, we set $K_1 = 2K_0M^3$, $H_1 = 2K_0M$, so that $\sqrt{K_1/H_1} = M$. Note that if $q_1 = M - 1$ or $q_1 = M + 1$, we get the following relations:

$$\frac{K_1}{M+1} + H_1(M+1) - \left(\frac{K_1}{M} + H_1 M\right) = \frac{H_1}{M+1} > \frac{K_0 M}{M} = K_0$$
$$\frac{K_1}{M-1} + H_1(M-1) - \left(\frac{K_1}{M} + H_1 M\right) = \frac{H_1}{M-1} > K_0,$$

and therefore

$$\frac{K_1}{q_1} + H_1 q_1 + q_2 > \left(\frac{K_1}{M} + H_1 M\right) + K_0 + 1 = K_0 + 4K_0M^2 + 1 \qquad (3)$$

for $q_1 = M - 1$ or $q_1 = M + 1$. Using that $K_1/q_1 + H_1 q_1$ is convex in $q_1 \in \mathbb{R}^+$ with minimum at $q_1 = M$, we obtain that Eq. (3) holds for any integer $q_1 \neq M$. Since $K_0 + 4K_0M^2 + 1$ is the objective value of Program (2) when $q_1 = M$ and $q_2 = 1$, we have proven that $q_1 = M$ in any optimal solution.

Now, Program (2) reduces to

$$\begin{aligned} \min \quad & \frac{M^2}{4}\left(\frac{1}{q} - \frac{1}{\operatorname{lcm}(M,q)}\right) + q \\ \text{s.t} \quad & q \in \mathbb{Z}_+, \end{aligned} \qquad (4)$$

where we eliminated $q_1$ from the program, renamed $q_2$ as $q$, and set $K_0 = M^2/4$. Let us define

$$A(q) = \frac{M^2}{4q} + q, \quad B(q) = \frac{M^2}{4\operatorname{lcm}(M,q)}, \quad F(q) = A(q) - B(q)$$

so the objective value of Program (4) for a given $q$ is equal to $F(q)$.

Let us assume that $M \geq 6$. We now prove that any optimal value $q$ for Program 4 is in $\{2, \ldots, M-1\}$. First, note that

$$F(1) = \frac{M^2}{4} + 1 - \frac{M}{4} \geq \frac{M^2 - M}{4} \geq M \qquad \text{and} \qquad F(M) = M.$$

Then, note that for $q \geq M$ we have that $A(q) \geq A(M)$ and $B(q) \leq B(M)$. They follow from the facts that $A(\cdot)$ is convex with minimum $A(M/2) = M$ and that $\operatorname{lcm}(M,q) \geq M$. Therefore, $F(q) \geq M$ for $q \notin \{2, \ldots, M-1\}$. Finally, since for $M \geq 6$:

$$F(\lfloor M/2 \rfloor) \leq A(\lfloor M/2 \rfloor) - \frac{M}{4\lfloor M/2 \rfloor} \leq A\left(\frac{M}{2}\right) + \frac{1}{M/2 - 1} - \frac{M}{2(M-1)} < M,$$

then any $q$ minimizing Program 4 should be in $\{2, \dots, M-1\}$. The second inequality follows from an argument similar to the one used to show that $q_1 = M$. In particular, any such $q$ is either relative prime with $M$, or it shares a non-trivial divisor of $M$. The next lemma shows that the latter is always the case when $M$ is odd and composite.

**Lemma 1.** *Suppose that $M \geq 6$ is an odd composite number. Then every $q' \in \mathbb{Z}_+$ minimizing $F(\cdot)$ satisfies* $\gcd(M, q') \neq 1, M$.

