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Infinite dimensional Banach spaces of functions with nonlinear properties

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The aim of this paper is to show that there exist infinite dimensional Banach spaces of functions that, except for 0, satisfy properties that apparently should be destroyed by the linear combination of two of them. Three of these spaces are: a Banach space of differentiable functions on $\mathbb{R}^n$ failing the Denjoy-Clarkson property; a Banach space of non Riemann integrable bounded functions, but with antiderivative at each point of an interval; a Banach space of infinitely differentiable functions that vanish at infinity and are not the Fourier transform of any Lebesgue integrable function.

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0 Introduction

A set M of functions enjoying some special property is said to be *lineable* if $M \cup \{0\}$ contains an infinite dimensional vector space and *spaceable* if $M \cup \{0\}$ contains a closed infinite dimensional vector space. More specifically, we will say that M is μ*-lineable* if $M \cup \{0\}$ contains a vector space of dimension μ, where μ is a cardinal number. Furthermore, $\lambda(M)$ will denote the maximum dimension (if it exists) of such a vector space.

In a similar way, we say that M is *algebrable* if $M \cup \{0\}$ contains an infinitely generated algebra. More specifically, we will say that M is (μ, s)*-algebrable* if $M \cup \{0\}$ contains an algebra $\mathcal{A}$ such that $\dim(\mathcal{A}) = \mu$ (as a vector space) and $\operatorname{card}(S) = s$, where μ and s are two cardinal numbers and S is a minimal system of generators of $\mathcal{A}$. Trivially, $s \leq \mu$ and if M is (μ, s)-algebrable then it is also μ-lineable. Similarly, we could also define the notions of *dense-lineability*, *dense-algebrability*. The density of functions with certain properties yields interesting results (e.g. [12]). The term *coneability* refers to the existence of a positive (or negative) cone containing an infinite linearly independent family of functions enjoying a certain special property.

These notions of *lineability* and *spaceability* were coined by V. I. Gurariy, first introduced in [11] and later in [2] and [3], while the word *algebrability* didn't appear until recently (see, for instance, [5]).

The earliest results of this type come from 1966, when V. I. Gurariy showed in [13] that the set of *continuous nowhere differentiable functions* on the interval $[0, 1]$ is lineable. He also showed that the set of *everywhere differentiable functions* on the interval $[0, 1]$ is lineable, but not spaceable in $(\mathcal{C}([0, 1]), \|\cdot\|_\infty)$. Other pathological properties have been studied by various authors, for instance *differentiable nowhere monotone functions* and *everywhere surjective functions* in [3]. Along these lines, in the last few years, the set of zeros of polynomials on Banach spaces has been explored: some authors, when working with infinite dimensional spaces, found large linear structures in these sets [1] (even though they have not explicitly used terms like lineability or spaceability), while others concentrated more on the finite dimensional case [4].

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Sometimes these algebraic structures can be endowed with different types of topologies. For example, in [15] the authors proved the (c, c)-algebrability of the set of $\mathcal{C}^\infty$ functions with constant Taylor expansion on $\mathbb{R}$. Following the line of the proof given there, it can be shown that the algebra in case is a Fréchet space.

The paper is organized in four sections. In the first we show the existence of a Banach space of continuous functions on $\mathbb{R}$ for which their sets of proper local minima and maxima, respectively, are dense subsets of $\mathbb{R}$. In the second section we construct a Banach space of infinitely differentiable functions that vanish at infinity and are not the Fourier transform of any Lebesgue integrable function. The third part deals with the Denjoy-Clarkson property for several real variables. Following the recent solution (in the negative) of the Weil Gradient Problem, we display a Banach space of differentiable functions on $\mathbb{R}^n$ which fail the Denjoy-Clarkson property. The last part of the article is dedicated to Riemann integrability. We find a Banach space of bounded functions that are not Riemann integrable and also a Banach algebra and another Banach space of function whose derivatives are bounded but not Riemann integrable.

1 Functions with proper local minima and maxima in each interval

In [18] it was proved that there exist continuous functions on $\mathbb{R}$ with a proper local maximum at each point of a dense subset of $\mathbb{R}$. One could ask whether the set of all functions enjoying this property, $CM(\mathbb{R})$, is lineable. Apparently this is not true, the problem being that the proper local maxima become proper local minima for any negative multiple of f, with $f \in CM(\mathbb{R})$. If f also had a dense set of proper local minima then this problem would not arise. Let us denote by $CMm(\mathbb{R})$ the nonempty (see [10]) set of continuous functions such that both of their sets of proper local minima *and* maxima are dense in $\mathbb{R}$. We will show that $CMm(\mathbb{R})$ is spaceable.

Theorem 1.1

1. *There exists an infinite dimensional Banach space of continuous functions on $\mathbb{R}$ for which (except for 0) their sets of proper local minima and maxima, respectively, are dense subsets of $\mathbb{R}$, i.e. $CMm(\mathbb{R})$ is spaceable. Moreover, $\lambda(CMm(\mathbb{R})) = c$.*
2. *$CMm(\mathbb{R})$ is (c, c)-algebrable.*

Proof. Without loss of generality we can assume that $f(\mathbb{R})$ is the interval with endpoints 0 and 1. Let $A = \{r_n\}_n$ and $B = \{s_n\}_n$ be the dense sets of proper local maxima and minima, respectively, for f. We define $\Phi : C[0, 1] \to C_b(\mathbb{R})$ by $\Phi(h) = h \circ f$.

