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Abstract We investigate conditions under which L-weakly compact operators and M-weakly compact operators must be compact.

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Compactness of L-weakly and M-weakly compact operators on Banach lattices

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1 Introduction and Definitions.

An operator T from a Banach lattice E into a Banach space X is *M-weakly compact* if $\lim_n \|T(x_n)\| = 0$ holds for every norm bounded disjoint sequence (x_n) in E. An operator T from a Banach space X into a Banach lattice E is called *L-weakly compact* if $\lim_n \|y_n\| = 0$ holds for every disjoint sequence (y_n) in the solid hull of $T(B_X)$ where B_X is the closed unit ball of X. Neither L-weakly nor M-weakly compact operators are necessary compact (see Proposition 3.6.20 of [9]) whilst compact operators need not be either L-weakly or M-weakly compact (see [2], page 322.)

On the other hand, Chen and Wickstead [7] showed that an M-weakly compact operator from an AL-space into a Banach space is compact ([7], Corollary 2.7) and each L-weakly compact operator from a Banach space into an AM-space is compact ([7], Corollary 2.8). Here, we generalize these two results, study the converse problem and seek conditions on the other space which force the same conclusion.

For any unexplained terms from Banach lattice theory and positive operators, we refer the reader to [2] and [9].

2 Some Preliminaries.

Our generalizations of the results from [7] depend on identifying a suitable class of Banach lattices containing the AM-spaces. Recall that if E is a Banach lattice then E^a

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is the maximal lattice ideal in E on which the norm is order continuous. A positive element in E_+ is *discrete* or an *atom* if its linear span is a lattice ideal in E. E is termed *discrete* or *atomic* if the band generated by the discrete elements is the whole space. It turns out that the class of Banach lattices E that we need are those such that E^a is discrete. Although a sublattice of a discrete vector lattice need not be discrete, an ideal must be, so that if E is discrete then so is E^a. In Corollary 2.3 of [8], it was shown that if E has an order continuous norm then E is discrete if and only if E has weakly sequentially continuous lattice operations. It is clear that if F is a closed sublattice of a Banach lattice with weakly sequentially continuous lattice operations then the same is true for F. This applies, in particular, when $F = E^a$ so that if E has weakly sequentially continuous lattice operations then E^a is discrete. AM-spaces E have weakly sequentially continuous lattice operations, Theorem 4.31 of [2], so that E^a is discrete and we see that this property does indeed generalize that of being an AM-space.

3 Compactness of L-weakly compact operators.

A subset A of a Banach lattice E is said to be approximately order bounded if for every $\varepsilon > 0$ there exists $u \in E_+$ such that $A \subset [-u,u] + \varepsilon B_E$ where $B_E = \{x \in E : \|x\| \leq 1\}$ is the closed unit ball of E.

The following result is a straightforward generalization of Corollary 2.8 of [7].

Theorem 1 *Let F be a Banach lattice and X a Banach space. If F^a is discrete, then every L-weakly compact operator $T : X \to F$ is compact.*

Proof If $T : X \to F$ is L-weakly compact then, by Proposition 3.6.2 of [9], $T(B_X)$ is approximately order bounded in F^a. Thus if $\varepsilon > 0$ there exists $x \in F^a_+$ such that

$$T(B_X) \subset [-x,x] + \varepsilon B_{F^a}. \qquad (*)$$

If F^a is discrete it follows from Theorem 6.1 of [12] that the order interval $[-x,x]$ is norm compact. Now $(*)$, combined with part (4) of Theorem 3.1 of [2], shows that $T(B_X)$ is norm totally bounded so that T is compact. □

As a partial converse, we have

Theorem 2 *Let E and F be Banach lattices and assume that E' does* not *have an order continuous norm. The following are equivalent:*

1. *Every L-weakly compact operator $T : E \to F$ is compact.*
2. *Each positive L-weakly compact operator $T : E \to F$ is compact.*
3. *F^a is discrete.*

Proof Clearly (1) implies (2) and Theorem 1 gives us that (3) implies (1). It remains to prove that (2) implies (3). Since the norm of E' is not order continuous then, by Theorem 116.1 of [13] or Theorem 2.4.14 of [9], there is a norm bounded disjoint sequence (u_n) of positive elements in E which does not converge weakly to zero.

Without loss of generality, we may assume that $\|u_n\| \leq 1$ for all n and that there are $\phi \in E'_+$ and $\varepsilon > 0$ such that $\phi(u_n) > \varepsilon$ for all n. It follows from Theorem 116.3 of Zaanen [13] that the components ϕ_n of ϕ, in the carriers C_{u_n}, form an order bounded disjoint sequence in E'_+ such that

$$\phi_n(u_n) = \phi(u_n) \text{ for all } n \text{ and } \phi_n(u_m) = 0 \text{ if } n \neq m. \tag{†}$$

Note that $0 \leq \phi_n \leq \phi$ for all n.

