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VECTOR LATTICES OF ALMOST POLYNOMIAL SEQUENCES.

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Abstract. We study the order structure, various duals and orthomorphisms on vector lattices of sequences that differ from a polynomial (either of bounded or unbounded degree) by a sequence in ℓ_p.

1. Introduction.

In [12] the suggestion was made of replacing a complete order unit norm on a vector lattice E by the rather weaker assumption that there is an element $a \in E$ such that any $\|\cdot\|_a$-Cauchy sequence is $\|\cdot\|_a$-convergent, where

$$\|x\|_a = \inf\{\lambda \in \mathbb{R}_+ : |x| \le \lambda |a|\},$$

with the convention that $\inf \emptyset = +\infty$. In [11] I pointed out the very high degree of generality of this notion and suggested restricting attention to the case that a was a weak order unit. I also suggested, as a non-trivial example, the space of sequences that differ from a polynomial (either of fixed or arbitrary degree) either by a bounded or a null sequence. These examples actually have rather more structure than I realized at the time, so we should expect particularly nice behaviour. It turns out not to be all that good after all. It does, however, seem that we can in some ways generalize my suggestion in [11] even more without losing whatever nice properties we do have.

In §2 we define the spaces that we will study, make a few simple comments on their topology and its relationship to the linear structure and then we look at the order, showing that we do indeed have a vector lattice structure but that not much more can be said about the order in general. The third section looks at linear functionals on the vector lattices that we define, in particular the relationship between continuity and regularity for them. In the final section we characterize the centre and orthomorphisms of these spaces and obtain some slightly surprising results.

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2. Definitions, Topology and Order Structure.

Before we start, let us point out that at various points below it is necessary to refer both to the set of positive integers and the set of non-negative integers. In order to avoid continual repetition, we define now $\mathbb{N} = \{1, 2, 3, \dots\}$ to be the set of positive integers and $\mathbb{N}^* = \{0, 1, 2, \dots\}$ to be the set of non-negative integers.

Our spaces are going to be defined as the sums of ℓ_p-spaces and certain polynomial sequences. As some features of our results become clearer if we allow the case $p < 1$, and proofs are no more complicated (although details of statements sometimes differ) we will work with ℓ_p for all $p > 0$. As a notational convenience, we will also define ℓ_0 to mean the usual space of null sequences, c_0, so that we will be allowing p to range over $[0, \infty]$.

If $0 < p < \infty$ then we define on ℓ_p

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}.$$

If $1 \leq p < \infty$ then $\|\cdot\|_p$ is a norm, whilst for $0 < p < 1$ it is a quasi-norm. On ℓ_∞ and on $\ell_0 = c_0$ we use the usual supremum norm

$$\|x\|_\infty = \sup\{|x_n|; n \in \mathbb{N}\}.$$

In all cases, the topology induced on ℓ_p is a complete metrizable one, so is of the second Baire category.

If $1 \leq p < \infty$ then the topological dual of ℓ_p, ℓ_p', may be identified with ℓ_q where $p^{-1} + q^{-1} = 1$, with the convention that if $p = 1$ then $q = \infty$. If $p = 0$ then ℓ_0' may be identified with ℓ_1, whilst if $0 < p \leq 1$ then Day proved in [5] that ℓ_p' may be identified with ℓ_∞. The topological dual in the case that $p = \infty$ is much larger than ℓ_1, but we will have no need of its detailed description. In all cases, it follows from the representation of the topological dual that all elements are *regular*, i.e. the difference of two positive linear functionals. The converse is also true, so that the *order dual* of ℓ_p, $\ell_p^\sim$, is equal to ℓ_p' in all these cases. This is well known for the Banach lattice cases, but actually follows in all the cases from Proposition 2.16 (c), in Chapter 2 of [9]. The *order continuous dual*, $\ell_p^\times$, is equal to $\ell_p^\sim$ in all cases except $p = \infty$ when $\ell_\infty^\times$ may be identified with ℓ_1.

We will use the same notation for such of the various duals as are defined in a general vector lattice E, and also the notation E^s for the band of *singular* linear functionals which is simply the complementary band in $E^\sim$ to $E^\times$.

Definition 2.1. If $k \in \mathbb{N}^*$ and $p \in [0, \infty]$ then

(1) $\mathfrak{p}_p(k)$ consists of all real sequences (a_n) such that there is a polynomial p of degree at most k such that $\big(a_n - p(n)\big) \in \ell_p$.
(2) $\mathfrak{p}_p = \bigcup_{k=1}^{\infty} \mathfrak{p}_p(k)$.

It is clear that all of these define vector spaces over the reals. Note, in particular, that $\mathfrak{p}_0(0) = c$ and that $\mathfrak{p}_\infty(0) = \ell_\infty$. If X_p denotes any of the spaces $\mathfrak{p}_p(k)$ or $\mathfrak{p}_p$ and if $p < \infty$ (including $p = 0$) then, as only the zero polynomial lies in ℓ_p, the decomposition of an element of X_p into a polynomial part and an ℓ_p-sequence is unique. If $p = \infty$ then the decomposition is unique to within a constant sequence.

If a sequence x does not lie in ℓ_p then we will set $\|x\|_p = \infty$ (note that $X_0 \cap \ell_\infty = c_0$.) The sets of the form $\{x : \|x - y\|_p < \epsilon\}$, for $y \in X_p$ and $\epsilon > 0$ form a base for a topology on X_p for which convergence is precisely $\|\cdot\|_p$-convergence. If $p < \infty$ and $\|x - y\|_p < \infty$ then the polynomial parts of x and of y must be the same whilst if $p = \infty$ those polynomial parts may differ by a constant. This topology is metrizable and the definitions make it clear that all of these spaces are complete and hence of the second Baire category. Unfortunately that is not as useful as might have been thought as open sets will not be absorbent.

