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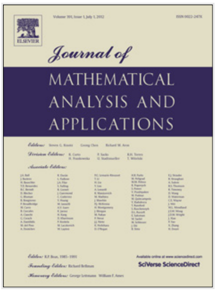

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Hypercyclic operators on countably dimensional spaces

Andre Schenke and Stanislav Shkarin

Abstract

According to Grivaux, the group $GL(X)$ of invertible linear operators on a separable infinite dimensional Banach space X acts transitively on the set $\Sigma(X)$ of countable dense linearly independent subsets of X. As a consequence, each $A \in \Sigma(X)$ is an orbit of a hypercyclic operator on X. Furthermore, every countably dimensional normed space supports a hypercyclic operator. Recently Albanese have extended this result to Fréchet spaces supporting a continuous norm.

We show that for a separable infinite dimensional Fréchet space X, $GL(X)$ acts transitively on $\Sigma(X)$ if and only if X possesses a continuous norm. We also prove that every countably dimensional metrizable locally convex space supports a hypercyclic operator.

MSC: 47A16
Keywords: Cyclic operators; hypercyclic operators; invariant subspaces; topological vector spaces

1 Introduction

All vector spaces in this article are over the field $\mathbb{K}$, being either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers. As usual, $\mathbb{N}$ is the set of positive integers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Throughout the article, all topological spaces *are assumed to be Hausdorff.* For a topological vector space X, $L(X)$ is the algebra of continuous linear operators on X, X' is the space of continuous linear functionals on X and $GL(X)$ is the group of $T \in L(X)$ such that T is invertible and $T^{-1} \in L(X)$. By saying 'countable', we always mean 'infinite countable'. Recall that a *Fréchet space* is a complete metrizable locally convex space. Recall also that the topology τ of a topological vector space X is called *weak* if τ is exactly the weakest topology making each $f \in Y$ continuous for some linear space Y of linear functionals on X separating points of X. It is well-known and easy to see that a topology of a metrizable infinite dimensional topological vector space X is weak if and only if X is isomorphic to a dense linear subspace of $\omega = \mathbb{K}^{\mathbb{N}}$.

Recall that $x \in X$ is called a *hypercyclic vector* for $T \in L(X)$ if the orbit

$$O(T,x) = \{T^n x : n \in \mathbb{Z}_+\}$$

is dense in X and T is called *hypercyclic* if it has a hypercyclic vector. It is easy to see that an orbit of a hypercyclic vector is always dense countable and linearly independent. For a topological vector space X, we denote the set of all countable dense linearly independent subsets of X by the symbol $\Sigma(X)$. Thus

$$O(T,x) \in \Sigma(X) \ \text{ if } x \text{ is a hypercyclic vector for } T.$$

For more information on hypercyclicity see books [2, 8] and references therein. For the sake of brevity, we shall say that a subset A of a topological vector space X is an *orbit* if there are $T \in L(X)$ and $x \in X$ such that $A = O(T,x)$. If a group G acts on a set X and Σ is a family of subsets of X, we say that G *acts transitively on* Σ if $T(A) \in \Sigma$ for every $A \in \Sigma$ and for each $A, B \in \Sigma$ there exists $T \in G$ such that $T(A) = B$.

The starting point for this article is the theorem by Grivaux [7] stating that every countable dense linearly independent subset of a separable infinite dimensional Banach space is an orbit of a hypercyclic operator and thus solving a problem of Halperin, Kitai and Rosenthal [6]. This result easily follows from another theorem in [7]:

Theorem G. *For every separable infinite dimensional Banach space X, $GL(X)$ acts transitively on $\Sigma(X)$.*

The above theorem leads to the following definition.

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Definition 1.1. A locally convex topological vector space X is called a *G-space* if $\Sigma(X)$ is non-empty and $GL(X)$ acts transitively on $\Sigma(X)$.

Thus Theorem G states that every separable infinite dimensional Banach space is a G-space. For the convenience of the reader we reproduce the derivation of the main result in [7] from Theorem G.

Lemma 1.2. *Let X be a G-space possessing a hypercyclic operator $T_0 \in L(X)$. Then every $A \in \Sigma(X)$ is an orbit.*

Proof. Let x_0 be a hypercyclic vector for $T_0 \in L(X)$ and $A \in \Sigma(X)$. Since X is a G-space and both A and $O(T_0, x)$ belong to $\Sigma(X)$, there is $J \in GL(X)$ such that $J^{-1}(A) = O(T_0, x_0)$. Let $T = JT_0J^{-1}$ and $x = Jx_0$. Then

$$O(T, x) = O(JT_0J^{-1}, Jx_0) = \{JT_0^n x : n \in \mathbb{Z}_+\} = J(O(T_0, x_0)) = A.$$

Thus A is an orbit. $\square$

Recently Albanese [1] extended the result of Grivaux to Fréchet spaces with continuous norms. Namely, she proved that if X is a separable infinite dimensional Fréchet space possessing a continuous norm then every $A \in \Sigma(X)$ is an orbit. Note that due to Bonet and Peris [5] every separable infinite dimensional Fréchet space supports a hypercyclic operator. Thus Lemma 1.2 implies that for every Fréchet space X, which is also a G-space, every $A \in \Sigma(X)$ is an orbit of a hypercyclic operator. Just as in [7], Albanese proves the result by means of showing that $GL(X)$ acts transitively on $\Sigma(X)$.

Theorem A. *For every separable infinite dimensional Fréchet space X possessing a continuous norm, $GL(X)$ acts transitively on $\Sigma(X)$. That is, X is a G-space.*

Note that Bonet, Frerick, Peris and Wengenroth [3] constructed $A \in \Sigma(\omega)$, which is not an orbit of a hypercyclic operator and therefore ω is not a G-space. Thus the natural question arises which Fréchet spaces are actually G-spaces. The following theorem gives an explicit answer to this question.

Theorem 1.3. *Let X be a separable infinite dimensional Fréchet space. Then the following statements are equivalent:*

(1.3.1) *X possesses a continuous norm;*
(1.3.2) *X is a G-space;*
(1.3.3) *every $A \in \Sigma(X)$ is an orbit.*

Note that the implication (1.3.2) $\iff$ (1.3.3) is the direct consequence of the above mentioned result of Bonet and Peris and the elementary Lemma 1.2, while the implication (1.3.1) $\Longrightarrow$ (1.3.2) is exactly Theorem A of Albanese. We include our proof of the last implication since it turns out to be an immediate consequence of the stronger Theorem 1.5 below, which we need anyway in order to prove the following result.

Theorem 1.4. *Every countably dimensional metrizable locally convex space possesses a hypercyclic operator.*

The above theorem answers a question raised in [9] and extends the above mentioned result of Grivaux stating that every countably dimensional normed space possesses a hypercyclic operator. It is worth noting that Bonet, Frerick, Peris and Wengenroth [3] constructed a countably dimensional locally convex space which supports no transitive (hence no hypercyclic) operators. It is also well-known (see, for instance, the same paper [3]) that there are separable (uncountably!) infinite dimensional metrizable locally convex spaces supporting no transitive operators. Theorem 1.4 is in sharp contrast with these results.

The following theorem is our main instrument. In order to formulate it we need to recall a few definitions. A subset D of a locally convex space X is called a *disk* if D is bounded, convex and balanced (=is stable under multiplication by $\lambda \in \mathbb{K}$ with $|\lambda| \leqslant 1$). The symbol X_D stands for the space $\mathtt{span}\,(D)$ endowed with the norm $p_{_D}$ being the Minkowski functional of the set D. Boundedness of D implies that the topology of X_D is stronger than the one inherited from X. A disk D in X is called a *Banach disk* if the normed space

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X_D is complete. It is well-known that a sequentially complete disk is a Banach disk, see, for instance, [4]. In particular, every compact or sequentially compact disk is a Banach disk.