If we assume that F^a is not discrete, it follows from Theorem 6.1 of [12] that there exists some $0 \leq y \in F^a$ such that $[0,y]$ is not norm compact. Now, fix a sequence (y_n) in $[0,y]$ which has no norm convergent subsequence in F^a and therefore in none in F.

Define an operator $T : E \to F$ by

$$T(x) = \sum_{k=1}^{\infty} \big(\phi_k(x)/\phi(u_k)\big) y_k$$

for $x \in E$. Note that in view of the inequality

$$\sum_{k=1}^{\infty} \big\| \big(\phi_k(x)/\phi(u_k)\big) y_k \big\| \leq \varepsilon^{-1} \|y\| \sum_{k=1}^{\infty} \phi_k(|x|) \leq \varepsilon^{-1} \|y\| \phi(|x|)$$

for each $x \in E$, the series defining T converges in norm for each $x \in E$. It follows from (†) that $T(u_n) = y_n$ for all n. Since (y_n) has no norm convergent subsequence in F, T is not compact. However, T is L-weakly compact. To see this, note that for all $x \in B_E$, we have

$$\begin{aligned} |T(x)| \leq T(|x|) &= \sum_{k=1}^{\infty} \big(\phi_k(|x|)/\phi(u_k)\big) y_k \\ &\leq \varepsilon^{-1} \left(\sum_{k=1}^{\infty} \phi_k(|x|) \right) y \\ &\leq \varepsilon^{-1} \phi(|x|) y \\ &\leq \varepsilon^{-1} \|\phi\| y. \end{aligned}$$

so that $T(B_E) \subset \varepsilon^{-1} \|\phi\| [-y,y]$. Since $y \in F^a$, $[-y,y]$ is an L-weakly compact subset of F, so that $T(B_E)$ is also L-weakly compact and hence T is L-weakly compact. □

Theorem 2.1 of [5] (which is essentially part of Theorem 1 in [6]) includes the statement that if E and F are non-zero Banach lattices then F has an order continuous norm if and only if every positive compact operator $T : E \to F$ is L-weakly compact. Combining that with Theorem 2 we see that:

Corollary 1 *For a Banach lattice F the following assertions are equivalent:*

1. *An operator $T : \ell_1 \to F$ is L-weakly compact if and only if it is compact.*
2. *F is discrete with an order continuous norm.*

The hypothesis that E' does not have an order continuous norm can certainly not be omitted from Theorem 2. In fact, if E' is discrete with an order continuous norm then we can deduce nothing about F.

Let us first note some sufficient conditions to force compactness of *regular* operators.

Proposition 1 *Let E and F be Banach lattices.*

1. *If F is discrete with an order continuous norm then every regular M-weakly compact operator $T : E \to F$ is compact.*
2. *If E' is discrete with an order continuous norm then each regular L-weakly compact operator $T : E \to F$ is compact.*

Proof If F is discrete with an order continuous norm then by Corollary 3.6.14 of [9] every regular M-weakly compact operator $T : E \to F$ is L-weakly compact[1] and hence is compact by Theorem 1. This establishes (1). Statement (2) follows from (1) by duality (see Proposition 3.6.11 of [9].) □

Example 1 The assumption that T is regular cannot be omitted from Proposition 1. In fact, let $E = l^2$ and $F = L^1[0,1]$. Note that E and E' are reflexive (so have order continuous norms) and discrete. By Corollary 2.7.7 of [9], F contains a closed subspace H which is isomorphic to l^2. The isomorphism $T : E \to H \subset F$ is weakly compact and hence, as F is an AL-space, L-weakly compact. As T is certainly not compact, (2) fails for non-regular operators. By Proposition 3.6.11 of [9] the adjoint $T' : F' \to E'$ is M-weakly compact but not compact, so that (1) also fails for non-regular operators.

In fact, the property in (2) gives a characterization of Banach lattices whose duals are discrete and order continuous.

Theorem 3 *The following conditions on a Banach lattice E are equivalent:*

1. *E' is discrete and its norm is order continuous.*
2. *For every Banach lattice F, every regular L-weakly compact operator $T : E \to F$ is compact.*
3. *For every Banach lattice F, every positive L-weakly compact operator $T : E \to F$ is compact.*

Proof That (1) implies (2) is Proposition 1, whilst (2) certainly implies (3).

Assume now that (3) holds. If we choose F such that F^a is not discrete, for example $F = L_1([0,1])$, Theorem 2 tells us that the norm of E' is order continuous.