It is easy to see that addition is continuous for this topology so that these spaces are commutative topological groups under addition. However if $\|x\|_p = \infty$ then $\|\frac{1}{n}x\|_p = \infty$ for all $n \in \mathbb{N}$ so that $\frac{1}{n}x \not\to 0$ and scalar multiplication is not continuous so that we do not have a topological vector space.

These spaces were introduced as examples of vector lattices, so it is natural that we investigate their order structure. First we must justify our implied claim that they are vector lattices.

Proposition 2.2. *For all $k \in \mathbb{N}^*$, and $p \in [0, \infty]$, $\mathfrak{p}_p(k)$ and $\mathfrak{p}_p$ are vector lattices under the pointwise partial order.*

Proof. If $a, b \in \mathfrak{p}_p(k)$ then there are polynomials q and r of degree at most k such that $\big(a_n - q(n)\big), \big(b_n - r(n)\big) \in \ell_p$. Without loss of generality we may make the choice of polynomials unique by specifying that the constant term is zero. As $\ell_p \subseteq \ell_\infty$ for all p, there is $M \in \mathbb{R}$ with $|a_n - q(n)|, |b_n - r(n)| \le M$ for all $n \in \mathbb{N}$.

If $q = r$ then

$$\begin{aligned}\big(a_n \vee b_n - q(n)\big) &= \big(a_n - q(n)\big) \vee \big(b_n - q(n)\big)\\ &= \big(a_n - q(n)\big) \vee \big(b_n - r(n)\big) \in \ell_p,\end{aligned}$$

so that $(a_n \vee b_n) \in \mathfrak{p}_p(k)$.

If $q \neq r$ then, given that neither has a non-zero constant term, $q - r$ is unbounded so that *either* $q(n) - r(n) \to \infty$ as $n \to \infty$ *or* $r(n) - q(n) \to \infty$ as $n \to \infty$. We consider the first case, the second being similar. Choose $N \in \mathbb{N}$ such that $n \geq N \Rightarrow q(n) - r(n) \geq 2M$. Then, if $n \geq N$ we have

$$a_n \geq p(n) - M \geq q(n) + M \geq b_n$$

so that $a_n \vee b_n = a_n$ if $n \geq N$. Hence $|a_n \vee b_n - q(n)| = |a_n - q(n)|$ if $n \geq N$. It follows that

$$(a_n) \vee (b_n) - \big(q(n)\big) \in \ell_p$$

so that $(a_n \vee b_n) \in \mathfrak{p}_p(k)$. This suffices to show that $\mathfrak{p}_p$ is a vector lattice.

It is clear that $\mathfrak{p}_p(k) \subset \mathfrak{p}_p(k+1)$ for all $k \in \mathbb{N}^*$, from which it follows easily that $\mathfrak{p}_p$ is also a vector lattice. $\square$

We noted above that $\mathfrak{p}_0(0) = c$, so that we should not expect the spaces $\mathfrak{p}_0(k)$ to enjoy any nice order theoretic completeness properties. The space $\mathfrak{p}_\infty(0)$, on the other hand is just ℓ_∞ which is Dedekind complete. However, we have:

Example 2.3. *For any $p \in [0, \infty]$, $\mathfrak{p}_p(1)$ is not Dedekind σ-complete.*

Proof. If we use e_n to denote the usual n'th basis vector then certainly $ne_n \in \mathfrak{p}_p(1)$ for all $n \in \mathbb{N}$. The sequence $i = (n)$ lies in $\mathfrak{p}_p(1)$ and is an upper bound for the set $\{2ne_{2n} : n \in \mathbb{N}\}$. It is routine, because of the presence of the basis vectors in the space, to see that if this set had a supremum then it would be the pointwise supremum. That would be the sequence (s_n) where $s_{2n} = 2n$ and $s_{2n-1} = 0$ for all $n \in \mathbb{N}$. No matter what first degree polynomial, $an + b$, we take, which we might hope had $\|\big(s_n - (an + b)\big)\|_p < \infty$ and hence (for some $M \in \mathbb{R}$) $|s_n - (an + b)| \leq M$ for all $n \in \mathbb{N}$, we would have

$$\begin{aligned} 2M &\geq \big|2n - \big(a(2n) + b\big)\big| + |a(2n+1) + b - 0| \\ &= \big|(2n + a) - \big(a(2n+1) + b\big)\big| + |a(2n+1) + b| \\ &\geq |2n + a| \to \infty \end{aligned}$$

as $n \to \infty$, which is impossible. $\square$

Using as upper bounds all the possible sequences $i - (2n+1)e_{2n+1}$ will similarly show that $\mathfrak{p}_p(1)$ does not have the countable interpolation property.

The bands in the spaces $\mathfrak{p}_p(k)$ and in $\mathfrak{p}_p$ are precisely the sequences supported by a fixed subset of $\mathbb{N}$ and, as elements of ℓ_p can be found which are non-zero precisely on any specified subset of $\mathbb{N}$, they are all principal bands. In the language of the last example, there is no band projection onto the band consisting of sequences supported by $\{2n : n \in \mathbb{N}\}$, as can be seen by considering the image of the sequence i. This shows that (at least for $k \geq 1$) $\mathfrak{p}_p(k)$ does not have the principal projection property.

The Dedekind completions of $\mathfrak{p}_p(k)$ or of $\mathfrak{p}_p$ may be identified with the ideal that they generate in the space of all real sequences, in which they are clearly *super order dense* (i.e. every positive element of the completion is the supremum of a countable family from the space.) This means that the spaces are *almost σ-Dedekind complete* in the terminology of [1].

Recall that a vector lattice is (o)*-complete* or *order Cauchy complete* (see, [4]) if every order Cauchy sequence is order convergent. Even c, which is precisely $\mathfrak{p}_0(1)$ fails to be (o)-complete.

The spaces $\mathfrak{p}_p(k)$ have a strong order unit, namely the sequence $(n^k)_{n=1}^{\infty}$, whilst the spaces $\mathfrak{p}_p$ do not, although they do have weak order units.