We say that a seminorm p on a vector space X is *non-trivial* if $X/\texttt{ker}\,p$ is infinite dimensional, where

$$\texttt{ker}\,p = \{x \in X : p(x) = 0\}.$$

Note that the topology of a locally convex space X is non-weak if and only if there is a non-trivial continuous seminorm on X.

If p is a seminorm on a vector space X, we say that $A \subset X$ is *p-independent* if $p(z_1a_1 + \ldots + z_na_n) \neq 0$ for any $n \in \mathbb{N}$, any pairwise different $a_1, \ldots, a_n \in A$ and any non-zero $z_1, \ldots, z_n \in \mathbb{K}$. In other words, vectors $x + \texttt{ker}\,p$ for $x \in A$ are linearly independent in $X/\texttt{ker}\,p$.

Theorem 1.5. *Let X be a locally convex space, p be a continuous seminorm on X, D be a Banach disk in X and A, B be countable subsets of X such that both A and B are p-independent and both A and B are dense subsets of the Banach space X_D. Then there exists $J \in GL(X)$ such that $J(A) = B$ and $Jx = x$ for every $x \in \texttt{ker}\,p$.*

We prove Theorem 1.5 in Section 2. In Section 3 we show that $GL(\omega)$ acts transitively on the set of dense countably dimensional subspaces of ω. Section 4 is devoted to the proof of Theorem 1.3. We prove Theorem 1.4 in Section 5 and discuss open problems in Section 6.

2 Proof of Theorem 1.5

For a continuous seminorm p on a locally convex space X, we denote

$$X'_p = \{f \in X' : p^*(f) = \sup\{|f(x)| : x \in X,\ p(x) \leqslant 1\} < \infty\}.$$

Note that $p^*(f)$ (if finite) is the smallest non-negative number c such that $f(x) \leqslant cp(x)$ for every $x \in X$.

Lemma 2.1. *Let X be a locally convex space, p be a continuous seminorm on X, D be a Banach disk in X and $\{x_n\}_{n\in\mathbb{N}}$ and $\{f_n\}_{n\in\mathbb{N}}$ be sequences in X_D and X' respectively such that $c = \sum\limits_{n=1}^{\infty} p^*(f_n)p_D(x_n) < \infty$. Then the formula $Tx = \sum\limits_{n=1}^{\infty} f_n(x)x_n$ defines a continuous linear operator on X. Furthermore, if p is bounded by 1 on D and $c < 1$, then the operator $I + T$ is invertible.*

Proof. Clearly $p_D(f_n(x)x_n) = |f_n(x)|p_D(x_n) \leqslant p(x)p^*(f_n)p_D(x_n)$. It follows that the series defining Tx converges in X_D and

$$p_D(Tx) \leqslant cp(x) \ \text{ for every } x \in X.$$

Thus T is a well-defined continuous linear map from X to X_D. Since the topology of X_D is stronger than the one inherited from X, $T \in L(X)$.

Assume now that p is bounded by 1 on D and $c < 1$. Then from the inequality $p(x) \leqslant p_D(x)$ and the above display it follows that $p_D(T^nx) \leqslant c^np(x)$ for every $x \in X$ and $n \in \mathbb{N}$. Since $c < 1$, the formula $Sx = \sum\limits_{n=1}^{\infty} (-T)^nx$ defines a linear map from X to X_D satisfying $p_D(Sx) \leqslant \frac{c}{1-c}p(x)$ for $x \in X$. Thus S is continuous as a map from X to X_D and therefore $S \in L(X)$. It is a routine exercise to check that $(I + S)(I + T) = (I + T)(I + S) = I$. That is, $I + T$ is invertible. ☐

Lemma 2.2. *Let $\varepsilon > 0$, X be a locally convex space, D be a Banach disk in X, Y be a closed linear subspace of X, $M \subseteq Y \cap X_D$ be a dense subset of Y such that M is p_D-dense in $Y \cap X_D$, p be a continuous seminorm on X, L be a finite dimensional subspace of X and $T \in L(X)$ be a finite rank operator such that $T(Y) \subseteq Y \cap X_D$, $T(\texttt{ker}\,p) \subseteq \texttt{ker}\,p$ and $\texttt{ker}\,(I + T) = \{0\}$. Then*

1. *for every $u \in Y \cap X_D$ such that $(u + L) \cap \texttt{ker}\,p = \varnothing$, there are $f \in X'$ and $v \in Y \cap X_D$ such that $p^*(f) = 1$, $f\big|_L = 0$, $p_D(v) < \varepsilon$, $(I + R)u \in M$ and $\texttt{ker}\,(I + R) = \{0\}$, where $Rx = Tx + f(x)v$;*

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(2) *for every* $u \in Y \cap X_D$ *such that* $(u + (I+T)(L)) \cap \ker p = \varnothing$, *there are* $f \in X'$, $a \in M$ *and* $v \in Y \cap X_D$ *such that* $p^*(f) = 1$, $f\big|_L = 0$, $p_D(v) < \varepsilon$, $(I+R)a = u$ *and* $\ker(I+R) = \{0\}$, *where* $Rx = Tx + f(x)v$.

Proof. Let $u \in Y \cap X_D$ be such that $(u+L) \cap \ker p = \varnothing$. The Hahn–Banach theorem provides $f \in X'$ such that $p^*(f) = 1$, $f(u) \neq 0$ and $f\big|_L = 0$. First, observe that there is $\delta > 0$ such that the only solution of the equation $Tx + x + f(x)v = 0$ is $x = 0$ whenever $v \in X_D$ and $p_D(v) < \delta$. Indeed, assume the contrary. Then there exist sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ in X_D such that $p_D(x_n) = 1$ for every $n \in \mathbb{N}$, $p_D(v_n) \to 0$ and $Tx_n + x_n + f(x_n)v_n = 0$ for each $n \in \mathbb{N}$. Since $\{Tx_n\}_{n\in\mathbb{N}}$ is a bounded sequence in the finite dimensional subspace $T(X)$ of X_D, passing to a subsequence, if necessary, we can without loss of generality assume that x_n converges to $x \in T(X) \subset X_D$ with respect to p_D. That is, $p_D(x) = 1$ and $p_D(x_n - x) \to 0$. Passing to the p_D-limit in $Tx_n + x_n + f(x_n)v_n = 0$, we obtain $Tx + x = 0$, which contradicts the equality $\ker(I+T) = \{0\}$. Thus there is $\delta > 0$ such that the only solution of the equation $Tx + x = f(x)v$ is $x = 0$ whenever $v \in X_D$ and $p_D(v) < \delta$. Since M is p_D-dense in $Y \cap X_D$ and $u + Tu \in Y \cap X_D$, there is $r \in M$ such that $p_D(r - u - Tu) < \min\{\delta|f(u)|, \varepsilon|f(u)|\}$. Define $v = \frac{1}{f(u)}(r - u - Tu) \in Y \cap X_D$ and $Rx = Tx + f(x)v$. Clearly, $p_D(v) < \varepsilon$. Since $p_D(v) < \delta$, $\ker(I+R) = \{0\}$. A direct computation gives $(I+R)u = u + Tu + \frac{1}{f(u)}f(u)(r - u - Tu) = r \in M$. Thus f and v satisfy all desired conditions.