*Proof.* We already proved that $\gcd(M, q') \neq M$ for any $q'$ minimizing $F(\cdot)$. Suppose $M$ and $q'$ are coprimes. Then $B(q') = \frac{M}{4q'}$, and therefore $F(q') \geq \min_{q \in \mathbb{R}}\{A(q) - \frac{M}{4q}\}$. Let us define $L$ to be this minimum value, and suppose that it is achieved at $q^*$, then it is easy to see that

$$L = M\sqrt{u}, \quad q^* = \frac{M}{2}\sqrt{u},$$

where $u = 1 - \frac{1}{M}$. To get a contradiction, we will prove that there exists $q$ near to $q^*$ such that $F(q) < L$. To see this, let $3 \leq p \leq \sqrt{M}$ be any non-trivial divisor of $M$ and let $q \in [q^* - p/2, q^* + p/2]$ be any integer divisible by $p$. Let us write $q = (1+\epsilon)q^*$, where $\epsilon$ may be negative. Using that $\operatorname{lcm}(M, q) \cdot \gcd(M, q) = Mq$, we have that

$$F(q) = \frac{M^2}{4q} + q - \frac{M \gcd(M, q)}{4q} \leq \frac{M^2}{4q} + q - \frac{Mp}{4q}$$

and therefore

$$F(q) - L \leq \frac{M^2 u}{4q} + q - L - \frac{M(p-1)}{4q}. \tag{5}$$

Using $\frac{M^2 u}{4q^*} = q^*$ we can simplify

$$\frac{M^2 u}{4q} + q - L = \frac{M^2 u}{4(1+\epsilon)q^*} + (1+\epsilon)q^* - 2q^* = \frac{q^*}{1+\epsilon} + \epsilon q^* - q^* = \frac{\epsilon^2}{1+\epsilon}q^*$$

which, combined with Eq. 5 and $|\epsilon| \leq \left|\frac{p}{2q^*}\right| \leq \frac{1}{\sqrt{Mu}} = \frac{1}{\sqrt{M-1}}$ gives:

$$F(q) - L \leq \frac{1}{1+\epsilon}\left(\epsilon^2 q^* - \frac{M(p-1)}{4q^*}\right) \leq \frac{1}{1+\epsilon}\left(\frac{M\sqrt{u}}{2(M-1)} - \frac{p-1}{2\sqrt{u}}\right)$$

Finally, since $1 + \epsilon > 0$ and $\sqrt{u} < 1$, we have that

$$(1+\epsilon)\left(F(q) - L\right) \leq \frac{M\sqrt{u}}{2(M-1)} - \frac{p-1}{2\sqrt{u}} < \frac{M}{2(M-1)} - \frac{p-1}{2},$$

and it is easy to see that the rightmost expression is negative for $M \geq 3, p \geq 3$, which are true by assumption. Hence, $F(q) < L$, which proves the desired contradiction. □

If 2 is not a divisor of a composite number $M \geq 6$, Lemma 1 guarantees that the greatest common divisor between $M$ and the solution to Program 4 is always a non-trivial divisor. Since we can check if $M$ is prime in polynomial time [AKS04], the following result holds:

**Theorem 1.** *Suppose that GI-CF2 is polynomial-time solvable. Then* Integer-Factorization *is polynomial-time solvable.*

## 3 Approximation algorithm for finite horizon

In this section we present a dynamic policy for the finite horizon case. Recall that if we replenish commodity $i$ at times $0 = t_1 < t_2 < \ldots < t_{n_i} < T$, then $C_{\text{ord}}^{\text{indiv}}(i) \equiv n_i K_i$ and $C_{\text{hold}}(i) \equiv \frac{d_i h_i}{2} \sum_{j=1}^{n_i} (t_{j+1} - t_j)^2$, where $t_{n_i+1} = T$. We call the values $t_{j+1} - t_j$ the inter-replenishment lengths.