Now assume, by way of contradiction, that E' is not discrete. Then there exists some $0 \le x' \in E'$ such that the order interval $[-x', x']$ is not norm compact in E' (see Theorem 6.1 of [12].) Let G denote the AL-space which is the completion of the quotient E/N, where $N = \{x \in E : x'(|x|) = 0\}$, under the norm induced on the quotient by the semi-norm $x \mapsto x'(|x|)$ on E. Let Q denote the quotient map of E into G. It follows from Chapter IV, Exercise 9 of [10] that the topological dual of (G, x')

[1] Corollary 3.6.14 of [9] actually claims this for any M-weakly compact operator, but the proof uses Proposition 3.6.13 and the proof of that, in turn, explicitly assumes that T is regular. Both results are false without the assumption of regularity as Example 1 shows.

is isomorphic to $(E')_{x'}$, the ideal generated by x' in E', under the order unit norm induced by x'. Furthermore, the adjoint of Q may then be identified with the natural embedding of $(E')_{x'}$ into E'.

This tells us that Q' maps the unit ball in G' onto the order interval $[-x',x']$ in E', which is not compact so that Q' is not a compact operator and therefore Q is not a compact operator. In order to show that Q is actually L-weakly compact it suffices, by Theorem 3.6.11 of [9], to show that Q' is M-weakly compact. But if (f_n) is any disjoint sequence in the unit ball of F' then $(Q'f_n)$ is a disjoint sequence in the order interval $[-x',x']$ and, as we already know that E' has an order continuous norm, we must have $\|Q'(f_n)\| \to 0$. Thus Q' is M-weakly compact. □

4 Compactness of M-weakly compact operators.

As an immediate consequence of Theorem 1 we have:

Corollary 2 *Let E be a Banach lattice and X a Banach space. If $(E')^a$ is discrete, then each M-weakly compact operator $T : E \to X$ is compact.*

Proof If $T : E \to X$ is M-weakly compact then by Theorem 3.6.11 of [9], $T' : X' \to E'$ is L-weakly compact. If $(E')^a$ is discrete then Theorem 1 tells us that T' is compact and Schauder's Theorem now shows that T is compact. □

Again, we have a partial converse. In the Banach lattice setting we have:

Theorem 4 *Let E and F be Banach lattices and suppose that F does not have an order continuous norm. The following are equivalent:*

1. *Every M-weakly compact operator $T : E \to F$ is compact.*
2. *Every positive M-weakly compact operator $T : E \to F$ is compact.*
3. *$(E')^a$ is discrete.*

Proof Again (1) certainly implies (2) and that (3) implies (1) follows from Corollary 2. We prove that (2) implies (3). We assume that $(E')^a$ is not discrete and construct a positive M-weakly compact operator from E into F which is not compact.

Since the norm of F is not order continuous then, by Theorem 2.4.2 of [9], there exists a disjoint order bounded sequence (y_n) in F_+ which does not converge to zero in norm. We may assume that $0 \le y_n \le y$ and $\|y_n\| = 1$ for all n and some $y \in F_+$.

As $(E')^a$ is not discrete, there is $0 \le \psi \in (E')^a$ such that the order interval $[0,\psi]$ is not norm compact, see Theorem 6.1 of [12]. Fix a sequence (ϕ_n) in $[0,\psi]$ which has no norm convergent subsequence in E'. Also, since $\psi \in (E')^a$, the order interval $[0,\psi]$ is an L-weakly compact subset of E' and hence $[0,\psi]$ is weakly compact in E' (see Proposition 3.6.5 of [9].) Thus, by the Eberlein-Šmulian Theorem ([2], Theorem 3.40), we may assume, by extracting a subsequence if necessary, that (ϕ_n) converges weakly to some $\phi \in [0,\psi]$. So (ϕ_n) converges weakly* to ϕ.

Now, define two operators $S,T : E \to F$ by

$$S(x) = \phi(x)y + \sum_{n=1}^{\infty} (\phi_n - \phi)(x)y_n \text{ and } T(x) = \psi(x)y \text{ for each } x \in E.$$

It follows from the proof of Theorem 1 of [11] that $0 \leq S \leq T$ and that S is not compact. To finish the proof, we have to show that S is M-weakly compact. By Proposition 3.6.11 of [9], it suffices to establish that the adjoint $S' : F' \to E'$ is L-weakly compact. For this, note that $0 \leq S' \leq T'$ and $T'(h) = h(y)\psi$ for all $h \in F'$. Then for every $h \in B_{F'}$, we have

$$\left|S'(h)\right| \leq S'(|h|) \leq T'(|h|) = |h|(y)\,\psi \leq \|y\|\,\psi,$$

so that $S'(B_{F'}) \subset \|y\|\,[-\psi,\psi]$. Since $[-\psi,\psi]$ is an L-weakly compact subset of E' so is $S'(B_{F'})$ and S' is L-weakly compact. $\square$

If we are willing to drop the second equivalence involving positive M-weakly compact operators then we can do slightly better.