Now assume that $u \in Y \cap X_D$ is such that $(u + (I+T)(L)) \cap \ker p = \varnothing$. The fact that T has finite rank and $\ker(I+T) = \{0\}$ implies that $I+T$ is invertible. Furthermore, the inclusion $T(X) \subseteq Y \cap X_D$ and the finiteness of the rank of T imply that $(I+T)^{-1}(Y) \subseteq Y$ and $(I+T)^{-1}(X_D) \subseteq X_D$. Thus there is a unique $w \in Y \cap X_D$ such that $(I+T)w = u$. Since $(u + (I+T)(L)) \cap \ker p = \varnothing$ and $T(\ker p) \subseteq \ker p$, we have $(w+L) \cap \ker p = \varnothing$. The Hahn–Banach theorem provided $f \in X'$ such that $p^*(f) = 1$, $f(w) \neq 0$ and $f\big|_L = 0$. Exactly as in the first part of the proof, we observe that there is $\delta > 0$ such that the only solution of the equation $Tx + x + f(x)v = 0$ is $x = 0$ whenever $v \in X_D$ and $p_D(v) < \delta$. Since $f(w) \neq 0$ and M is p_D-dense in $Y \cap X_D$, we can find $a \in M$ (close enough to w with respect to p_D) such that $p_D(v) < \delta$ and $p_D(v) < \delta$, where $v = \frac{1}{f(a)}(I+T)(w-a)$. Now set $Rx = Tx + f(x)v$. Since $p_D(v) < \delta$, $\ker(I+R) = \{0\}$. A direct computation gives $(I+R)a = a + Ta + \frac{1}{f(a)}f(a)(I+T)(w-a) = (I+T)w = u$. Thus f, a and v satisfy all desired conditions. ☐

Now we are ready to prove Theorem 1.5. Without loss of generality, we may assume that p is bounded by 1 on D. Equivalently, $p(x) \leqslant p_D(x)$ for every $x \in X_D$.

Fix arbitrary bijections $a : \mathbb{N} \to A$ and $b : \mathbb{N} \to B$ and a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive numbers such that $\sum\limits_{n=1}^{\infty} \varepsilon_n < 1$. We shall construct inductively sequences $\{n_k\}_{k\in\mathbb{N}}$ and $\{m_k\}_{k\in\mathbb{N}}$ of positive integers, $\{v_k\}_{k\in\mathbb{N}}$ in X_D, $\{f_k\}_{k\in\mathbb{N}}$ in X'_p and $\{T_n\}_{n\in\mathbb{Z}_+}$ in $L(X)$ such that $T_0 = 0$ and for every $k \in \mathbb{N}$,

$$m_j \neq m_l \ \text{ and } \ n_j \neq n_l \ \text{ for } 1 \leqslant j < l \leqslant 2k; \tag{2.1}$$

$$\{1, \dots, k\} \subseteq \{n_1, \dots, n_{2k}\} \cap \{m_1, \dots, m_{2k}\} \ \text{ for } k \geqslant 1; \tag{2.2}$$

$$p_D(v_j) < \varepsilon_j \text{ and } p^*(f_j) \leqslant 1 \text{ for } 1 \leqslant j \leqslant 2k; \tag{2.3}$$

$$(T_k - T_{k-1})x = f_{2k-1}(x)v_{2k-1} + f_{2k}(x)v_{2k}; \tag{2.4}$$

$$(I + T_k)a(n_j) = b(m_j) \ \text{ for } 1 \leqslant j \leqslant 2k. \tag{2.5}$$

$T_0 = 0$ serves as the basis of induction. Let $q \geqslant 1$ and assume that m_j, n_j, v_j, f_j, T_j for $j \leqslant 2q-2$ satisfying (2.1–2.5) are already constructed. By (2.4) with $k < q$,

$$T_{q-1}x = \sum_{j=1}^{2q-2} f_j(x)v_j.$$

According to (2.3) with $k = q-1$, $\sum\limits_{j=1}^{2q-2} p^*(f_j)p_D(v_j) < \sum\limits_{j=1}^{2n-2} \varepsilon_j < 1$. By Lemma 2.1, $I + T_{q-1}$ is invertible. Since each $p^*(f_j)$ is finite, T_{q-1} vanishes on $\ker p$. In particular, $T_{q-1}(\ker p) \subseteq \ker p$. Since each v_j belongs to X_D, $T_{q-1}(X) \subseteq X_D$. Clearly T_{q-1} has finite rank. Set $Y = X$, $n_{2q-1} = \min(\mathbb{N} \setminus \{n_1, \dots, n_{2q-2}\})$ and $m_{2q} = \min(\mathbb{N} \setminus \{m_1, \dots, m_{2q-2}\})$. Since A is p-independent $(u+L) \cap \ker p = \varnothing$, where $u = a(n_{2q-1})$

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and $L = \texttt{span}\{a(n_1), \dots, a_{n_{2q-2}}\}$. Applying the first part of Lemma 2.2 with the just defined u, Y and L and with $T = T_{q-1}$, $\varepsilon = \varepsilon_{2q-1}$ and $M = B \setminus \{b(m_{2q}), b(m_1), b(m_2), \dots, b(m_{2q-2})\}$, we find $f_{2q-1} \in X'$ and $v_{2q-1} \in X_D$ such that $p^*(f_{2q-1}) = 1$, $f_{2q-1}\big|_L = 0$, $p_D(v_{2q-1}) < \varepsilon_{2q-1}$ and $(I+S)u \in M$, where $Sx = T_{q-1}x + f_{2q-1}(x)v_{2q-1}$. The inclusion $(I+S)u \in M$ means that $(I+S)u = b(m_{2q-1})$ for some $m_{2q-1} \in \mathbb{N} \setminus \{m_{2q}, m_1, m_2, \dots, m_{2q-2}\}$. Since $u = a(n_{2q-1})$ and $f_{2q-1}\big|_L = 0$, (2.5) with $k = q-1$ implies that

$$(I+S)a(m_j) = b(n_j) \ \text{ for } \ 1 \leqslant j \leqslant 2q-1.$$

By definition of S, $Sx = \sum\limits_{j=1}^{2q-1} f_j(x)v_j$ with $\sum\limits_{j=1}^{2q-1} p^*(f_j)p_D(v_j) < \sum\limits_{j=1}^{2q-1} \varepsilon_j < 1$. By Lemma 2.1, $I+S$ is invertible. Since each $p^*(f_j)$ is finite and each v_j belongs to X_D, S vanishes on $\texttt{ker}\, p$ and $S(X) \subseteq X_D$. Clearly S has finite rank. Since B is p-independent the above display ensures that $(u + (I+S)(L)) \cap \texttt{ker}\, p = \varnothing$, where $u = b(m_{2q})$ and $L = \texttt{span}\{a(n_j) : 1 \leqslant j \leqslant 2q-1\}$. Applying the second part of Lemma 2.2 with $Y = X$ and the just defined u and L and with $T = S$, $\varepsilon = \varepsilon_{2q}$ and $M = A \setminus \{a(n_j) : 1 \leqslant j \leqslant 2q-1\}$, we find $f_{2q} \in X'_p$, $v_{2q} \in X_D$ and $w \in M$ such that $p^*(f_{2q}) = 1$, $f_{2q}\big|_L = 0$, $p_D(v_{2q}) < \varepsilon_{2q}$ and $(I + T_q)w = u$, where $T_qx = Sx + f_{2n}(x)v_{2n}$. The inclusion $w \in M$ means that $w = a(n_{2q})$ for some $n_{2q} \in \mathbb{N} \setminus \{n_1, \dots, n_{2q-1}\}$. Since $u = b(m_{2q})$ and $f_{2n}\big|_L = 0$, the above display yields

$$(I + T_q)a(m_j) = b(n_j) \ \text{ for } \ 1 \leqslant j \leqslant 2q.$$

Since $n_{2q-1} \neq n_{2q}$, $m_{2q-1} \neq m_{2q}$, $n_{2q-1}, n_{2q} \notin \{n_1, \dots, n_{2q-2}\}$ and $m_{2q-1}, m_{2q} \notin \{n_1, \dots, m_{2q-2}\}$, (2.1) with $k = q$ follow from (2.1) with $k = q-1$. By construction, (2.3), (2.4) and (2.5) with $k = q$ are satisfied. Since $n_{2q-1} = \min(\mathbb{N} \setminus \{n_1, \dots, n_{2q-2}\})$ and $m_{2q} = \min(\mathbb{N} \setminus \{m_1, \dots, m_{2q-2}\})$, (2.2) for $k = q$ follows from (2.2) with $k = q-1$. Thus (2.1–2.5) are all satisfied for $k = q$. This concludes the inductive construction of m_j, n_j, v_j, f_j, T_j for $j \in \mathbb{N}$.