We temporarily assume that the approximation algorithm has oracle access to $N$, the total number of joint orders in some optimal solution for JRP. We briefly describe how to remove this assumption in Sect. 4. The description of the algorithm (Alg. 1) is a simple two-step process. In the first step, the algorithm places joint ordering points at every multiple of $T/N$, starting at $t = 0$. In the second step, each commodity places its orders on a subset of those joint orders in such a way that the individual ordering and holding costs are minimized. Note that this can be carried out separately for each commodity. A similar observation has been used to define an algorithm for GI policies [WFD97].

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**Algorithm 1**

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1: **Approx-JRP** $(T, h_i, d_i, K_i, K_0)$
2: Guess $N$, the number of joint orders in an optimal solution.
3: Set $p = T/N$ to be the joint inter-replenishment length.
4: Set $J = \{jp : j = 0, \ldots, N-1\}$, the set of joint order positions.
5: **for** $i \in \mathcal{I}$ **do**
6: Choose a subset of $J$ to be the orders of commodity $i$ such that $C_{\text{ord}}^{\text{indiv}}(i) + C_{\text{hold}}(i)$ is minimal.
7: **return** the schedule obtained.

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*Running time.* We have to be careful in how to execute the algorithm. The set $J$ may have $\Omega(T)$ elements, while the input size is proportional to $\log T$. However, we can explicitly define this set by giving $T$ and $N$, and it is easy to check that the size of $N$ is polynomial in the input size.

The same difficulty arises in Step 6, but a similar representation can be applied to keep the space polynomial: if the individual schedule for some commodity $i$ minimizes $C_{\text{ord}}^{\text{indiv}}(i) + C_{\text{hold}}(i)$, a simple convexity argument implies that the inter-replenishment lengths can take at most two values and they are consecutive multiples of $p$. Therefore, we can define this individual schedule by giving the (at most) two inter-replenishment lengths and their frequencies.

It follows that the only step where polynomiality can fail is Step 6. The following lemma establishes its complexity. The proof is omitted due to lack of space.

**Lemma 2.** *Suppose that commodity $i$ can be ordered only at multiples of some fixed period $p$. Moreover, assume that $T$ is a multiple of $p$. Then, it is possible to compute the schedule minimizing $C_{\text{hold}}(i) + C_{\text{ord}}^{\text{indiv}}(i)$ in polynomial time with respect to the input size.*

*Approximation analysis.* Given an instance of JRP, let OPT be any optimal solution having exactly $N$ joint orders, where $N$ is the value guessed by Alg. 1. For $i \in \mathcal{I}$, let $n_i$ be the total number of individual orders of commodity $i$ in OPT and let **OPT** be the optimal cost. In this section, we may emphasize the dependency on the schedule by including the schedule in brackets. For example, we may write $C[\text{OPT}] = \mathbf{OPT}$.

If a commodity is ordered exactly $m$ times, it is easy to show that its holding cost is minimized when the replenishments occur at $\{jT/m : j = 0, \ldots, m-1\}$. We say that $m$ orders are *evenly distributed* when they are placed according to this configuration. This optimality property for the holding cost of evenly distributed orders is the basis for a lower bound on **OPT** we use to prove the approximation guarantee. Our first step in this direction is to define two feasible solutions for the problem:

- The virtual schedule (or VS) places exactly $(1 + \beta_i)n_i$ evenly distributed orders of commodity $i$, for every $i \in \mathcal{I}$. Each $\beta_i$ is a parameter to be defined.
- The real schedule (or RS) allows joint orders in $J = \{jp : j = 0, \ldots, N-1\}$. For each commodity $i$ we place exactly $(1+\beta_i)n_i$ orders, that are obtained by shifting each individual order in the virtual schedule to the closest point in $J$. If there are two closest joint orders, we choose the closest one backwards in time.

Note that both schedules are not defined algorithmically. The real schedule is defined from the virtual schedule, and there is a one-to-one correspondence between their individual orders through the shifting process. We use the term *shifted order* to indicate this correspondence.