If $h \in C[0, 1]$ is a nonconstant analytic function on $(0, 1)$ then, since h' is also analytic, it must have at most a countable number of zeroes and the only accumulation points for its zeroes could be 0 or 1. Thus we can write $(0, 1) \setminus \{t : h'(t) = 0\} = \bigcup I_n$, with I_n disjoint open intervals such that h' vanishes at their endpoints and has constant sign inside. Let us show that $A_h = A \cap f^{-1}\left(\bigcup I_n\right)$ is dense in $\mathbb{R}$.

Assume that we have $\mathbb{R} \setminus \overline{A_h} \neq \emptyset$. Then there exists an interval $(\alpha, \beta) \subset \mathbb{R} \setminus \overline{A_h}$. If $x \in (\alpha, \beta)$ then $f(x)$ must be either one of the endpoints of I_n or 0 or 1. Thus $f((\alpha, \beta))$ is a connected subset of a countable set and so it must consist of only one element. This implies that f is constant on (α, β), a fact which contradicts the existence of proper local maxima in this interval.

In the same way we have that $B_h = B \cap f^{-1}(\bigcup I_n)$ is dense in $\mathbb{R}$.

Suppose that h' is positive on the interval I_n. For every $r_k \in f^{-1}(I_n)$ and x in a neighborhood of r_k contained in $f^{-1}(I_n)$ we can write $h(f(x)) = h(f(r_k)) + h'(\gamma)(f(x) - f(r_k))$ with $\gamma \in f^{-1}(I_n)$ and so each such r_k is a proper local maximum for $h \circ f$. In the same way it follows that every $s_k \in f^{-1}(I_n)$ is a proper local minimum for $h \circ f$.

If h' is negative on the interval I_n then each r_k in $f^{-1}(I_n)$ is a proper local minimum for $h \circ f$ and each s_k in $f^{-1}(I_n)$ is a proper local maximum for $h \circ f$.

Furthermore, $f^{-1}(I_n) \cap A$ and $f^{-1}(I_n) \cap B$ are dense sets in $f^{-1}(I_n)$ for every n.

Now let us show that the set of proper local maxima for $h \circ f$ is dense in $\mathbb{R}$. Let $x \in \mathbb{R}$ and $\varepsilon > 0$. There exists an $r_k \in A_h$ such that $|r_k - x| < \varepsilon$. If r_k is a proper local maximum for $h \circ f$ then we are done. If not, it means that there exists $I_n \ni f(r_k)$ such that $h' < 0$ on I_n. Since $f^{-1}(I_n) \cap B$ is dense in $f^{-1}(I_n)$, there is $s_l \in f^{-1}(I_n)$ (therefore a proper local maximum for $h \circ f$) such that $|s_l - r_k| < \varepsilon$ and so $|s_l - x| < 2\varepsilon$.

Likewise the set of proper local minima for $h \circ f$ is dense in $\mathbb{R}$.

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1. Let $\{n_k\}_k$ be a lacunary sequence of positive integers (i.e. with $n_{k+1}/n_k \geq d > 1$). Let F be the closure in $C[0,1]$ of the linear span of $\{t^{n_k}\}_k$. By [8], F consists of functions which are analytic on (0,1) and continuous on $[0,1]$. Let us note that $h(0) = 0$ for all h in F and so the only constant function in F is the zero function. Then $\Phi(F) \subset CMm(\mathbb{R}) \cup \{0\}$ and, since $\|\Phi(h)\| = \|h\|$ for all h, is a closed infinite dimensional subspace. Since we are working with continuous functions, we necessarily have $\lambda(CMm(\mathbb{R})) = c$.

2. Since Φ is an injective algebra homomorphism, it follows that $CMm(\mathbb{R}) \cup \{0\}$ contains an injective image of any algebra of (except for zero) nonconstant analytic functions on $(0,1)$. Consider $\mathcal{H}$ a Hamel basis of $\mathbb{R}$ as a $\mathbb{Q}$-vector space and the algebra $A = \mathcal{A}(\{f_\beta\}, \beta \in \mathcal{H})$ of analytic functions in $(0,1)$, where $f_\beta(x) = e^{\beta x}$. As in [15], A has a minimal system of generators of cardinality c and using the ideas in the proof there it can easily be shown that the only constant function in A is the zero function. Thus $CMm(\mathbb{R}) \cup \{0\}$ contains a (c,c) subalgebra. □

Remark 1.2 The method used does not allow us to find a nontrivial *Banach algebra* inside of $CMm(\mathbb{R}) \cup \{0\}$ because there does not exist a Banach subalgebra of $C([0,1])$ which consists of analytic functions on $(0,1)$. Indeed, assume that such an algebra $\mathcal{B}$ exists and take g an element in $\mathcal{B}$ whose range is the interval $[0,1]$. Consider the continuos function

$$s(t) = \begin{cases} 0 & \text{if} \quad t \leq 1/3, \\ 3t-1 & \text{if} \quad 1/3 < t < 2/3, \\ 1 & \text{if} \quad t \geq 2/3. \end{cases}$$

Let W_n be the Bernstein polynomials which approximate s. Since $s(0) = 0$, we also have $W_n(0) = 0$ and so $W_n \circ g \in \mathcal{B}$. Consequently $s \circ g \in \mathcal{B}$ is analytic on $(0,1)$ and vanishes on the open set $g^{-1}(0,1/3)$ and so $s \circ g$ must be identically 0. That leads to $g(x) \leq 1/3$ for all $x \in [0,1]$, a contradiction.