By (2.1) and (2.2), the map $n_j \mapsto m_j$ is a bijection from $\mathbb{N}$ to itself. By (2.3) and (2.4), the sequence $\{T_n\}$ converges pointwise to the operator $T \in L(X)$ given by the formula $Tx = \sum\limits_{j=1}^{\infty} f_j(x)v_j$. Since $\sum \varepsilon_j < 1$, (2.3) and Lemma 2.1 imply that $J = I + T$ is invertible. Since $p^*(f_j) < \infty$ for every j, T vanishes on $\texttt{ker}\, p$ and therefore $Jx = x$ for $x \in \texttt{ker}\, p$. Passing to the limit in (2.5), we obtain that $Ja(n_j) = b(m_j)$ for every $j \in \mathbb{N}$. Since $n_j \mapsto m_j$ is a bijection from $\mathbb{N}$ to itself, we get $J(A) = B$. Thus J satisfies all required conditions. The proof of Theorem 1.5 is now complete.

3 Countably dimensional subspaces of ω

The main result of this section is the following theorem.

Theorem 3.1. *$GL(\omega)$ acts transitively on the set of dense countably dimensional linear subspaces of ω.*

Note that according to [3, Proposition 3.3], $GL(\omega)$ does not act transitively on $\Sigma(\omega)$. In order to prove the above result we need few technical lemmas. As usual, we identify ω with $\mathbb{K}^{\mathbb{N}}$. For $n \in \mathbb{N}$, the symbol δ_n stands for the n^{th} coordinate functional on ω. That is, $\delta_n \in \omega'$ is defined by $\delta_n(x) = x_n$. By φ we denote the linear subspace of ω consisting of sequences with finite support. That is, $x \in \varphi$ precisely when there is $n \in \mathbb{N}$ such that $x_m = 0$ for all $m \geqslant n$.

Lemma 3.2. *Let $f_1, \dots, f_{n+1}$ be linearly independent functionals on a vector space E, $A \subseteq E$ be such that $\texttt{span}(A) = E$ and $x_1, \dots, x_n \in E$ be such that the matrix $\{f_j(x_k)\}_{1 \leqslant j,k \leqslant n}$ is invertible. Then there exists $x_{n+1} \in A$ such that $\{f_j(x_k)\}_{1 \leqslant j,k \leqslant n+1}$ is invertible.*

Proof. Since the matrix $B = \{f_j(x_k)\}_{1 \leqslant j,k \leqslant n}$ is invertible, the vector $(f_{n+1}(x_1), \dots, f_{n+1}(x_n)) \in \mathbb{K}^n$ is a linear combination of the rows of B. That is, there exist $c_1, \dots, c_n \in \mathbb{K}$ such that $g(x_j) = 0$ for $1 \leqslant j \leqslant n$, where $g = f_{n+1} - \sum\limits_{j=1}^{n} c_j f_j$. Since f_j are linearly independent, $g \neq 0$. Since $\texttt{span}(A) = E$, we can find $x_{n+1} \in A$ such that $g(x_{n+1}) \neq 0$. Consider the $(n+1) \times (n+1)$ matrix $C = \{\gamma_{j,k}\}_{1 \leqslant j,k \leqslant n+1}$ defined by $\gamma_{j,k} = f_j(x_k)$ for $1 \leqslant j \leqslant n$, $1 \leqslant k \leqslant n+1$ and $\gamma_{n+1,k} = g(x_k)$ for $1 \leqslant k \leqslant n+1$. Since $\{\gamma_{j,k}\}_{1 \leqslant j,k \leqslant n} = B$

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is invertible, $\gamma_{n+1,k} = g(x_k) = 0$ for $1 \leqslant k \leqslant n$ and $\gamma_{n+1,n+1} = g(x_{n+1}) \neq 0$, C is invertible. Indeed, $\mathtt{det}\, C = g(x_{n+1})\mathtt{det}\, B$. It remains to notice that $B^+ = \{f_j(x_k)\}_{1\leqslant j,k\leqslant n+1}$ is obtained from C by adding a linear combination of the first n rows to the last row. Hence $\mathtt{det}\, B^+ = \mathtt{det}\, C \neq 0$ and B^+ is invertible as required. ☐

Applying Lemma 3.2 and treating the elements of a vector space E as linear functionals on a space of linear functionals on E, we immediately get the following result.

Lemma 3.3. *Let $x_1, \dots, x_{n+1}$ be linearly independent elements of a vector space E, A be a collection of linear functionals on E separating the points of E and $f_1, \dots, f_n$ be linear functionals on E such that the matrix $\{f_j(x_k)\}_{1\leqslant j,k\leqslant n}$ is invertible. Then there exists $f_{n+1} \in A$ such that $\{f_j(x_k)\}_{1\leqslant j,k\leqslant n+1}$ is invertible.*

Lemma 3.4. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a Hamel basis in a vector space E and $\{f_n\}_{n\in\mathbb{N}}$ be a linearly independent sequence of linear functionals on E separating the points of E. Then there exist bijections $\alpha : \mathbb{N} \to \mathbb{N}$ and $\beta : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$, the matrix $\{f_{\alpha(j)}(x_{\beta(k)})\}_{1\leqslant j,k\leqslant n}$ is invertible.*

Furthermore, there exist complex numbers $c_{j,k}$ for $j \leqslant k$ such that $c_{j,j} \neq 0$ and $f_{\alpha(j)}(v_j) = 1$ for $j \in \mathbb{N}$ and $f_{\alpha(j)}(v_k) = 0$ for $j, k \in \mathbb{N}$ and $j < k$, where $v_k = \sum\limits_{m=1}^{k} c_{m,k}u_{\beta(m)}$.

Proof. We shall construct inductively two sequences $\{\alpha_j\}_{j\in\mathbb{N}}$ and $\{\beta_k\}_{k\in\mathbb{N}}$ of natural numbers such that

$$\{1, \dots, n\} \subseteq \{\alpha_1, \dots, \alpha_{2n}\} \cap \{\beta_1, \dots, \beta_{2n}\} \text{ for each } n \in \mathbb{N}; \tag{3.1}$$

$$\{f_{\alpha_j}(x_{\beta_k})\}_{1\leqslant j,k\leqslant n} \text{ is invertible for every } n \in \mathbb{N}. \tag{3.2}$$

Basis of induction. Take $\alpha_1 = 1$. Since $f_1 \neq 0$ and the vectors u_n span E, there is $\beta_1 \in \mathbb{N}$ such that $f_{\alpha_1}(u_{\beta_1}) \neq 0$. Now we take $\beta_2 = \min(\mathbb{N}\setminus\{\beta_1\})$. By Lemma 3.3, there is $\alpha_2 \in \mathbb{N}$ such that $\{f_{\alpha_j}(x_{\beta_k})\}_{1\leqslant j,k\leqslant 2}$ is invertible. Clearly, $1 \in \{\alpha_1, \alpha_2\} \cap \{\beta_1, \beta_2\}$. Thus $\alpha_1, \alpha_2, \beta_1, \beta_2$ satisfy (3.1) and (3.2).