Loosely speaking, the cost of the real schedule is closely related to the cost of the schedule output by Alg. 1, while the virtual schedule is related to a lower bound on **OPT**. Both are used as a bridge that relates **OPT** with the cost of the schedule returned by Alg. 1.

**Proposition 1.** *If $\beta_i \leq 1/8$ for every $i \in \mathcal{I}$, then $C_{\text{hold}}[\text{RS}] \leq \frac{9}{8} C_{\text{hold}}[\text{VS}]$.*

*Proof.* Consider any commodity $i \in \mathcal{I}$. For simplicity, we omit subindices and write $n$ instead of $n_i$ and $\beta$ instead of $\beta_i$. Let $q = T/(1 + \beta)n$ be the inter-replenishment length of the commodity in the virtual schedule. Let $p = T/N$ be the joint inter-replenishment length for the real schedule. Note that $q \geq p/(1+\beta)$.

Suppose first that $p \geq q$. In RS, the commodity is replenished in every joint-order position. Directly evaluating the holding costs gives

$$C_{\text{hold}}[\text{RS}](i) = \frac{T^2hd}{2N} \leq \frac{T^2hd}{2n} = (1+\beta)C_{\text{hold}}[\text{VS}](i) \leq \frac{9}{8}C_{\text{hold}}[\text{VS}](i).$$

On the other hand, if $p < q$, let $k$ be the only integer satisfying $kp \leq q < (k+1)p$. Clearly, the inter-replenishment lengths in the real schedule can only take the values $kp$ or $(k+1)p$. Let $a$ be the number of orders of length $kp$ and let $b$ the number of orders of length $(k+1)p$ in the real schedule. We have the relations:

$$a + b = (1+\beta)n \quad \text{and} \quad a(kp) + b(k+1)p = q(1+\beta)n,$$

from where we get, in particular, that $bp = (1+\beta)n(q-kp)$. Using these three relations, and evaluating the holding cost, we obtain:

$$\frac{C_{\text{hold}}[\text{RS}](i)}{C_{\text{hold}}[\text{VS}](i)} = \frac{a(kp)^2 + b(k+1)^2p^2}{q^2(1+\beta)n} \leq \frac{(1+\beta)n(kp)^2 + b(2k+1)p^2}{q^2(1+\beta)n}.$$

which can be written after some additional manipulation as

$$\frac{C_{\text{hold}}[\text{RS}](i)}{C_{\text{hold}}[\text{VS}](i)} \leq (-k^2-k)\left(\frac{p}{q}\right)^2 + (2k+1)\frac{p}{q}.$$

To conclude, note that $-k(k+1)x^2 + (2k+1)x$, as a function of $x$, has maximum value $\frac{(2k+1)^2}{4k(k+1)}$, which is at most $9/8$ when $k \geq 1$. □

The next proposition shows that the individual ordering and holding costs in RS are within a constant factor of the respective costs in OPT. The proof (not included due to lack of space) uses Prop. 1 and some simple relations among RS, VS and OPT.

**Proposition 2.** *Let $\gamma = \sqrt{9/8}$. Then for every $\epsilon > 0$ we can choose $\{\beta_i\}_{i\in\mathcal{I}}$ so that the real schedule satisfies the following properties for $T$ sufficiently large:*

- $C_{\text{hold}}[\text{RS}] \leq (1+\epsilon)\gamma \cdot C_{\text{hold}}[\text{OPT}]$
- $C_{\text{ord}}^{\text{indiv}}[\text{RS}] \leq (1+\epsilon)\gamma \cdot C_{\text{ord}}^{\text{indiv}}[\text{OPT}]$