3. Linear Functionals.

There are three classes of linear functionals on our spaces in which we have an obvious interest. We know that for Banach lattices, as well as for other topological vector lattices including all the ℓ_p-spaces, the toplogical and order duals coincide. That turns out not to be the case here, although we do have, in the positive direction:

Proposition 3.1. *For all $k \in \mathbb{N}^*$, and $p \in [0, \infty]$, every regular linear functional on any of the spaces $\mathfrak{p}_p(k)$ or $\mathfrak{p}_p$ is continuous.*

Proof. It suffices to prove continuity at the origin. Sequences which converge to the origin are eventually in the corresponding ℓ_p space, on which the restriction of a regular functional is again regular. Now use the continuity of regular linear functionals on ℓ_p. □

For a start, we describe the topological duals.

Proposition 3.2. *For any $p \in [0, \infty)$, $k \in \mathbb{N}^*$ and any $f \in \mathfrak{p}_p(k)'$ there are $g \in \ell_p'$ and $a_j \in \mathbb{R}$, for $0 \leq j \leq k$ such that if $\mathfrak{p}_p(k) \ni x = z + (\sum_{j=0}^{k} c_j n^j)_{n=1}^{\infty}$ with $z \in \ell_p$ and $c_j \in \mathbb{R}$ for $0 \leq j \leq k$ then*

$$f(x) = g(z) + \sum_{j=0}^{k} a_j c_j.$$

Proof. Algebraically, we may write $\mathfrak{p}_p(k)$ as a direct sum of ℓ_p and the polynomial sequences of degree at most k. A sequence (x_m) of elements of $\mathfrak{p}_p(k)$, each of which may be written as $x_m = z_m + \big(p_m(n)\big)$ with $z_m \in \ell_p$ and p_m a polynomial, is convergent if and only if the sequence of polynomials (p_m) is eventually constant and (z_m) converges in ℓ_p. This shows that functionals of the given form are continuous. Conversely, for algebraic reasons any functional may be written in the given form except for not assuming that g is continuous. Continuity of f applied to sequences in ℓ_p shows that g is continuous. □

If $p = \infty$ the description needs to be modified slightly to take into account the fact that constant polynomials lie in ℓ_∞, but the proof is no different.

Proposition 3.3. *For any $k \in \mathbb{N}^*$ and any $f \in \mathfrak{p}_\infty(k)'$ there are $g \in \ell'_\infty$ and $a_j \in \mathbb{R}$, for $1 \le j \le k$ such that if $\mathfrak{p}_\infty(k) \ni x = z + (\sum_{j=1}^k c_j n^j)_{n=1}^\infty$ with $z \in \ell_p$ and $c_j \in \mathbb{R}$ for $1 \le j \le k$ then*

$$f(x) = g(z) + \sum_{j=1}^{k} a_j c_j.$$

Similarly there are descriptions of $\mathfrak{p}'_p$ which involve infinite sequences of reals (a_j). There are no convergence considerations as we are only considering polynomials of finite, although arbitrary, order.

Theorem 3.4. *If $p \in [0, \infty)$ and $k \in \mathbb{N}^*$ then*

(1) *If $f \in \mathfrak{p}_p(k)^\times$ then there is a sequence (a_n) with $(n^k a_n) \in \ell_1$ such that $f(x) = \sum_{n=1}^\infty a_n x_n$ for all $x \in \mathfrak{p}_p(k)$. Conversely, every such sequence (a_n) defines an order continuous linear functional f.*

(2) *If $g \in \mathfrak{p}_p(k)^s$ then there is $\alpha \in \mathbb{R}$ such that if we write $x = z + (\sum_{j=0}^k c_j n^j)_{n=1}^\infty$, with $z \in \ell_p$, then $g(x) = \alpha c_k$. Conversely, this formula always defines a singular functional on $\mathfrak{p}_p(k)$.*

Proof. It suffices to prove these claims for positive functionals and then apply them to the positive and negative parts in general.

If $f \in \mathfrak{p}_p(k)_+^\times$ and we define $a_n = f(e_n)$, then for any $x \in \mathfrak{p}_p(k)$ we have $x = \sup\{\sum_{j=1}^n x_j e_j : n \in \mathbb{N}\}$ so that

$$f(x) = \sup\{\sum_{j=1}^{n} x_j a_j : n \in \mathbb{N}\} = \sum_{j=1}^{\infty} x_j a_j.$$

The condition that $(n^k a_n) \in \ell_1$ is because the sequence $(n^k) \in \mathfrak{p}_p(k)$. The converse is clear.

In particular, we may identify each e_n with an element of $\mathfrak{p}_p(k)^\times$, which we will denote by $\tilde{e}_n$ to distinguish it from the corresponding member of $\mathfrak{p}_p(k)$. If $g \in \mathfrak{p}_p(k)^s$ then $g \wedge \tilde{e}_n = 0$ for all n. As

$$\begin{aligned}
0 &= (g \wedge \tilde{e}_n)(e_n) \\
&= \inf\{g(x) + \tilde{e}_n(y) : 0 \le x, y \text{ with } x + y = e_n\} \\
&\qquad \text{(using the Riesz-Kantorovich formula for the infimum,} \\
&\qquad \text{ see e.g. Theorem 83.6 (5) of [13])} \\
&= \inf\{g(\lambda e_n) + \tilde{e}_n(\mu e_n) : 0 \le \lambda, \mu \in \mathbb{R} \text{ with } \lambda + \mu = 1\} \\
&= \inf\{\lambda g(e_n) + \mu : 0 \le \lambda, \mu \in \mathbb{R} \text{ with } \lambda + \mu = 1\}
\end{aligned}$$

we see that $g(e_n) = 0$. Given that regular linear functionals are continuous it follows that g is zero on ℓ_p. Thus, to describe g completely we need only specify its value on the polynomial sequences (n^j) for $1 \le j \le k$. The claim in the theorem is that g vanishes at all of these except for $j = k$. If $j < k$ then, for all $N \in \mathbb{N}$, there is $n_0 \in \mathbb{N}$ such that $n^k \ge Nn^j$ for $n > n_0$. Thus there is a sequence (x_n) with only finitely many non-zero terms such that $n^k + x_n \ge Nn^j$ for *all* $n \in \mathbb{N}$. I.e. in terms of sequences we have $(n^k) + x \ge N(n^j)$. As $g(x) = 0$, we see that $g\big((n^k)\big) \ge Ng\big((n^j)\big) \ge 0$ for all $N \in \mathbb{N}$. This certainly shows that $g\big((n^j)\big) = 0$ as claimed. Again, the converse is routine to verify. □