Remark 1.3 1. Let us denote by $Cm(\mathbb{R})$ the set of continuous functions on $\mathbb{R}$ with a proper local minimum at each point of a dense subset of $\mathbb{R}$. Since $CMm(\mathbb{R}) \subset Cm(\mathbb{R})$ and $CMm(\mathbb{R}) \subset CM(\mathbb{R})$, it follows that $CM(\mathbb{R})$ and $Cm(\mathbb{R})$ are both spaceable and (c,c)-algebrable.

2. If we consider the set of all continuous functions on $\mathbb{R}$ with proper local extrema at each point of a dense subset of the real line, then $CE(\mathbb{R})$ is certainly spaceable and (c,c)-algebrable. In fact, the proof of this result can be obtained in a simpler way. Indeed, once we know that $CE(\mathbb{R})$ contains a function f, then, with the notations of the previous proof, we can argue in the following way: if h is a nonconstant analytic function on $(0,1)$, given r_n, a proper local extreme for f, there exists a minimal positive integer l such that $h(f(x)) = h(f(r_n)) + (f(x) - f(r_n))^l(\alpha_l + P_l(x))$ with $\alpha_l \neq 0$ and P_l continuous with $\lim_{x \to r_n} P_l(x) = 0$. Thus there exists a neighborhood of r_n in which $\Phi(h)(x) - \Phi(h)(r_n)$ has constant sign. Therefore $\Phi(h)$ belongs to $CE(\mathbb{R})$ and the rest of the proof follows as above.

2 Infinitely differentiable functions that vanish at infinity and are not the Fourier transform of any Lebesgue integrable function

We can make further use of this method of embedding a Banach space in a set of continuous functions. *Let $\mathcal{C}^\infty_{0,NF}$ be the set of infinitely differentiable functions that vanish at infinity and are not the Fourier transform of any Lebesgue integrable function.* One class of examples (see [14, Chap. 6] for details) is given by the functions of the form $f(x) = \sum_{-\infty}^{\infty} c_n g(x-n)$, where

1. $(c_n)_n$ is any double infinite sequence such that $\sum_{-\infty}^{\infty} c_n e^{inx}$ converges for all x but c_n are not the Fourier coefficients of a Lebesgue-integrable function on $[-\pi, \pi]$,

2. g is any infinitely differentiable function that vanishes outside $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and with $g(0) \neq 0$.

It is easy to see that $f(x) = c_n g(x-n)$ in the interval $\left[n - \frac{1}{2}, n + \frac{1}{2}\right]$, and thus if f vanishes everywhere then so does g. All these facts allow us to prove the following result:

Theorem 2.1 *There is an infinite dimensional Banach space of infinitely differentiable functions that vanish at infinity and (except for 0) are not the Fourier transform of any Lebesgue integrable function, i.e. $\mathcal{C}^\infty_{0,NF}$ is spaceable and $\lambda\big(\mathcal{C}^\infty_{0,NF}\big) = c$.*

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P r o o f. Without loss of generality we can assume that $g\big([-\frac{1}{2},\frac{1}{2}]\big)=[0,1]$. Let us now define Φ on $\mathcal{C}^\infty[0,1]$ by $\Phi(h)(x)=\sum_{-\infty}^{\infty}c_n(h\circ g)(x-n)$. Clearly $h\circ g$ is an infinitely differentiable function. If we also ask that $h(0)=0$ then $h\circ g$ also vanishes outside $\big[-\frac{1}{2},\frac{1}{2}\big]$. If $\Phi(h)\equiv 0$, then $h\circ g$ must be identically 0 and it follows that $h=0$. Thus, Φ is injective on $\{h\in\mathcal{C}^\infty[0,1] : h(0)=0\}$.

It remains to show that $\Phi(h)$ cannot be a Fourier transform if $h\neq 0$. Since $h\circ g\neq 0$, there exists an $x_0\in\big(-\frac{1}{2},\frac{1}{2}\big)$ such that $h\circ g(x_0)\neq 0$. If there exists a Lebesgue integrable function H such that $\int_{-\infty}^{\infty}H(t)e^{-itx}\,dt=\Phi(h)(x)$ then, taking $G(t)=e^{-itx_0}\sum_{-\infty}^{\infty}H(t+2\pi m)$, we obtain

$$\int_{-\pi}^{\pi}G(t)e^{-int}\,dt=\int_{-\infty}^{\infty}H(t)e^{-i(n+x_0)t}\,dt=\Phi(h)(x_0+n)=\sum_{-\infty}^{\infty}c_k(h\circ g)(x_0+n-k)=c_n(h\circ g)(x_0),$$

which yields that c_n are the Fourier coefficients of the Lebesgue integrable function $\frac{2\pi}{(h\circ g)(x_0)}G$, a contradiction.