The induction step. Assume that $m \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_{2m}, \beta_1, \dots, \beta_{2m}$ satisfying (3.1) with $n \leqslant m$ and (3.2) with $n \leqslant 2m$ are already constructed. The latter implies that β_j are pairwise distinct and α_j are pairwise distinct. First, take $\alpha_{2m+1} = \min(\mathbb{N} \setminus \{\alpha_1, \dots, \alpha_{2m}\})$. By Lemma 3.2, there is $\beta_{2m+1} \in \mathbb{N}$ such that $\{f_{\alpha_j}(x_{\beta_k})\}_{1\leqslant j,k\leqslant 2m+1}$ is invertible. Automatically, $\beta_{m+1} \notin \{\beta_1, \dots, \beta_{2m}\}$. Next, we take $\beta_{2m+2} = \min(\mathbb{N} \setminus \{\beta_1, \dots, \beta_{2m+1}\})$. By Lemma 3.3, there is $\alpha_{2m+2} \in \mathbb{N}$ such that $\{f_{\alpha_j}(x_{\beta_k})\}_{1\leqslant j,k\leqslant 2m+2}$ is invertible. Since $\{1, \dots, m\} \subseteq \{\alpha_1, \dots, \alpha_{2m}\} \cap \{\beta_1, \dots, \beta_{2m}\}$, $\alpha_{2m+1} = \min(\mathbb{N} \setminus \{\alpha_1, \dots, \alpha_{2m}\})$ and $\beta_{2m+2} = \min(\mathbb{N} \setminus \{\beta_1, \dots, \beta_{2m+1}\})$, we have $\{1, \dots, m+1\} \subseteq \{\alpha_1, \dots, \alpha_{2m+2}\} \cap \{\beta_1, \dots, \beta_{2m+2}\}$. Thus $\alpha_1, \dots, \alpha_{2m+2}, \beta_1, \dots, \beta_{2m+2}$ satisfy (3.1) with $n \leqslant m+1$ and (3.2) with $n \leqslant 2m+2$.

This concludes the inductive construction of $\{\alpha_j\}_{j\in\mathbb{N}}$ and $\{\beta_k\}_{k\in\mathbb{N}}$ satisfying (3.1) and (3.2). According to (3.2), α_j are pairwise distinct and β_j are pairwise distinct. By (3.1), $\{\alpha_j : j \in \mathbb{N}\} = \{\beta_k : k \in \mathbb{N}\} = \mathbb{N}$. Hence the maps $\alpha, \beta : \mathbb{N} \to \mathbb{N}$ defined by $\alpha(j) = \alpha_j$ and $\beta(j) = \beta_j$ are bijections. By (3.2), the matrix $\{f_{\alpha(j)}(x_{\beta(k)})\}_{1\leqslant j,k\leqslant n}$ is invertible for every $n \in \mathbb{N}$.

Now let $m \in \mathbb{N}$. Since $A_m = \{f_{\alpha(j)}(x_{\beta(k)})\}_{1\leqslant j,k\leqslant m}$ is invertible, we can find $c_{1,m}, c_{2,m}, \dots, c_{m,m}$ such that the linear combination of the columns of A_m with the coefficients $c_{1,m}, \dots, c_{m,m}$ is the vector $(0, \dots, 0, 1)$. Note that $c_{m,m}$ can not be 0. Indeed, otherwise a non-trivial linear combination of the columns of the invertible matrix A_{m-1} is 0. The fact that the linear combination of the columns of A_m with the coefficients $c_{1,m}, \dots, c_{m,m}$ is $(0, \dots, 0, 1)$ can be rewritten as $f_{\alpha(m)}(v_m) = 1$ and $f_{\alpha(j)}(v_m) = 0$ for $j < m$, where $v_m = \sum\limits_{j=1}^{m} c_{j,m}u_{\beta(j)}$. Doing this for every $m \in \mathbb{N}$, we obtain the numbers $\{c_{j,m}\}$ and the vectors v_m satisfying all desired conditions. ☐

Lemma 3.5. *Let E be a dense countably dimensional linear subspace of ω. Then there is a Hamel basis $\{v_n\}_{n\in\mathbb{N}}$ in E and a bijection $\alpha : \mathbb{N} \to \mathbb{N}$ such that $\delta_{\alpha(n)}(v_n) = 1$ and $\delta_{\alpha(k)}(v_n) = 0$ whenever $n \in \mathbb{N}$ and $k < n$.*

Proof. Take an arbitrary Hamel basis $\{u_n\}_{n\in\mathbb{N}}$ in E. Applying Lemma 3.4 with $f_n = \delta_n$, we find bijections $\alpha, \beta : \mathbb{N} \to \mathbb{N}$ and complex numbers $c_{j,k}$ for $j \leqslant k$ such that $c_{j,j} \neq 0$ and $f_{\alpha(j)}(v_j) = 1$ for $j \in \mathbb{N}$ and $f_{\alpha(j)}(v_k) = 0$ for $j, k \in \mathbb{N}$ and $j < k$, where $v_k = \sum\limits_{m=1}^{k} c_{m,k}u_{\beta(m)}$.

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It remains to notice that since $\{u_n\}$ is a Hamel basis in E, $\{v_n\}$ is also a Hamel basis in E. Indeed, it is straightforward to verify that $u_{\beta(n)} \in \mathtt{span}\,\{v_1,\dots,v_n\} \setminus \mathtt{span}\,\{v_1,\dots,v_{n-1}\}$ for every $n \in \mathbb{N}$. Thus the Hamel basis $\{u_n\}$ and the bijection α satisfy all desired conditions. $\square$

Proof of Theorem 3.1. Let E be a dense countably dimensional subspace of ω. By Lemma 3.5, there is a Hamel basis $\{v_n\}_{n\in\mathbb{N}}$ in E and a bijection $\alpha : \mathbb{N} \to \mathbb{N}$ such that $\delta_{\alpha(n)}(v_n) = 1$ and $\delta_{\alpha(k)}(v_n) = 0$ whenever $n \in \mathbb{N}$ and $k < n$. Consider $T : \omega \to \omega$ defined by the formula

$$Tx = \sum_{n=1}^{\infty} x_{\alpha(n)} v_n = \sum_{n=1}^{\infty} \delta_{\alpha(n)}(x) v_n.$$

If $\{e_j\}_{j\in\mathbb{N}}$ is the standard basis of ω, then it is easy to see that the matrix of T with respect to the 'shuffled' basis $\{e_{\alpha(j)}\}_{j\in\mathbb{N}}$ is lower-triangular with all entries 1 on the main diagonal. It follows that T is a well-defined invertible continuous linear operator on ω. It remains to observe that $T(\varphi) = E$. Hence each dense countably dimensional subspace of ω is the image of φ under an isomorphism of ω onto itself. Hence isomorphisms of ω act transitively on the set of dense countably dimensional linear subspaces of ω. $\square$

4 Proof of Theorem 1.3

The most of the following lemma (density bit excluded) is a particular case of a number of well-known stronger results, see, for instance, [4, Section 3.2]. For example, it is known that in a sequentially complete locally convex space X the closed balanced convex hull D of a pre-compact metrizable subset A is compact and metrizable and therefore is a Banach disk. In this generality though the linear span of A may turn out to be non-dense in X_D. We include the complete proof of the particular case when A is a convergent to 0 sequence for the sake of the reader's convenience.