Notice, in particular, that the order duals of $\mathfrak{p}_p(k)$ do not depend on p, as long as $p \ne \infty$. In this case, the order continuous dual is linearly order isomorphic to ℓ_1, so that its order dual and order continuous dual coincide and may be identified with ℓ_∞. It follows that:

Corollary 3.5. *If $p \in [0, \infty)$ and $k \in \mathbb{N}^*$ then $\mathfrak{p}_p(k)^{\times\sim} = \mathfrak{p}_p(k)^{\times\times}$ and may be identified with the sequences (b_n) such that $(n^{-k}b_n) \in \ell_\infty$. This is precisely the order ideal generated in the space of all real sequences by $\mathfrak{p}_p(k)$, which may be identified with the Dedekind completion of $\mathfrak{p}_p(k)$.*

We can extend the descriptions of the duals to the case of polynomials of arbitrary degree as well.

Corollary 3.6. *If $p \in [0, \infty)$ then $\mathfrak{p}_p^\sim = \mathfrak{p}_p^\times$ and both may be identified with the sequences $\{(a_n) : (n^k a_n) \in \ell_1 \text{ for all } k \in \mathbb{N}^*\}$.*

Proof. This description of $\mathfrak{p}_p^\times$ is immediate from the preceding theorem. If $g \in \mathfrak{p}_p^s$ then the argument in the proof of the theorem (using the fact that each $\tilde{e}_n \in \mathfrak{p}_p^\times$) shows that g vanishes at each of the e_n. It follows from the theorem that $g_{|\mathfrak{p}_p(k)} \in \mathfrak{p}_p(k)^s$ so that g vanishes on the sequence (n^j) for any $j < k$. But k was arbitrary, so that g vanishes on all sequences (n^k). As these, together with ℓ_p, span $\mathfrak{p}_p$ algebraically, that shows that $g = 0$ and the proof is complete. $\square$

The next result is very similar to Corollary 3.5, although the proof is not quite so straightforward.

Corollary 3.7. *If $p \in [0, \infty)$ then $\mathfrak{p}_p^{\sim\sim}$ may be identified with the sequences $\{(a_n) : (n^{-k} a_n) \in \ell_\infty$ for all $k \in \mathbb{N}^*\}$. This is precisely the order ideal generated by $\mathfrak{p}_p$ in the space of all real sequences, which may be identified with the Dedekind completion of $\mathfrak{p}_p$.*

Proof. We know that $\mathfrak{p}_p^\sim$ may be identified with the space of sequences (x_n) such that $n^k x_n \in \ell_1$ for all $k \in \mathbb{N}^*$. We can topologize $\mathfrak{p}_p^\sim$ using the infinite sequence of norms $\|(x_n)\|_{k+1} = \|(n^k x_n)\|_1$, where $\|\cdot\|_1$ denotes the usual ℓ_1-norm. Our labeling of the norms makes the notation $\|\cdot\|_1$ unambiguous. This topology, which is precisely $|\sigma|(\mathfrak{p}_p^\sim, \mathfrak{p}_p)$, is certainly metrizable. We claim that $\mathfrak{p}_p^\sim$ is complete under $|\sigma|(\mathfrak{p}_p^\sim, \mathfrak{p}_p)$. Indeed, if we have a sequence (x^m) of elements of $\mathfrak{p}_p^\sim$ which is Cauchy for each of the norms then $\left((n^k x_n^m)_{n=1}^\infty\right)_{m=1}^\infty$ will be Cauchy in ℓ_1 for each $k \in \mathbb{N}^*$ so converges to a limit $y^k \in \ell_1$. As $\|(z_n)\|_1 \le \|(nz_n)\|_1$ if $(nz_n) \in \ell_1$, we see that these limits must be consistent in the sense that $y_n^{k+1} = n y_n^k$. I.e. $y_n^k = n^k y_n^0$. This means that $\|x^m - y^0\|_k \to 0$, as $m \to \infty$, for all $k \in \mathbb{N}^*$ so that $x^m \to y^0$ for $|\sigma|(\mathfrak{p}_p^\sim, \mathfrak{p}_p)$. It is clear also that the positive cone in $\mathfrak{p}_p^\sim$ is closed under each of the norms $\|\cdot\|_k$ and is therefore closed so is itself complete. It now follows from Proposition 2.16 (c), in Chapter 2 of [9] that every positive (and therefore every regular) linear functional on $\mathfrak{p}_p^\sim$ is continuous. As the sequences with finitely many non-zero terms are dense in $\mathfrak{p}_p^\sim$, this means that such functionals may certainly be described by real sequences acting in the usual duality. The fact that the sequences which do so are precisely those that we claim is now routine. $\square$

Again note that the description of these duals does not involve the number p. Notice also that there will certainly be continuous linear functionals which are not regular, either because sequences representing elements of the dual of ℓ_p need not satisfy the condition on elements of the order continuous dual or because we can give (n^j) a non-zero image when working on $\mathfrak{p}_p(k)$ with $k > j$. The order dual of $\mathfrak{p}_p$ is certainly non-trivial, including not only sequences of finite support, but also sequences like (n^{-n}).