Thus $\Phi(h)\in\mathcal{C}^\infty_{0,NF}$ for all $h\neq 0$ and since

$$\|\Phi(a)\|_\infty=(\sup_n|c_n|)\|h\|_\infty,$$

we obtain spaceability, since $\mathcal{C}^\infty_{0,NF}\cup\{0\}$ contains the isometric image of the Banach space F introduced in the proof of the previous theorem (and whose members are analytic and vanish at 0). Since we are working with continuous functions, $\lambda\big(\mathcal{C}^\infty_{0,NF}\big)=c$. □

3 Differentiable functions on $\mathbb{R}^n$ which fail the Denjoy-Clarkson property

It is well-known that derivatives of functions of one real variable satisfy the Denjoy-Clarkson property: if $u:\mathbb{R}\to\mathbb{R}$ is everywhere differentiable, then the counterimage through u' of any open subset of $\mathbb{R}$ is either empty or has positive Lebesgue measure. Extending this result to several real variables is known as the Weil Gradient Problem [19] and, after being an open problem for almost 40 years, was eventually solved in the negative for $\mathbb{R}^2$ by Buczolich in 2002 [7]. His example was simplified in some subsequent articles, in particular very recently by Deville and Matheron [9]. They constructed an everywhere differentiable function on $Q=[0,1]^n$ and then extended it through $\mathbb{Z}^n$-periodicity to the whole of $\mathbb{R}^n$ obtaining a bounded, everywhere differentiable function $f:\mathbb{R}^n\to\mathbb{R}$ such that

1. f and ∇f vanish on the boundary of Q,
2. $\|\nabla f\|=1$ almost everywhere in $\mathbb{R}^n$ and $\|\nabla f(x)\|\leq 1$ for all $x\in\mathbb{R}^n$.

Thus it is clear that f fails the Denjoy-Clarkson property, since $(\nabla f)^{-1}(B(0,1))$ is a nonempty set of zero Lebesgue measure.

We will use this function (which we call the Deville-Matheron function) to prove the following result:

Theorem 3.1 *For every $n\geq 2$ there exists an infinite dimensional Banach space of differentiable functions on $\mathbb{R}^n$ which (except for 0) fail the Denjoy-Clarkson property.*

P r o o f. For the sake of simplicity we will work with $\mathbb{R}^2$, but everything is also valid for $\mathbb{R}^n$ with $n\geq 2$.

For every $(k,l)\in\mathbb{Z}^2$ let $Q_{k,l}=[k,k+1]\times[l,l+1]$ and let $f_{k,l}:\mathbb{R}^2\to\mathbb{R}$ be defined by the restriction of f on $Q_{k,l}$ and 0 everywhere else. Of course, now $(\nabla f_{k,l})^{-1}(B(0,1))$ has infinite Lebesgue measure, but the function still fails the Denjoy-Clarkson property. For this we just need to show that there exist an $0<\alpha<1$ and a point $a_\alpha\in Q_{k,l}$ with $\|\nabla f_{k,l}(a_\alpha)\|=\alpha$.

For the sake of completeness we give a very simple proof of the fact that for every $\alpha\in[0,\|f\|_\infty]$, there exists a point $a_\alpha\in Q_{k,l}$ with $\|\nabla f_{k,l}(a_\alpha)\|=\alpha$. This is all we need to know about the intermediate values of the gradient in order to reach the conclusion of our theorem. By Weierstrass theorem and connectedness we have $|f|(Q)=[0,\|f\|_\infty]$. Hence, by changing f by $-f$ if necessary, we have $[0,\|f\|_\infty]\subset f(Q)$. Let $\alpha\in(0,\|f\|_\infty]$. Consider $b\in Q_{k,l}$ with $f_{k,l}(b)=\alpha$. Of course, b does not belong to the boundary of $Q_{k,l}$. Define the differentiable function $h:Q_{k,l}\to\mathbb{R}$ by $h(x,y)=f_{k,l}(x,y)+\alpha(x-k)$. Since $h(b)>f_{k,l}(b)=\alpha=\max_{\partial Q_{k,l}}h$, it follows that h attains its maximum in the interior of $Q_{k,l}$. Thus, there exists a point a_α with $0=\nabla h(a_\alpha)=\nabla f_{k,l}(a_\alpha)+(\alpha,0)$ and so $\|\nabla f(a_\alpha)\|=\alpha$.

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Now we choose $0 < \delta < \min\{\alpha, 1-\alpha\}$ and we have that

$$(\nabla f_{k,l})^{-1}(B(\nabla f(a_\alpha), \delta)) \subset (\nabla f)^{-1}(B(0,1)) \cap Q$$

is a nonempty set of zero Lebesgue measure.