Theorem 5 *Let E be a Banach lattice and Y a Banach space which contains an isomorphic copy of c_0 then the following are equivalent:*

1. *Every M-weakly compact operator $T : E \to Y$ is compact.*
2. *$(E')^a$ is discrete.*

Proof Again (2) implies (1) follows from Corollary 2. We again prove that (1) implies (2) by contradiction. We assume that $(E')^a$ is not discrete and construct an M-weakly compact operator from E into Y which is not compact.

Let $J : c_o \to H$ be an isomorphism, where H is a closed subspace of Y.

As $(E')^a$ is not discrete, there is $0 \leq \psi \in (E')^a$ such that the order interval $[0,\psi]$ is not norm compact (see Theorem 6.1 of [12]). Fix a sequence (ϕ_n) in $[0,\psi]$ which has no norm convergent subsequence. Also, since $\psi \in (E')^a$, the order interval $[0,\psi]$ is relatively weakly compact. Thus, by the Eberlein-Šmulian Theorem ([2], Theorem 3.40), we may assume, by extracting a subsequence if necessary, that (ϕ_n) converges weakly, and hence weak*, to some $\phi \in [0,\psi]$.

Define a positive operator $R : E \to c_0$ by

$$R(x) = \big((\phi - \phi_n)(x)\big)_{n=1}^{\infty} \text{ for each } x \in E.$$

We show that the composed operator $T = J \circ R : E \to Y$ is M-weakly compact but it is not compact or, equivalently, R is M-weakly compact but it is not compact.

If R were compact then Schauder's Theorem would show that $R' : \ell_1 \to E'$ is compact. But if e_n is the usual basis element in ℓ_1 then $R'(e_n) = \phi - \phi_n$ so that (ϕ_n) would have a norm convergent subsequence. This contradicts the choice of (ϕ_n).

To show that R is M-weakly compact, it suffices to establish that $R' : \ell_1 \to E'$ is L-weakly compact, by Proposition 3.6.11 of [9]. For this, note that each $\phi - \phi_n \in [-\psi,\psi]$. For every $(\lambda_n) \in B_{l^1}$ we have

$$\left|R'\big((\lambda_n)\big)\right| = \left|\sum_{n=1}^{\infty} \lambda_n (\phi - \phi_n)\right| \leq \left(\sum_{n=1}^{\infty} |\lambda_n|\right)\psi \leq \psi.$$

Hence $R'(B_{l^1}) \subset [-\psi,\psi]$. Since $[-\psi,\psi]$ is an L-weakly compact subset of E' (because $\psi \in (E')^a$) so is $R'(B_{l^1})$. Hence R' is L-weakly compact. $\square$

In particular, by Theorem 2.4.12 of [9], the conclusion of Theorem 5 holds if Y is a Banach lattice which is not a KB-space.

Using Theorem 5 and Theorem 2.2 of [4] we have:

Corollary 3 *For a Banach lattice E the following assertions are equivalent:*

1. *An operator $T : E \to c_0$ is M-weakly compact if and only if it is compact.*
2. *E' is discrete with an order continuous norm.*

As with L-weakly compact operators, we also have a condition on the range space which gives the conclusion we seek for regular operators.

Theorem 6 *The following conditions on a Banach lattice F are equivalent:*

1. *F is discrete and its norm is order continuous.*
2. *For every Banach lattice E, every regular M-weakly compact operator $T : E \to F$ is compact.*
3. *For every Banach lattice E, every positive M-weakly compact operator $T : E \to F$ is compact.*

Proof That (1) implies (2) follows from Proposition 1 whilst (2) implies (3) is clear.

Let us suppose that (3) holds. If we choose E such that $(E')^a$ is not discrete (for example $E = l^\infty$) we obtain from Theorem 4 that F has an order continuous norm. Next, we claim that F is discrete. Assume by way of contradiction that F is not discrete. Then there exists some $0 \leq y \in F$ such that the order interval $[-y, y]$ is not norm compact in F (Theorem 6.1 of [12].) Take $E = F_y$, the principal ideal in F generated by y, with the order unit norm generated by y, and let $i : E \to F$ denote the canonical imbedding.

The image under i of the unit ball in E is precisely the order interval $[-y, y]$ which is not norm compact in F, so that i is not compact. However, i is M-weakly compact. To see this, let (x_n) be a disjoint sequence in the closed unit ball of E. Then $|x_n| \leq y$ for all n. Hence the sequence (x_n) is disjoint and order bounded in F and, since the norm of F is order continuous, converges to 0 for the norm of F. I.e. i is M-weakly compact as claimed. □

Notice that, by combining Theorem 5 with Theorem 6, we see that every regular M-weakly compact operator from any Banach lattice E into c_0 is compact but that, if $(E')^a$ is not discrete, there is a (non-regular) M-weakly compact operator from E into c_0 which is not compact.

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