4. Centre and Orthomorphisms.

The *centre* of an Archimedean vector lattice E consists of all linear operators T on E for which there is $\lambda \in \mathbb{R}$ such that $|Tx| \leq \lambda|x|$ for all $x \in E$. In a concrete setting, these operators are precisely the operators of pointwise multiplication by bounded functions which leave E invariant. The *orthomorphisms* of E are the order bounded operators such that $x \perp y \Rightarrow Tx \perp y$. In concrete cases, these are similar to central operators except that the boundedness condition on the multiplier function is removed. The centre of E is denoted by $Z(E)$ and the space of orthomorphisms on E by $\mathrm{Orth}(E)$. We always have $Z(E) \subset \mathrm{Orth}(E)$. For any $p \in [0, \infty]$ we have identifications $\mathrm{Orth}(\ell_p) = Z(\ell_p) = \ell_\infty$.

We start by dealing with the space $\mathfrak{p}_p(k)$.

Theorem 4.1. *If $p \in \{0\} \cup (1, \infty]$ and $k \in \mathbb{N}^*$ then the centre of $\mathfrak{p}_p(k)$, $Z\big(\mathfrak{p}_p(k)\big)$, coincides with $\mathrm{Orth}\left(\mathfrak{p}_p(k)\right)$ and may be identified with the space of all sequences (z_n) such that $(n^k z_n) \in \mathfrak{p}_p(k)$ under the action $(a_n) \mapsto (a_n z_n)$.*

Proof. It is routine to verify that orthomorphisms on $\mathfrak{p}_p(k)$ may be identified with a certain sequence space acting in this manner. We need to verify exactly which sequences (z_n) arise in this way.

First, note that $(n^k) \in \mathfrak{p}_p(k)$ so that we certainly need $(z_n n^k) \in \mathfrak{p}_p(k)$ if multiplication by (z_n) does define a orthomorphism.

Now suppose that (z_n) does satisfy this condition. Notice first that this certainly entails that (z_n) is bounded, so that if multiplication by (z_n) does leave $\mathfrak{p}_p(k)$ invariant then that will certainly define a central operator. If $(a_n) \in \mathfrak{p}_p(k)$ then we need to verify that $(a_n z_n) \in \mathfrak{p}_p(k)$. There are polynomials p and q of degree at most k and sequences $(c_n), (d_n) \in \ell_p$ such that $a_n = c_n + p(n)$ and $n^k z_n = d_n + q(n)$. As (z_n) is bounded, $(z_n c_n) \in \ell_p$. Also, $(d_n/n^k)\big(p(n)\big) = (d_n)\big(p(n)/n^k\big)$ and $p(n)/n^k$ is bounded as p is of degree at most k so that $(d_n/n^k)\big(p(n)\big) \in \ell_p$. Finally, the expression $\frac{q(n)}{n^k} \times p(n)$ may be written as a polynomial of degree k in n, for which the corresponding sequence certainly lies in $\mathfrak{p}_p(k)$, plus a polynomial in n^{-1} with no constant term. The sequence defined by the polynomial in n^{-1} will certainly lie in c_0, taking care of the cases $p = 0$ and $p = \infty$ and will behave like $(1/n)$ so will lie in ℓ_p for $p > 1$. $\square$

It is clear that the final stage in this proof fails for $0 < p \leq 1$, but does the result? Yes!

Theorem 4.2. *If $k \in \mathbb{N}^*$ and $0 < p \leq 1$ then the centre of $\mathfrak{p}_p(k)$, $Z\big(\mathfrak{p}_p(k)\big)$, coincides with* $\mathrm{Orth}\,\big(\mathfrak{p}_p(k)\big)$ *and may be identified with the space of all sequences (z_n), which may be written as a constant sequence plus a sequence (w_n) such that $(n^k w_n) \in \ell_p$, under the action $(a_n) \mapsto (a_n z_n)$.*

Proof. The proof of the preceding theorem certainly shows that such sequences (w_n) define central operators on $\mathfrak{p}_1(k)$ as do the constant sequences. If (z_n) does define an orthomorphism on $\mathfrak{p}_p(k)$ then by applying it to the sequence (n^k) we see that $(n^k z_n)_{n=1}^{\infty} \in \mathfrak{p}_p(k)$. As the ℓ_p part of this sequence does act on $\mathfrak{p}_p(k)$, the difference, which is multiplication by a polynomial of degree at most k divided by n^k, will also do so. This difference may be written as $\sum_{j=0}^{k} c_j n^{-j}$. How does this act on the sequence $(n^i) \in \mathfrak{p}_p(k)$, for $0 \leq i < k$? We obtain the sequence with n'th term $\sum_{j=0}^{k} c_j n^{i-j} = \sum_{j=0}^{i} c_j n^{j-i} + \sum_{j=i+1}^{k} c_j n^{j-i}$ where the first sum is a polynomial and the second behaves like c_{i+1}/n if $c_{i+1} \neq 0$. This would prevent the non-polynomial part of the sequence from being in ℓ_p so that $c_{i+1} = 0$ if $0 \leq i < k$. Thus the difference sequence is actually constantly c_0. ☐

Alternatively we could describe these sequences (z_n) by the property that $(n^k z_n) \in \ell_p + \mathbb{R}(n^k)$ which simultaneously shows how much smaller the centre is in this case than in the case $p > 1$ and is also somewhat reminiscent of the description of regular functionals on the space.

We haven't actually seen the following circle of ideas in the literature, although it does seems a natural one. We use the notation $\hat{E}$ for the Dedekind completion of E and if $x \in E_+$ then we use the notations $[0, x]_E$ and $[0, x]_{\hat{E}}$ to denote order intervals in E and in $\hat{E}$ respectively.

Definition 4.3. If E is an Archimedean vector lattice let $D(E)$ denote the linear span of the cone $D(E)_+ = \{x \in E : [0, x]_E = [0, x]_{\hat{E}}\}$.