We define $\Phi : c_0(\mathbb{N}^2) \to C_b(\mathbb{R}^2)$ to be the function that associates to every double sequence $d = (d_{k,l})_{k,l}$ the function $\Phi_d(x,y) = \sum d_{k,l} f_{k,l}(x,y)$. Since $f_{k,l} = \nabla f_{k,l} = 0$ on $\mathbb{R}^2 \setminus \operatorname{int}(Q_{k,l})$, the pieces glue together well and so Φ_d is an everywhere differentiable function. Clearly we have $\nabla\Phi_d(x,y) = \sum d_{k,l} \nabla f_{k,l}(x,y)$ and so $\|\nabla\Phi_d(x,y)\| \in \{|d_{k,l}|, (k,l) \in \mathbb{N}^2\}$ for almost all $(x,y) \in \mathbb{R}^2$. Given $d \neq 0$, there exists $d_{k_0,l_0} \neq 0$. Choose $\alpha \in (0, \|f\|_\infty]$ such that $\alpha|d_{k_0,l_0}| \notin K = \{|d_{k,l}| : (k,l) \in \mathbb{N}^2\} \cup \{0\}$. Since $|d_{k_0,l_0}|\alpha$ does not belong to the compact set K the distance δ from $|d_{k_0,l_0}|\alpha$ to K is positive. Consider $a_\alpha \in Q_{k_0,l_0}$ with $\|\nabla f_{k_0,l_0}(a_\alpha)\| = \alpha$. Then $\|\nabla\Phi_d(a_\alpha)\| = |d_{k_0,l_0}|\alpha$. If $x \in (\nabla\Phi_d)^{-1}\big(B(\nabla\Phi_d(a_\alpha), \delta)\big)$ then $\|\nabla\Phi_d(x)\| \neq |d_{k,l}|$ for all $(k,l) \in \mathbb{N}^2$ and so $(\nabla\Phi_d)^{-1}(B(\nabla\Phi_d(a_\alpha), \delta))$ is a nonempty set of zero Lebesgue measure.

All this shows that $\Phi(c_0(\mathbb{N}^2))$ is a vector space of differentiable functions which (except for 0) do not have the Denjoy-Clarkson property. But $\Phi(c_0(\mathbb{N}^2))$ is also a closed subspace of $(C_b(\mathbb{R}^2), \|\cdot\|_\infty)$ since $\|\Phi_d\|_\infty = \|(d_{k,l})_{k,l}\|_\infty \|f\|_\infty$ for all $d \in c_0(\mathbb{N}^2)$ and so, $\big(\Phi(c_0(\mathbb{N}^2)), \|.\|_\infty\big)$ is an infinite dimensional Banach space isometric to $c_0(\mathbb{N}^2)$. □

Remark 3.2 1. The proof can be simplified if, instead of using the simple intermediate value result for gradients that we proved, one uses the much stronger Darboux property for gradients [17].

2. The result can be reformulated in the following way. If we denote by $NDC(\mathbb{R}^n)$ the set of everywhere differentiable functions on $\mathbb{R}^n$ which do not have the Denjoy-Clarkson property then $NDC(\mathbb{R}^n)$ is spaceable and $\lambda(NDC(\mathbb{R}^n)) = c$.

3. It is possible to construct another infinite dimensional Banach space of differentiable functions on $\mathbb{R}^n$ which (except from 0) fail the Denjoy-Clarkson property and which contains the Deville-Matheron function. Indeed, it is enough to replace $c_0(\mathbb{N}^n)$ in the proof above with the Banach space $c(\mathbb{N}^n)$ of all convergent sequences in $\mathbb{R}$ endowed with the supremum norm. In that case f is the image of the sequence $(1, 1, \dots)$.

4. It is also possible to construct a non separable infinite dimensional normed space of differentiable functions on $\mathbb{R}^n$ which (except from 0) fail the Denjoy-Clarkson property and which contains the Deville-Matheron function too. For that, consider the non-separable norm space $X = \operatorname{span}\{\chi_P : P \subset \mathbb{N}^n\}$, endowed with the supremum norm, instead of $c_0(\mathbb{N}^n)$ in the proof above. Our method cannot be extended to $\ell_\infty(\mathbb{N}^n)$, the completion of X. To check that (we write it only for the case $n = 2$), consider $c = (c_{k,l})_{k,l}$ the sequence of all rational numbers in $[0,1]$. The function Φ_c actually does satisfy the Denjoy-Clarkson property.

4 Bounded functions which are Lebesgue but not Riemann integrable

One of the key points that led Lebesgue to his theory of integration was the existence of two examples, one given by Volterra in 1881 and another by Brodén in 1896 which showed that, at least from the point of view of Riemann integration, the process of obtaining antiderivatives of a function and the integration theory were not equivalent. Our aim in sequel is to show that it is possible to find infinite dimensional Banach spaces of functions having Brodén (Theorem 4.1) and Volterra (Remark 4.2) properties. Actually, the derivatives of the Brodén-type functions will allow us to build (Theorem 4.3) an infinite dimensional Banach space of bounded functions on an interval which are Lebesgue integrable, have antiderivative at any point and (except for 0) are not Riemann integrable.