Let S be the schedule returned by Alg. 1. Recall that its output is a schedule S that minimizes $C_{\text{ord}}^{\text{indiv}} + C_{\text{hold}}$ restricted to use $N$ evenly distributed joint orders. This and Prop. 2 give the following inequalities for large $T$:

$$\left(C_{\text{ord}}^{\text{indiv}} + C_{\text{hold}}\right)[\text{S}] \leq \left(C_{\text{ord}}^{\text{indiv}} + C_{\text{hold}}\right)[\text{RS}] \leq (1+\epsilon)\gamma\left(C_{\text{ord}}^{\text{indiv}} + C_{\text{hold}}\right)[\text{OPT}].$$

Since $N$ is the number of joint orders in OPT, then $C_{\text{ord}}^{\text{joint}}[\text{S}] \leq C_{\text{ord}}^{\text{joint}}[\text{OPT}]$. Adding up, we obtain $C[\text{S}] \leq (1+\epsilon)\gamma C[\text{OPT}]$ which is an approximation guarantee asymptotically equal to $\gamma$ for Alg. 1. For small $T$, a closer look at our analysis gives an approximation factor of $9/8$.

**Theorem 2.** *Alg. 1 is a $9/8$-approximation algorithm ($\sqrt{9/8}$ for large $T$) for dynamic policies in the finite horizon case.*

## 4 GI model

We can easily adapt the algorithm described in Sect. 3 to the GI model with variable base (see Alg. 2). We now guess $p$, the optimal joint inter-replenishment length. Note that Step 5 is simpler, since $q_i$ is always one of the two multiples of $p$ closest to $\sqrt{K_i/h_i}$.

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**Algorithm 2** GI model algorithm

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1: **Approx-JRP** $(T, h_i, d_i, K_i, K_0)$
2: Guess $p$, the optimal renewal interval in an optimal solution.
3: Set $J = \{jp : j = 0, \dots, N-1\}$, the set of joint order positions.
4: **for** $i \in \mathcal{I}$ **do**
5: Choose $q_i$ as a multiple of $p$ such that $K_i/q_i + h_i q_i$ is minimum.
6: **return** the schedule obtained.

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Note that Alg. 2 finds the best value of $q_i$ for the optimal $p$, and therefore computes the optimal GI policy. Since GI policies approximate unrestricted policies by a factor of $1/(\sqrt{2}\log 2) \approx 1.02$ [MR93,TB01], our algorithm achieves these guarantees. The bound in Sect. 3 ($\approx 1.06$) is slightly worse since we are not using the powerful machinery available for the stationary case.

From this observation we can obtain a fully polynomial-time approximation scheme for GI policies by exhaustively searching $p$ in powers of $(1+\epsilon)$. The range of search can be $[p_{\min}, p_{\max}]$, which are the values defined in Eq. (1). The total running time is polynomial in the size of the input and $\frac{1}{\log(1+\epsilon)} = O(1/\epsilon)$. The only thing we need to prove is that choosing $p'$ in the range $p \leq p' \leq p(1+\epsilon)$ is enough to get a $(1+\epsilon)$-approximation. This follows from the fact that if $(p, \{k_i\}_{i\in\mathcal{I}})$ defines an optimal schedule with value **OPT**, then $(p/(1+\epsilon), \{k_i\}_{i\in\mathcal{I}})$ has cost

$$\frac{K_0}{p(1+\epsilon)} + \sum_{i\in\mathcal{I}} \frac{K_i}{(1+\epsilon)k_i p} + \frac{1}{2}h_i d_i k_i (1+\epsilon) \leq (1+\epsilon)\mathbf{OPT}.$$

Essentially the same idea can be used to remove the guessing assumption in Alg. 1. We just exhaustively search $N$ in (aproximated) powers of $\gamma$.

Finally, Alg. 2 can be extended to the fixed base GI model. The only difference is that we guess $p$ assuming it is a multiple of the base $B$. The exhaustive search in powers of $(1+\epsilon)$ has to carefully round the values of $p$ to be multiples of $B$.

**Theorem 3.** *Alg. 2 (properly modified) is an FPTAS in the class of GI policies and in the class of fixed base GI policies.*

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