Theorem 4.4. *For any Archimedean vector lattice E, $D(E)$ is the largest ideal in E which is Dedekind complete in itself.*

Proof. It is clear that $D(E)_+$ is closed under multiplication by non-negative scalars. It is closed under addition as if $x_1, x_2 \in D(E)_+$ and $y \in [0, x_1 + x_2]_{\hat{E}}$ then the Riesz Decomposition property in $\hat{E}$ gives us $y_1, y_2 \in \hat{E}$ with $y_i \in [0, x_i]_{\hat{E}}$ for $i = 1, 2$. By hypothesis, $y_i \in [0, x_i]_E$, so that $y = y_1 + y_2 \in E$ also. If $0 \leq z \leq x \in D(E)_+$ and $y \in [0, z]_{\hat{E}}$ then also $y \in [0, x]_{\hat{E}} = [0, x]_E$, so that $y \in [0, z]_E$ and hence $z \in D(E)_+$. The linear span of $D(E)$ is now clearly an ideal in E. To see that it is Dedekind complete it suffices to consider non-empty subsets A of the positive cone that are bounded above in $D(E)$. For some $x \in E$,

$A \subset [0,x]_E = [0,x]_{\hat{E}}$, so A has a supremum in $[0,x]_{\hat{E}}$ which also lies in $[0,x]$.

Conversely, if J is a Dedekind complete ideal in E let $x \in J_+$ and $y \in [0,x]_{\hat{E}}$ then the set $\{z \in E : 0 \le z \le y\}$ has supremum y in $\hat{E}$ and also has a supremum in J as J is Dedekind complete. As the Dedekind completion preserves arbitrary suprema and infima ([8], Theorem 32.2) these suprema coincide, showing that $y \in [0,x]_E$ and hence that $x \in D(E)_+$. As $J_+ \subset D(E)_+$ we certainly have $J \subset D(E)$. □

Corollary 4.5. *In every vector lattice there is a largest ideal which is Dedekind complete in itself.*

Proof. Theorem 4.4 establishes this in the Archimedean case. Recall that a Dedekind complete vector lattice must be Archimedean, so that any Dedekind complete ideal in a vector lattice E will be Archimedean. But E contains a largest Archimedean ideal (see Theorem 1.5 of [6], Corollary 2.2 of [10] or look at [7] together with the *MathSciNet* review of [10]) and the largest Dedekind complete ideal in that will necessary be the largest Dedekind complete ideal in the original vector lattice. □

We will term this largest Dedekind complete ideal in E the *Dedekind complete kernel* of E. It is compatible with our previous definition if we also denote this by $D(E)$ in general, although the characterization in the Archimedean case will not work in general. In view of the fact that the result does not depend on E being Archimedean, our detour *via* the Archimedean case seems rather unnatural so that we would like to see a direct proof of this result.

In our context we have already commented on the description of the Dedekind completion of $\mathfrak{p}_p(k)$ and of $\mathfrak{p}_p$. These make the following result almost obvious.

Proposition 4.6. *If $p \in \{0\} \cup [1,\infty)$ and $k \in \mathbb{N}^*$ then the Dedekind complete kernel of $\mathfrak{p}_p(k)$, and of $\mathfrak{p}_p$, is precisely ℓ_p.*

Definition 4.7. The *non-Dedekind dimension* of a vector lattice is the linear dimension of the quotient space $E/D(E)$.

This is an isomorphic invariant of an Archimedean vector lattice, which will rarely be of any interest! However, in our setting the fact that the non-Dedekind dimension of $\mathfrak{p}_p(k)$ is precisely $k+1$, except when $p=\infty$ when it is k, is of some use to us.

Coming back to the centres of $\mathfrak{p}_p(k)$ we now note the following facts.

Proposition 4.8. *The spaces $\mathfrak{p}_p(k)$ and $Z\big(\mathfrak{p}_p(k)\big)$ are order isomorphic precisely when $p \in \{0\} \cup (1,\infty]$ or when $0 < p \le 1$ and $k = 0$.*

Proof. Theorem 4.1 proves the isomorphism when $p = 0$ or $p > 1$, *via* the map $(z_n) \mapsto (n^k z_n) : \mathfrak{p}_p(k) \to Z\big(\mathfrak{p}_p(k)\big)$. When $0 < p \le 1$ and $k = 0$ the description of the centre in Theorem 4.2 shows that $Z\big(\mathfrak{p}_p(0)\big)$ may be identified with the linear span of ℓ_p and the constant sequences, which is precisely $\mathfrak{p}_p(0)$. The description of $Z\big(\mathfrak{p}_p(k)\big)$ for $0 < p \le 1$ shows that it has non-Dedekind dimension of 1, whilst $\mathfrak{p}_p(k)$ has non-Dedekind dimension of $k+1$ so these spaces are not order isomorphic if $k \neq 0$. □

Even though $\mathfrak{p}_p(k) \subsetneqq \mathfrak{p}_p(k+1)$ and we have just seen that $Z\big(\mathfrak{p}_p(k)\big)$ is order isomorphic to $\mathfrak{p}_p(k)$ provided $p \neq 1$, we have the reverse inclusion between the centres.

Proposition 4.9. *For any $p \in [0, \infty]$ and $k \in \mathbb{N}^*$, $\mathfrak{p}_p(k)$ is invariant under the action of $Z\big(\mathfrak{p}_p(k+1)\big)$.*

Proof. We deal first with the case that $p = 0$ or $p > 1$. If (z_n) acts on $\mathfrak{p}_p(k+1)$ then $n^{k+1} z_n = x_n + p_{k+1}(n)$ where $(x_n) \in \ell_p$ and p_{k+1} is a polynomial of degree at most $k+1$. We may write $p_{k+1}(n) = n p_k(n) + \alpha$ where p_k is a polynomial of degree at most k and $\alpha \in \mathbb{R}$. Now $(n^k z_n) = (x_n/n) + \big(p_k(n)\big) + (\alpha/n)$. As $p = 0$ or $p > 1$ the first and third term in this sum lie in ℓ_p (remembering that (x_n) is bounded) whilst the centre one is a polynomial, so that $(n^k z_n) \in \mathfrak{p}_p(k)$ showing that $(z_n) \in Z\big(p_p(k)\big)$.