4.1 Brodén-type functions

In 1896, Brodén (see, e.g. [6, sect. 4.4]) gave an example of a homeomorphism $g : [a,b] \longrightarrow [-1,1]$ such that g is differentiable in $[a,b]$, with g' being bounded and vanishing on a dense subset of $[a,b]$. He considered a function $f(x) = \sum_{n=1}^{\infty} \frac{(x-a_n)^{1/3}}{2^n}$, where (a_n) is a dense sequence in $[-1,1]$, with $a_1 = -1$, $a_2 = 1$, and satisfying that:

1. $f'(a_n) = +\infty$ for all $n \in \mathbb{N}$, and
2. $f'(x) \geq \frac{1}{12}$ for every $x \neq a_n$.

This function is a homeomorphism $f : [-1, 1] \to [f(-1), f(1)] = [a, b]$, strictly increasing and taking $g := f^{-1}$, we obtain that

$$g'(x) = \frac{1}{f'(f^{-1}(x))},$$

which takes the values $g'(x) \in (0, \frac{1}{12}]$ if $x \neq f(a_n)$, and 0 if $x = f(a_n)$ for some n. Since $a = f(a_1)$ and $b = f(a_2)$, we have that $g'(a) = g'(b) = 0$. Let $c = 2b - a$. The function g can be modified in the following way

$$s(x) = \begin{cases} g(x) + 1 & \text{if} \quad a \leq x \leq b, \\ g(2b - x) + 1 & \text{if} \quad b < x \leq c, \\ 0 & \text{if} \quad x \notin [a, c], \end{cases}$$

such that s is differentiable, with bounded derivative on $\mathbb{R}$, and $s(x) = 0$ if either $x \leq a$, or $x \geq c$. Thus $s'(x) = 0$ if either $x \leq a$, or $x \geq c$, and $s'(x) = g'(x)$ if $x \in (a, b)$ and $-g'(2b - x)$ if $x \in (b, c)$. So, s is continuous, non-constant, differentiable, and s' bounded and zero in a dense subset of $\mathbb{R}$. This kind of function will be called a Brodén-type function.

In the proofs of the following two theorems we are going to use the following classical result, which we call

Theorem A *If a sequence of differentiable functions $f_n : (a, b) \to \mathbb{R}$ converges at some point $t_0 \in (a, b)$ and if the sequence of derivatives f_n' converges uniformly on (a, b) to a function g, then there exists a differentiable function $f : (a, b) \to \mathbb{R}$ such that the sequence f_n converges uniformly on (a, b) to f and $f' = g$.*

Theorem 4.1 *There exists an infinite dimensional Banach algebra of (except for 0) Brodén-type functions. In particular, the set of Brodén-type functions is spaceable and algebrable.*

Proof. Let us proceed with the Brodén-type function s, which we gave earlier. Let $[\alpha, \beta]$ be any closed bounded interval, $\alpha < \beta$, and let $I_n = [\alpha_n, \beta_n]$ be a sequence of disjoint proper subintervals of $[\alpha, \beta]$. Now, consider the linear mappings given by

$$\begin{array}{rcccc} \phi_n & : & [\alpha_n, \beta_n] & \longrightarrow & [a, c] \\ & & t & \longmapsto & \dfrac{t - \alpha_n}{\beta_n - \alpha_n}(c - a) + a \end{array}$$

and call

$$s_n := s \circ \phi_n : \mathbb{R} \longrightarrow \mathbb{R}.$$

We have that, for every $n \in \mathbb{N}$, s_n is continuous, non-constant, differentiable, and s_n' vanishes in a dense subset in $\mathbb{R}$. We have

$$s_n(t) = \begin{cases} 0 & \text{if} \quad t \leq \alpha_n, \\ s\left(\frac{t - \alpha_n}{\beta_n - \alpha_n}(c - a) + a\right) & \text{if} \quad \alpha_n < t < \beta_n, \\ 0 & \text{if} \quad t \geq \beta_n, \end{cases}$$

and

$$s_n'(t) = \begin{cases} 0 & \text{if} \quad t \leq \alpha_n, \\ \dfrac{c - a}{\beta_n - \alpha_n} g'\left(\frac{t - \alpha_n}{\beta_n - \alpha_n}(c - a) + a\right) & \text{if} \quad \alpha_n < t < \beta_n, \\ 0 & \text{if} \quad t \geq \beta_n, \end{cases}$$

and we have obtained a sequence of functions with disjoint supports, each one of them vanishing in a dense set in $\mathbb{R}$. Moreover, we have that $\operatorname{supp}(s_n') \subset [\alpha_n, \beta_n]$.

Since the functions $\{s_n : n \in \mathbb{N}\}$ have pairwise disjoint supports, they form a minimal set of generators for an algebra that we call $\mathcal{A}$. Let $h \in \mathcal{A}$. Then h is necessarily of the form

$$h = \sum_{i=1}^{k_1} c_{1,i} s_1^i + \sum_{i=1}^{k_2} c_{2,i} s_2^i + \cdots + \sum_{i=1}^{k_l} c_{l,i} s_l^i := p_1 + p_2 + \cdots + p_l.$$

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Suppose that $h \equiv 0$. Then, h restricted to I_j is equal to p_j, $j \in \{1, \ldots, l\}$. Clearly, if each $p_j \equiv 0$, then it follows that $c_{j,i} = 0$ for every $i \in \{1, \ldots, k_j\}$, since we would have that the polynomial $\sum_{i=0}^{k_j} c_{j,i} z^j \equiv 0$. This shows the algebrability of the set of Brodén-type functions. Note also that all functions in $\mathcal{A}$ vanish at all points α_n and β_n.