Lemma 4.1. *Let $\{x_n\}_{n\in\mathbb{Z}_+}$ be a sequence in a sequentially complete locally convex space X such that $x_n \to 0$. Then the set*

$$K = \left\{ \sum_{n=0}^{\infty} a_n x_n : a \in \ell_1,\ \|a\|_1 \leqslant 1 \right\}$$

is a Banach disk. Moreover, $E = \mathtt{span}\,\{x_n : n \in \mathbb{Z}_+\}$ is a dense linear subspace of the Banach space X_K.

Proof. Let $Q = \{a \in \ell_1 : \|a\|_1 \leqslant 1\}$ be endowed with the coordinatewise convergence topology. Then Q is metrizable and compact as a closed subspace of the compact metrizable space $\mathbb{D}^{\mathbb{Z}_+}$, where $\mathbb{D} = \{z \in \mathbb{K} : |z| \leqslant 1\}$. Obviously, the map $\Phi : Q \to K$, $\Phi(a) = \sum\limits_{n=0}^{\infty} a_n x_n$ is onto. Moreover, Φ is continuous. Indeed, let p be a continuous seminorm on X, $a \in Q$ and $\varepsilon > 0$. Since $x_n \to 0$, there is $m \in \mathbb{Z}_+$ such that $p(x_n) \leqslant \varepsilon$ for $n > m$. Set $\delta = \frac{\varepsilon}{1+p(x_0)+\dots+p(x_m)}$ and $W = \{b \in Q : |a_j - b_j| < \delta \ \text{ for } \ 0 \leqslant j \leqslant m\}$. Then W is a neighborhood of a in Q and for each $b \in W$, we have

$$p(\Phi(b) - \Phi(a)) = p\biggl(\sum_{n=0}^{\infty} (b_n - a_n) x_n \biggr) \leqslant \sum_{n=0}^{\infty} |b_n - a_n| p(x_n).$$

Since $p(x_n) < \varepsilon$ for $n > m$, $|a_n - b_n| < \delta$ for $n \leqslant m$ and $\|a\|_1 \leqslant 1$, $\|b\|_1 \leqslant 1$, we obtain

$$p(\Phi(b) - \Phi(a)) \leqslant \delta \sum_{n=0}^{m} p(x_m) + \varepsilon \sum_{n=m+1}^{\infty} |b_n - a_n| \leqslant 2\varepsilon + \delta \sum_{n=0}^{m} p(x_m).$$

Using the definition of δ, we see that $p(\Phi(b) - \Phi(a)) \leqslant 3\varepsilon$. Since a, p and ε are arbitrary, Φ is continuous. Thus K is compact and metrizable as a continuous image of a compact metrizable space. Obviously, K is convex and balanced. Hence K is a Banach disk (any compact disk is a Banach disk). It remains to show that E is dense in X_K. Take $u \in X_K$. Then there is $a \in \ell_1$ such that $u = \sum\limits_{k=0}^{\infty} a_k x_k$. Clearly, $u_n = \sum\limits_{k=0}^{n} a_k x_k \in E$. Then $p_K(u - u_n) = p_K\Bigl(\sum\limits_{k=n+1}^{\infty} a_k x_k \Bigr) \leqslant \sum\limits_{k=n+1}^{\infty} |a_k| \to 0$ as $n \to \infty$. Hence E is dense in X_K. $\square$

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Lemma 4.2. *Let X be a Fréchet space and A and B be dense countable subsets of X. Then there exists a Banach disk D in X such that both A and B are dense subsets of the Banach space (X_D, p_D).*

Proof. Let C be the set of all linear combinations of the elements of $A \cup B$ with rational coefficients. Obviously, C is countable. Pick a map $f : \mathbb{N} \to C$ such that $f^{-1}(x)$ is an infinite subset of $\mathbb{N}$ for every $x \in C$. Since A and B are dense in X, we can find maps $\alpha : \mathbb{N} \to A$ and $\beta : \mathbb{N} \to B$ such that $4^m(f(m) - \alpha(m)) \to 0$ and $4^m(f(m) - \beta(m)) \to 0$. Since A and B are countable, we can write $A = \{x_m : m \in \mathbb{N}\}$ and $B = \{y_m : m \in \mathbb{N}\}$. Using metrizability of X, we can find a sequence $\{\gamma_m\}_{m\in\mathbb{N}}$ of positive numbers such that $\gamma_m x_m \to 0$ and $\gamma_m y_m \to 0$. Enumerating the countable set

$$\{2^m(f(m) - \alpha(m)) : m \in \mathbb{N}\} \cup \{2^m(f(m) - \beta(m)) : m \in \mathbb{N}\} \cup \{\gamma_m x_m : m \in \mathbb{N}\} \cup \{\gamma_m y_m : m \in \mathbb{N}\}$$

as one (convergent to 0) sequence and applying Lemma 4.1 to this sequence, we find that there is a Banach disk D in X such that X_D contains A and B, the linear span of $A \cup B$ is p_D-dense in X_D and $f(m) - \alpha(m) \to 0$ and $f(m) - \beta(m) \to 0$ in X_D. The p_D-density of the linear span of $A \cup B$ in X_D implies the p_D-density of C in X_D. Taking into account that $f^{-1}(x)$ is infinite for every $x \in C$ and that α takes values in A, the p_D-density of C in X_D and the relation $p_D(f(m) - \alpha(m)) \to 0$ implies that A is p_D-dense in X_D. Similarly, B is p_D-dense in X_D. Thus D satisfies all required conditions. $\square$

Lemma 4.3. *Let X be a separable Fréchet space and p be a non-trivial continuous seminorm on X. Then for every dense countable set $A \subset X$, there is $B \subseteq A$ such that B is p-independent and dense in X.*

Proof. Let $\{U_n\}_{n\in\mathbb{N}}$ be a countable basis of the topology of X. We shall construct (inductively) a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of A such that for every $n \in \mathbb{N}$,

$$x_n \in U_n \ \text{ and } \ x_1, \dots, x_n \text{ are } p\text{-independent.} \tag{4.1}$$

Note that in every topological vector space, the linear span of a dense subset of a non-empty open set is a dense linear subspace. It follows that for each $n \in \mathbb{N}$,

$$\text{a proper closed linear subspace of } X \text{ can not contain } A \cap U_n. \tag{4.2}$$

Hence $A \cap U_1 \not\subseteq \mathtt{ker}\, p$. Thus we can pick $x_1 \in (A \cap U_1) \setminus \mathtt{ker}\, p$, which will serve as the basis of induction. Assume now that $m \in \mathbb{N}$ and $x_1, \dots, x_m$ satisfying (4.1) for $n \leqslant m$ are already constructed. Let L be the linear span of $x_1, \dots, x_m$. Since the sum of a closed subspace of a topological vector space and a finite dimensional subspace is always closed and the codimension of $\mathtt{ker}\, p$ in X is infinite, $L + \mathtt{ker}\, p$ is a proper closed linear subspace of X. By (4.2), we can pick $x_{m+1} \in (A \cap U_{m+1}) \setminus (\mathtt{ker}\, p + L)$. It is easy to see that $x_1, \dots, x_m, x_{m+1}$ satisfy (4.1) for $n \leqslant m + 1$, which concludes the inductive construction of $\{x_n\}_{n\in\mathbb{N}}$ satisfying (4.1) for every $n \in \mathbb{N}$. It remains to observe that $B = \{x_n : n \in \mathbb{N}\} \subseteq A$, B is dense in X since it meets each U_n and B is p-independent. $\square$

4.1 Proof of the implications (1.3.1)$\Longrightarrow$(1.3.2) and (1.3.2)$\Longrightarrow$(1.3.3)

Assume that a separable infinite dimensional Fréchet space X possesses a continuous norm p and that $A, B \in \Sigma(X)$. By Lemma 4.2, there is a Banach disk D in X such that both A and B are dense subsets of the Banach space X_D. By Theorem 1.5, there exists $J \in GL(X)$ such that $J(A) = B$. Thus $GL(X)$ acts transitively on $\Sigma(X)$. Since X is separable and metrizable, $\Sigma(X)$ is non-empty. Hence X is a G-space, which proves the implication (1.3.1)$\Longrightarrow$(1.3.2). Since every separable infinite dimensional Fréchet space supports a hypercyclic operator [5], Lemma 1.2 provides the implication (1.3.2) $\Longrightarrow$ (1.3.3).