If $0 < p \le 1$ then the condition for a sequence to act on $\mathfrak{p}_p(k+1)$ is that we can write it as constant sequence, which will certainly act on $\mathfrak{p}_p(k)$, plus a term (z_n) with $(n^{k+1} z_k) \in \ell_p$. Clearly we will have $(n^k z_n) \in \ell_p$ so that the sequence also acts on $\mathfrak{p}_p(k)$. □

It is clear that the restriction map from $Z\big(\mathfrak{p}_p(k+1)\big)$ to $Z\big(\mathfrak{p}_p(k)\big)$ is never onto, so that we may also write $Z\big(\mathfrak{p}_p(k+1)\big) \subsetneqq Z\big(\mathfrak{p}_p(k)\big)$.

Although it is clear that none of the spaces $\mathfrak{p}_p(k)$ are closed under pointwise multiplication, it is now clear that we do have a product structure on them if $p > 1$ or $p = 0$.

Recall that a *Riesz algebra* is a vector lattice, which is also an associative algebra under a multiplication $\star$, such that $x, y \ge 0 \Rightarrow x \star y \ge 0$, whilst and *$f$-algebra* is a Riesz algebra such that if $y \perp z$ then $x \star y \perp z$ and $y \star x \perp z$. See [2] and [3] for two recent relevant survey articles.

Proposition 4.10. *If $p \in \{0\} \cup (1, \infty]$ and $k \in \mathbb{N}^*$ then $\mathfrak{p}_p(k)$ is an f-algebra under the multiplication $a \star_k b = (a_n b_n / n^k)$ with identity $(n^k)_{n=1}^{\infty}$.*

Proof. The centres are algebras under composition, and this translates into pointwise multiplication of the representing sequences. Combining this with the order isomorphism of the centre with the original space shows that $\mathfrak{p}_p(k)$ is $\star_k$-closed. The remainder of the proof is routine. □

This is *not* true when $0 < p \leq 1$ and $k > 0$. For example, (1), $(n^{k-1}) \in \mathfrak{p}_p(k)$ but $(1 \times n^{k-1} \times n^{-k}) = (1/n) \notin \mathfrak{p}_p(k)$. Of course, in the case $k = 0$, $\mathfrak{p}_p(0)$ is just $\ell_p \oplus \mathbb{R}(1)$ which is certainly closed under pointwise multiplication.

The description of the centres of $\mathfrak{p}_p$ are rather more interesting. Before we deal with the detailed representations in the two cases that we need to separate, we see what we can prove in general.

Theorem 4.11. *If $p \in [0, \infty]$ then the following conditions on a sequence $z = (z_n)$ are equivalent:*

(1) $z \in \mathrm{Orth}(\mathfrak{p}_p)$,
(2) $z \in Z(\mathfrak{p}_p)$,
(3) $z \in \bigcap_{k=1}^{\infty} Z\big(\mathfrak{p}_p(k)\big)$,

Proof. As central operators are always orthomorphisms, to show that (1) $\Leftrightarrow$ (2), we need only show that if $(z_n) \in \mathrm{Orth}(\mathfrak{p}_p)$ then (z_n) is bounded. As $(1) \in \mathfrak{p}_p$ we see that $(z_n) = (z_n \times 1) \in \mathfrak{p}_p$. Thus there is a polynomial q and a constant M (as $\ell_p \subset \ell_\infty$) such that $|z_n - q(n)| \leq M$ for all $n \in \mathbb{N}$. If (z_n) were not bounded then neither would q be and, in particular, q is not constant. Choose a subsequence (n_k) such that $q(n_k) \to \infty$ fast enough that the sequence $\big(k/q(n_k)\big) \in \ell_p$. Define $(x_n) \in \ell_p$ by $x_n = k/q(n_{2k})$ if $n = 2k$ and $x_n = 0$ otherwise. As (z_n) acts on $\mathfrak{p}_p$ the product $(x_n y_n)$ lies in $\mathfrak{p}_p$. As $\big((z_n - q(n))x_n\big) \in \ell_p$, we see that $\big(q(n)x_n\big) \in \mathfrak{p}_p$. But $q(n)x_n = k$ if $n = n_{2k}$ and $q(n)x_n = 0$ otherwise. If we set $\big(q(n)x_n\big) = \big(r(n)\big) + (b_n)$ where r is a polynomial and $(b_n) \in \ell_p \subset \ell_\infty$ then looking at the terms with $n \neq n_{2k}$ shows that the polynomial part is bounded and therefore the entire sequence must be bounded, which is not the case.

To see that (2) $\Rightarrow$ (3), suppose that $(z_n) \in Z(\mathfrak{p}_p)$. If $(z_n) \notin \bigcap_{k=1}^{\infty} Z\big(\mathfrak{p}_p(k)\big)$, let m be the smallest integer such that $(z_n) \notin \mathfrak{p}_p(m)$, noting that $(1) \in \mathfrak{p}_p$ and that $(z_n) = (z_n)(1)$. It would follow that $(z_n n^m)_{n=1}^{\infty} \in \mathfrak{p}_p \setminus \mathfrak{p}_p(m)$. Thus there is a polynomial, q, of degree higher than m, such that $\big(z_n n^m - q(n)\big)_{n=1}^{\infty}$ lies in ℓ_p and is therefore bounded. If we write $q(n) = \alpha n^r + \dots$, where $r > m$ and the remaining terms are of lower degree, then $z_n - \alpha n^{r-m}$ tends to zero as $n \to \infty$. It follows that (z_n) is not bounded, contrary to hypothesis.