By Theorem A, the set X of all functions $h : \mathbb{R} \to \mathbb{R}$ which are differentiable, have compact support and bounded derivative, endowed with the norm $\|h\|_1 = \sup_{x\in\mathbb{R}}|h(x)| + \sup_{x\in\mathbb{R}}|h'(x)|$ is a Banach space. Now consider the closed subset of X given by $\mathcal{B} = \overline{\mathcal{A}}^{\|\cdot\|_1}$. Clearly $\mathcal{B}$ is a Banach algebra and so it only remains to show that all functions in $\mathcal{B}$ are of Brodén-type.

Let $h \in \mathcal{B}$ and take $h_k := \sum_{j=1}^{n_k} p_j^{(k)} \to h$ with the $\|\cdot\|_1$. Thus h is differentiable, h_k converges uniformly to h and h'_k converges uniformly to h' on $[\alpha, \beta]$. Now, h_k restricted to I_n (with n arbitrary) gives us $p_n^{(k)} = \sum_{i=1}^{l_k} c_{k,i} s_n^i$, and thus $h'_k = \sum_{i=1}^{l_k} c_{k,i} s_n^{i-1} s'_n$. For all n and k, whenever $s'_n(t) = 0$, we have that $h'_k(t) = 0$ and consequently $h'(t) = 0$ and so h' vanishes on a dense subset of $\mathbb{R}$. Since $h(\alpha_n) = 0$, if h is constant then it has to be the zero function. Thus $\mathcal{B}$ is a Banach algebra of (except for 0) Brodén-type functions. In particular, this shows that the set of Brodén-type functions is spaceable. □

Remark 4.2 In 1881, Volterra (see, e.g. [16, p. 100]) gave an example of a differentiable function on $\mathbb{R}$ whose derivative is bounded but not Riemann-integrable. Let $[a, b]$ be an interval in $\mathbb{R}$, and

$$F_{a,b}(x) = \begin{cases} 0 & \text{if} \quad x \notin (a, b), \\ \phi_a(x) & \text{if} \quad a < x < a + c, \\ \phi_a(a + c) & \text{if} \quad a + c \leq x \leq b - c, \\ \phi_b(x) & \text{if} \quad b - c < x < b, \end{cases}$$

where $\phi_d(x) = (x - d)^2 \sin\left(\frac{1}{x-d}\right)$ and $a + c = \sup\left\{x \in (a, \frac{a+b}{2}] : \phi'_a(x) = 0\right\}$. Let $(r_n)_n = \mathbb{Q} \cap (0, 1)$ and $I_n = \left(r_n - \alpha_n, r_n + \alpha_n\right)$ such that $\sum_n \mu(I_n) < \frac{1}{2}$ with μ being the Lebesgue measure on $\mathbb{R}$. Let $K = [0, 1] \setminus \bigcup_{n=1}^{\infty} I_n$, which is compact with empty interior. Clearly, $\mu(K) \geq \frac{1}{2}$. We can write $\bigcup_{n=1}^{\infty} I_n$ as an infinite union of connected components $J_n = [a_n, b_n]$. Volterra's function with support in $[0, 1]$ is given by $F = \sum_{n=1}^{\infty} F_{a_n, b_n}$ pointwise.

Choosing a collection of pairwise disjoint intervals $[\alpha_n, \beta_n]$ and taking

$$E = \overline{\text{span}\{G_n : n \in \mathbb{N}\}}^{\|\cdot\|_1},$$

where G_n is the Volterra function with support in the interval $[\alpha_n, \beta_n]$, we have another way of proving that *there is an infinite dimensional Banach space of differentiable functions on $\mathbb{R}$ whose derivatives are bounded and (except for 0) not Riemann-integrable.*

Let us notice that the E is not an algebra since the derivative of the square of Volterra function, F^2, is continuous, a fact which follows from $F(x) = 0$ for all $x \in K$.

4.2 Bounded derivatives which are not Riemann-integrable

Let us notice that the derivative of any Brodén-type function is bounded but not Riemann-integrable. We say that a bounded function f has property (P) if there exists a function F such that $F'(x) = f(x)$ for all $x \in \mathbb{R}$ but f is not Riemann-integrable on any compact interval of $\mathbb{R}$. In particular, any function enjoying (P) does not verify the Fundamental Theorem of Calculus. We will show that the set of functions enjoying (P) is spaceable.