4.2 Proof of the implication (1.3.3)$\Longrightarrow$(1.3.1)

Let X be a separable Fréchet space possessing no continuous norm. The implication (1.3.3)$\Longrightarrow$(1.3.1) will be verified if we show that there exists $A \in \Sigma(X)$, which is not an orbit of a continuous linear operator. If X is isomorphic to ω, the job is already done by Bonet, Frerick, Peris and Wengenroth [3, Proposition 3.3]. It remains to consider the case of X non-isomorphic to ω. Since X is a Fréchet space possessing no continuous

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norm and non-isomorphic to ω, the topology of X can be defined by an increasing sequence $\{p_n\}_{n\in\mathbb{N}}$ of seminorms such that p_1 is non-trivial and $\texttt{ker}\,p_n/\texttt{ker}\,p_{n+1}\neq\{0\}$ for each $n\in\mathbb{N}$. By Lemma 4.3, there is a dense in X countable p_1-independent set B. Since $\texttt{ker}\,p_n/\texttt{ker}\,p_{n+1}\neq\{0\}$ for each $n\in\mathbb{N}$, for each $n\in\mathbb{N}$, we can pick $x_n\in\texttt{ker}\,p_n\setminus\texttt{ker}\,p_{n+1}$. Let $C=\{x_n:n\in\mathbb{N}\}$ and $A=B\cup C$. Obviously A is a countable subset of X. Since B is dense in X and $B\subseteq A$, A is dense in X. Finally, the p_1-independence of B and the inclusions $x_n\in\texttt{ker}\,p_n\setminus\texttt{ker}\,p_{n+1}$ imply that A is linearly independent. Thus $A\in\Sigma(X)$. It suffices to verify that A is not an orbit. Assume the contrary. Then there are $T\in L(X)$ and $x\in X$ such that $A=O(T,x)$. Let $M=\{n\in\mathbb{Z}_+:T^nx\in C,\ T^{n+1}x\in B\}$. Since B does not meet $\texttt{ker}\,p_1$, $p_1(T^{n+1}x)>0$ for every $n\in M$. Thus we can consider the (finite or countable) series $S=\sum\limits_{n\in M}\frac{T^nx}{p_1(T^{n+1}x)}$. Since T^nx for $n\in M$ are pairwise distinct elements of C and every p_k vanishes on all but finitely many elements of C, the series S converges absolutely in X. Since $T:X\to X$ is a continuous linear operator and every continuous linear operator on a locally convex space maps an absolutely convergent series to an absolutely convergent series, the series $T(S)=\sum\limits_{n\in M}\frac{T^{n+1}x}{p_1(T^{n+1}x)}$ is also absolutely convergent. Hence the application of p_1 to the terms of $T(S)$ gives a convergent series of non-negative numbers. But the latter series is $\sum\limits_{n\in M}\frac{p_1(T^{n+1}x)}{p_1(T^{n+1}x)}=\sum\limits_{n\in M}1$. Its convergence is equivalent to the finiteness of M. Thus M is finite. Let $m=\max(M)$ if $M\neq\varnothing$ and $m=0$ if $M=\varnothing$. Since $C\subset O(T,x)$ and C is infinite, there is $k\in\mathbb{Z}_+$ such that $k>m$ and $T^kx\in C$. Since $M\cap\{j\in\mathbb{Z}_+:j\geqslant k\}=\varnothing$, from the definition of M it follows that $T^jx\in C$ for every $j\geqslant k$. Hence $T^jx\in C$ for all but finitely many j. It follows that $B=O(T,x)\setminus C$ is finite, which is a contradiction. This contradiction shows that A is not an orbit and completes the proof of the implication (1.3.3)$\Longrightarrow$(1.3.1) and that of Theorem 1.3.

5 Proof of Theorem 1.4

Lemma 5.1. *Let p be a continuous seminorm on a locally convex space X and E be a countably dimensional subspace of X such that $E\cap\texttt{ker}\,p=\{0\}$. Then there exist a Hamel basis $\{u_n\}_{n\in\mathbb{N}}$ in E and a sequence $\{f_n\}_{n\in\mathbb{N}}$ in X'_p such that $f_n(u_m)=\delta_{n,m}$ for every $m,n\in\mathbb{N}$.*

Proof. Begin with an arbitrary Hamel basis $\{y_n\}_{n\in\mathbb{N}}$ in E. The proof is a variation of the Gramm–Schmidt procedure. Clearly, it suffices to construct (inductively) two sequences $\{u_n\}_{n\in\mathbb{N}}$ in E and $\{f_n\}_{n\in\mathbb{N}}$ in X'_p such that for every $n\in\mathbb{N}$,

$$u_n\in y_n+\texttt{span}\,\{y_j:j<n\};\tag{5.1}$$

$$f_j(u_k)=\delta_{j,k}\ \text{ for } j,k\leqslant n.\tag{5.2}$$

Indeed, (5.1) ensures that $\{u_n:n\in\mathbb{N}\}$ is also a Hamel basis in E.

First, we set $u_1=y_1$ and note that $p(u_1)\neq 0$. Then we use the Hahn–Banach theorem to find $f_1\in X'_p$ such that $f_1(u_1)=1$. This gives us the basis of induction. Assume now that $m\geqslant 2$ and u_n,f_n satisfying (5.1) and (5.2) for $n<m$ are already constructed. Condition (5.2) for $n<m$ allows us to pick $u_m\in y_m+\texttt{span}\,\{y_n:n<m\}$ such that $f_j(u_n)=0$ for every $j<n$. Since y_n are linearly independent, $u_m\in E\setminus\{0\}$. Since $E\cap\texttt{ker}\,p=\{0\}$, $p(u_m)\neq 0$. Since $u_1,\dots,u_m$ are linearly independent elements of E and $p(u_m)\neq 0$, the Hahn–Banach theorem allows us to choose $f_m\in X'_p$ such that $f_m(u_m)=1$ and $f_m(u_j)=0$ for $j<m$. Clearly, u_n and f_n for $n\leqslant m$ satisfy (5.1) and (5.2) for $n\leqslant m$. This completes the inductive procedure of constructing the sequences $\{u_n\}_{n\in\mathbb{N}}$ in E and $\{f_n\}_{n\in\mathbb{N}}$ in X'_p satisfying (5.1) and (5.2) for every $n\in\mathbb{N}$. ☐

The following lemma features as [2, Theorem 2.2].