Clearly, if $(z_n) \in \bigcap_{k=1}^{\infty} Z\big(\mathfrak{p}_p(k)\big)$ then it certainly leaves each $\mathfrak{p}_p(k)$ invariant so also leaves their union, which is precisely $\mathfrak{p}_p$, invariant. Clearly (z_n) will act as an orthomorphism on $\mathfrak{p}_p$. I.e. (3) $\Rightarrow$ (1). □

Theorem 4.12. *If $p \in \{0\} \cup (1, \infty]$ then the following conditions on a sequence $z = (z_n)$ are equivalent:*

1. $z \in \bigcap_{k=1}^{\infty} Z\big(\mathfrak{p}_p(k)\big)$,
2. *There is a (unique) real sequence $(a_k)_{k=0}^{\infty}$, such that $(n^k z_n - \sum_{j=0}^{k} a_j n^{k-j}) \in \ell_p$ for all $k \in \mathbb{N}^*$.*

Proof. It is clear from Theorem 4.1 that (2) $\Rightarrow$ (1). The same theorem tells us that if $(z_n) \in \bigcap_{k=1}^{\infty} Z\big(\mathfrak{p}_p(k)\big)$ then for all k we may write

$$z_n = \sum_{j=0}^{k} a_j^k n^{-j} + b_n^k n^{-k}$$

where, for each k, the sequence $(b_n^k) \in \ell_p$. If we temporarily fix k and let $n \to \infty$ we see that $\lim_{n\to\infty} z_n = a_0^k$ so that we may drop the superscript and write a_0 irrespective of k. Now,

$$n(z_n - a_0) = \sum_{j=1}^{k} a_j^k n^{1-j} + b_n^k n^{-k}$$

so that again $a_1^k = \lim_{n\to\infty} n(z_n - a_0)$ independently of k. Proceeding inductively we have an infinite sequence (a_j) such that

$$z_n = \sum_{j=0}^{k} a_j n^{-j} + b_n^k n^{-k}$$

where, for each k, the sequence $(b_n^k)_{n=1}^{\infty} \in \ell_p$. This is precisely (2). The uniqueness of the sequence (a_n) is clear from the description of the elements as limits. □

It is tempting to think that we actually have convergence of the series $\sum_{j=0}^{\infty} a_j n^{-j}$ to z_n for this or some other sequence (a_j). This need not be the case. If there were then the sequence (a_j) will certainly satisfy (2). For example, if δ_m denotes the sequence with 0 for all terms except the m'th which is 1, then this lies in $Z(\mathfrak{p}_0)$. Taking (a_j) to be the zero sequence, the difference between δ_m and the k'th approximation using the sequence (a_j) is precisely δ_m which is in c_0 for all k. By the uniqueness of the sequence (a_j) there is no other possible choice.

Interestingly, the centres of $\mathfrak{p}_0$ and of $\mathfrak{p}_\infty$ coincide, in spite of the apparently different descriptions that are given in the preceding theorem.

Theorem 4.13. *The following conditions on a sequence (z_n) are equivalent:*

1. $(z_n) \in Z(\mathfrak{p}_\infty)$,
2. $(z_n) \in Z(\mathfrak{p}_0)$,

(3) *There is a (unique) real sequence* $(a_k)_{k=0}^{\infty}$, *such that* $(n^k z_n - \sum_{j=0}^{k} a_j n^{k-j}) \in \ell_\infty$ *for all* $k \in \mathbb{N}^*$,
(4) *There is a (unique) real sequence* $(a_k)_{k=0}^{\infty}$, *such that* $(n^k z_n - \sum_{j=0}^{k} a_j n^{k-j}) \in c_0$ *for all* $k \in \mathbb{N}^*$.

Proof. That (1) is equivalent to (3), and (2) to (4), are immediate from Theorem 4.11 and Theorem 4.12. Clearly (4) implies (3). If (a_n) is a sequence satisfying (3) and $k \in \mathbb{N}^*$, then

$$z_n = \sum_{j=0}^{k+1} a_j n^{-j} + b_n n^{-(k+1)}$$

where $(b_n) \in \ell_\infty$. Now

$$z_n = \sum_{j=1}^{k} a_j n^{-j} + \big((a_{k+1} + b_n)/n\big) n^{-k}$$

and it is clear that $\big((a_{k+1} + b_n)/n\big)_{n=1}^{\infty} \in c_0$, establishing (4). □

In this case, where $p \in \{0\} \cup (1, \infty]$, we have the following rather unlikely looking diagram which is, of course, not commutative.

$$\begin{array}{ccccccccc}
\cdots & \hookrightarrow & \mathfrak{p}_\infty(k) & \hookrightarrow & \mathfrak{p}_\infty(k+1) & \hookrightarrow & \cdots & \hookrightarrow & \mathfrak{p}_\infty \\
 & & \updownarrow & & \updownarrow & & & & \\
\cdots & \hookleftarrow & Z\big(\mathfrak{p}_\infty(k)\big) & \hookleftarrow & Z\big(\mathfrak{p}_\infty(k+1)\big) & \hookleftarrow & \cdots & \hookleftarrow & Z(\mathfrak{p}_\infty)
\end{array}$$

where $\updownarrow$ denotes order isomorphism. This diagram forces upon us the question of whether or not $\mathfrak{p}_p$ is order isomorphic to its centre. This is not the case as $Z(\mathfrak{p}_p)$ has a strong order unit, the identity operator, whilst $\mathfrak{p}_p$ certainly does not have a strong order unit.

Finally, let us look at the centre of $\mathfrak{p}_p$ where $p \in (0, 1]$, which turns out to be rather simpler than our first case.

Theorem 4.14. *If* $p \in (0, 1]$ *then the sequence* (z_n) *acts centrally on* $\mathfrak{p}_p$ *if and only if* (z_n) *can be written as the sum of a constant sequence plus a sequence* (w_n) *such that* $(n^k w_n) \in \ell_p$ *for all* $k \in \mathbb{N}^*$.

Proof. Immediate from Theorems 4.2 and 4.11. □

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