Theorem 4.3 *Given an interval $[a, b]$, $a < b$, there exists an infinite dimensional Banach space of bounded functions which are Lebesgue integrable, have antiderivatives at any point of $[a, b]$ but (except for 0) are not Riemann-integrable on $[a, b]$.*

P r o o f. Let g be the original Brodén function, and $I_n = [\alpha_n, \beta_n]$ a collection of pairwise disjoint intervals in $\mathbb{R}$. Let us define, for every $n \in \mathbb{N}$, the function $g_n : \mathbb{R} \to \mathbb{R}$ given by:

$$g_n(t) = \begin{cases} -1 & \text{if} \quad t \leq \alpha_n, \\ g\left(\frac{t-\alpha_n}{\beta_n-\alpha_n}(b - a) + a\right) & \text{if} \quad \alpha_n < t < \beta_n, \\ 1 & \text{if} \quad t \geq \beta_n. \end{cases}$$

It is clear that g'_n has property (P), they have pairwise disjoint supports, and as in the proof of Theorem 4.1, they are linearly independent. Let

$$E = \overline{\text{span}\{g'_n : n \in \mathbb{N}\}}^{\|\cdot\|_\infty}.$$

Pick $h \in E$, and let h_k be a sequence of functions in E such that h_k converges to h. We choose $t_0 \in \mathbb{R} \setminus \bigcup_n I_n$, and denote by H_k the antiderivative of h_k which vanishes at t_0. There exists a bounded sequence $(c_n)_n$ such that $h = \sum_{n=1}^{\infty} c_n g'_n$ point-wise. Putting $H(x) = \sum_{n=1}^{\infty} c_n g_n(x)$, we have

$$H_k(t_0) = H(t_0) = 0,$$

$$H'_k = h_k, \quad \text{and}$$

$$h_k \longrightarrow h \quad \text{uniformly.}$$

By Theorem A, it follows that H is differentiable and $H' = h$. Nevertheless h is not Riemann-integrable. If it were then, since $h = g'_n$ on the interval I_n, we would have

$$2 = g_n(\beta_n) - g_n(\alpha_n) = H(\beta_n) - H(\alpha_n) = \int_{\alpha_n}^{\beta_n} h(x)\,dx = \int_{\alpha_n}^{\beta_n} g'_n(x)\,dx = 0,$$

which is a contradiction. □

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References

[1] R. M. Aron, C. Boyd, R. Ryan, and I. Zalduendo, Zeros of polynomials on Banach spaces: the real story, Positivity **7**, No. 4, 285–295 (2003).

[2] R. M. Aron, D. García, and M. Maestre, Linearity in non-linear problems, RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **95**, No. 1, 7–12 (2001).

[3] R. M. Aron, V. I. Gurariy, and J. B. Seoane-Sepúlveda, Lineability and spaceability of sets of functions on $\mathbb{R}$, Proc. Amer. Math. Soc. **133**, 795–803 (2005).

[4] R. M. Aron and P. Hájek, Zero sets of polynomials in several variables, Arch. Math. (Basel) **86**, No. 6, 561–568 (2006).

[5] R. M. Aron, D. Pérez-García, and J. B. Seoane-Sepúlveda, Algebrability of the set of non-convergent Fourier series, Studia Math. **175**, 83–90 (2006).

[6] H. J. M. Bos, R. Bunn, J. W. Dauben, I. Grattan-Guinness, T. W. Hawkins, and K. M. Pedersen, From the calculus to set theory, in: An Introductory History, edited by I. Grattan-Guinness (Gerald Duckworth & Co. Ltd., London, 1980), pp. 1630–1910.

[7] Z. Buczolich, Solution to the gradient problem of C. E. Weil, Rev. Mat. Iberoamericana **21**, No. 3, 889–910 (2005).

[8] J. A. Clarkson and P. Erdös, Approximation by polynomials, Duke Math. J. **10**, 5–11 (1943).

[9] R. Deville and É. Matheron, Infinite games, Banach space geometry and the eikonal equation, Proc. Lond. Math. Soc. (3) **95**, No. 1, 49–68 (2007).

[10] V. Drobot and M. Morayne, Continuous functions with a dense set of proper local maxima, Amer. Math. Monthly **92**, No. 3, 209–211 (1985).

[11] P. Enflo and V. I. Gurariy, On lineability and spaceability of sets in function spaces, preprint (2000).

[12] J. Ferrer, An approximate Rolle's theorem for polynomials of degree four in a Hilbert space, Publ. Res. Inst. Math. Sci. **41**, No. 2, 375–384 (2005).

[13] V. I. Gurariy, Subspaces and bases in spaces of continuous functions, Dokl. Akad. Nauk SSSR **167**, 971–973 (1966) (in Russian).

[14] B. R. Gelbaum and J. M. H. Olmsted, Counterexamples in Analysis (Dover Publications, New York, 2003).

[15] F. J. García-Pacheco, N. Palmberg, and J. B. Seoane-Sepúlveda, Lineability and algebrability of pathological phenomena in analysis, J. Math. Annal. Appl. **326**, No. 2, 929–939 (2007).

[16] H. Lebesgue, Leçons sur l'Intégration (Gauthier-Villars, Paris, 1904).

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[17] J. Malý, The Darboux property for gradients, Real Anal. Exchange **22**, No. 1, 167–173 (1996/97).
[18] E. E. Posey and J. E. Vaughan, Functions with a proper local maximum in each interval, Amer. Math. Monthly **90**, No. 4, 281–282 (1983).
[19] C. E. Weil, On properties of derivatives, Trans. Amer. Math. Soc. **114**, 363–376 (1965).

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