Lemma 5.2. *Let X be a separable Fréchet space and $T\in L(X)$ be such that the linear span of the union of $T^n(X)\cap\texttt{ker}\,T^n$ for $n\in\mathbb{N}$ is dense in X. Then $I+T$ is hypercyclic.*

Lemma 5.3. *Let p be a non-trivial continuous seminorm on a separable locally convex space X for which there exists a Banach disk D in X such that X_D is a dense subspace of X and the Banach space (X_D,p_D) is separable. Then there exists $T\in L(X)$ such that T is hypercyclic and $Tx=x$ for every $x\in\texttt{ker}\,p$.*

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Proof. Since X_D is dense in X and the Banach space topology on X_D is stronger than the one inherited from X, the restriction of p to X_D is a non-trivial continuous seminorm on the Banach space X_D. By Lemma 4.2, there is a dense countable subspace A of the Banach space X_D such that A is p-independent. Let $E = \mathtt{span}\,(A)$. Then E is a dense in (X_D, p_D) and therefore in X countably dimensional subspace of X_D. Since A is p-independent, $E \cap \mathtt{ker}\,p = \{0\}$. By Lemma 5.1, there is a Hamel basis $\{u_n\}_{n\in\mathbb{N}}$ in E and a sequence $\{f_n\}_{n\in\mathbb{N}}$ in X'_p such that $f_n(u_m) = \delta_{n,m}$ for every $m, n \in \mathbb{N}$.

Consider the linear map $S : X \to X_D$ defined by the formula:

$$Sx = \sum_{n=1}^{\infty} \frac{2^{-n} f_{n+1}(x)}{p_D(u_n)p^*(f_{n+1})} u_n.$$

The series in the above display converges absolutely in X_D since $|f_{n+1}(x)| \leqslant p(x)p^*(f_{n+1})$. Furthermore $p_D(Sx) \leqslant p(x)$ for every $x \in X$. Hence S is a well-defined continuous linear map from X to X_D. In particular, $S \in L(X)$ and the restriction S_D of S to X_D is a continuous linear operator on the Banach space X_D. Moreover, analyzing the action of S on u_k, it is easy to see that $S(E) = S_D(E) = E$ and therefore $E \subseteq S_D^n(X_D)$ for every $n \in \mathbb{N}$. Furthermore, $u_n \in \mathtt{ker}\,S_D^n$ for every $n \in \mathbb{N}$. Hence $E \subseteq \bigcup\limits_{n\in\mathbb{N}} \mathtt{ker}\,S_D^n$. Since E is dense in X_D, Lemma 5.2 implies that $T_D = I + S_D$ is a hypercyclic operator on the Banach space X_D. Since the topology of X_D is stronger than the one inherited from X and X_D is dense in X, every hypercyclic vector for T_D is also hypercyclic for $T = I + S \in L(X)$. Thus $T = I + S$ is hypercyclic. Next, p-boundedness of each f_k implies that each f_k vanishes on $\mathtt{ker}\,p$. Hence $\mathtt{ker}\,p \subseteq \mathtt{ker}\,S$ and therefore $Tx = x$ for every $x \in \mathtt{ker}\,p$. ☐

Now we are ready to prove Theorem 1.4. Let E be a countably dimensional metrizable locally convex space. Denote the completion of E by the symbol X. That is, X is a separable infinite dimensional Fréchet space and E is a dense countably dimensional subspace of E.

Case 1: X is non-isomorphic to ω. In this case the topology of X is non-weak and therefore X supports a non-trivial continuous seminorm p. By Lemma 4.3, there is a countable dense in X p-independent set B such that $B \subseteq E$. A standard application of Zorn's lemma provides a maximal by inclusion p-independent subset A of E containing B. Since $B \subseteq A$, A is dense in X. Since E is countably dimensional, A is countable (p-independence implies linear independence). By Lemma 4.2, every separable infinite dimensional Fréchet space contains a Banach disk K such that X_K is a separable Banach space and X_K is dense in X. Now by Lemma 5.3 there is a hypercyclic $T \in L(X)$ such that $Tx = x$ for every $x \in \mathtt{ker}\,p$. Let u be a hypercyclic vector for T. First, we shall verify that $O(T, u)$ is p-independent. Assume the contrary. Then there exists a non-zero polynomial r such that $r(T)u \in \mathtt{ker}\,p$. Then for every $n \in \mathbb{Z}_+$, we can write $t^n = r(t)q(t)+v(t)$, where q and v are polynomials and $\deg v < \deg r = d$. Hence $T^n u = q(T)r(T)u+v(T)u$. Since $r(T)u \in \mathtt{ker}\,p$ and $\mathtt{ker}\,p$ is invariant for T, $q(T)r(T)u \in \mathtt{ker}\,p$. Hence $O(T, u) \subseteq L + \mathtt{ker}\,p$, where $L = \mathtt{span}\,\{u, Tu, \dots, T^{d-1}u\}$. Since L is finite dimensional and $\mathtt{ker}\,p$ is a closed subspace of X of infinite codimension, $L + \mathtt{ker}\,p$ is a proper closed subspace of X. We have obtained a contradiction with the density of $O(T, u)$. Thus the countable dense in X set $O(T, u)$ is p-independent. Recall that A is also countable, dense in X and p-independent. By Lemma 4.2, there is a Banach disk D in X such that both A and $O(T, u)$ are dense subsets of the Banach space (X_D, p_D). By Theorem 1.5, there exists $J \in GL(X)$ such that $J(O(T, u)) = A$ and $Jx = x$ for every $x \in \mathtt{ker}\,p$. Let $S = JTJ^{-1}$. Exactly as in the proof of Lemma 1.2, one easily sees that Ju is a hypercyclic vector for S and that $O(S, Ju) = A$. In particular, $Ju \in A \subset E$. It remains to verify that $S(E) \subseteq E$. Indeed, in this case the restriction of S to E provides a continuous linear operator on E with Ju being its hypercyclic vector.

Let $x \in E$. It suffices to show that $Sx \in E$. The maximality of A implies that we can write $x = y + z$, where $y \in \mathtt{span}\,(A)$ and $z \in \mathtt{ker}\,p$. Since $A \subset E$, $y \in E$ and therefore $z = x - y \in E$. Since $A = O(S, Ju)$, $S(A) \subseteq A$. Hence $S(\mathtt{span}\,(A)) \subseteq \mathtt{span}\,(A) \subseteq E$. It follows that $Sy \in E$. Since $Tv = Jv = v$ for $v \in \mathtt{ker}\,p$, we have $Sv = v$ for $v \in \mathtt{ker}\,p$ and therefore $Sz = z$. Thus $Sx = Sy + Sz = Sy + z \in E$, as required. This completes the proof for Case 1.

Case 2: X is isomorphic to ω. It is well-known (see, for instance, [5]) that ω supports a hypercyclic operator. Actually, it is easy to see that the shift $S \in L(\mathbb{K}^{\mathbb{N}})$, $(Sx)_n = x_{n+1}$ is hypercyclic. Thus, we can take $S \in L(X)$ and $x \in X$ such that x is a hypercyclic vector for S and let $F = \mathtt{span}\,(O(S, x))$. Then

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F is another dense countably dimensional subspace of X. Obviously F supports a hypercyclic operator (the restriction of S to F). By Theorem 3.1, E and F are isomorphic. Hence E supports a hypercyclic operator. The proof of Theorem 1.4 is now complete.

6 Open problems and remarks

Note that the locally convex direct sum φ of countably many copies of the one-dimensional space $\mathbb{K}$ is a complete countably dimensional locally convex space. A number of authors, see, for instance, [5], have observed that φ supports no hypercyclic operators.

Problem 6.1. *Characterize countably dimensional locally convex spaces supporting a hypercyclic operator.*

The following is an interesting special case of the above problem.

Problem 6.2. *Are there any complete countably dimensional locally convex spaces supporting a hypercyclic operator?*

The following question also seems to be interesting.

Problem 6.3. *Characterize complete G-spaces. Characterize complete G-spaces supporting a hypercyclic operator.*

Note that although ω is not a G-space, Theorem 3.1 shows that $GL(\omega)$ acts transitively on the set of dense countably dimensional subspaces of ω.

Problem 6.4. *Characterize complete locally convex spaces X with the property that $GL(X)$ acts transitively on the set of dense countably dimensional subspaces of X.*

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