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SK_1 OF GRADED DIVISION ALGEBRAS

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ABSTRACT. The reduced Whitehead group SK_1 of a graded division algebra graded by a torsion-free abelian group is studied. It is observed that the computations here are much more straightforward than in the non-graded setting. Bridges to the ungraded case are then established by the following two theorems: It is proved that SK_1 of a tame valued division algebra over a henselian field coincides with SK_1 of its associated graded division algebra. Furthermore, it is shown that SK_1 of a graded division algebra is isomorphic to SK_1 of its quotient division algebra. The first theorem gives the established formulas for the reduced Whitehead group of certain valued division algebras in a unified manner, whereas the latter theorem covers the stability of reduced Whitehead groups, and also describes SK_1 for generic abelian crossed products.

1. INTRODUCTION

Let D be a division algebra with a valuation. To this one associates a graded division algebra $\mathrm{gr}(D) = \bigoplus_{\gamma \in \Gamma_D} \mathrm{gr}(D)_\gamma$, where Γ_D is the value group of D and the summands $\mathrm{gr}(D)_\gamma$ arise from the filtration on D induced by the valuation (see §2 for details). As is illustrated in [HwW$_2$], even though computations in the graded setting are often easier than working directly with D, it seems that not much is lost in passage from D to its corresponding graded division algebra $\mathrm{gr}(D)$. This has provided motivation to systematically study this correspondence, notably by Boulagouaz [B], Hwang, Tignol and Wadsworth [HwW$_1$, HwW$_2$, TW], and to compare certain functors defined on these objects, notably the Brauer group.

In particular, the associated graded ring $\mathrm{gr}(D)$ is an Azumaya algebra ([HwW$_2$], Cor. 1.2); so the reduced norm map exists for it, and one defines the reduced Whitehead group SK_1 for $\mathrm{gr}(D)$ as the kernel of the reduced norm modulo the commutator subgroup of $\mathrm{gr}(D)^*$ and SH^0 as the cokernel of the reduced norm map (see (3.1) below). In this paper we study these groups for a graded division algebra.

Apart from the work of Panin and Suslin [PS] on SH^0 for Azumaya algebras over semilocal regular rings and [H$_4$] which studies SK_1 for Azumaya algebras over henselian rings, it seems that not much is known about these groups in the setting of Azumaya algebras. Specializing to division algebras, however, there is an extensive literature on the group SK_1. Platonov [P$_1$] showed that SK_1 could be non-trivial for certain division algebras over henselian valued fields. He thereby provided a series of counter-examples to questions raised in the setting of algebraic groups, notably the Kneser-Tits conjecture. (For surveys on this work and the group SK_1, see [P$_3$], [G], [Mer] or [W$_2$], §6.)

In this paper we first study the reduced Whitehead group SK_1 of a graded division algebra whose grade group is totally ordered abelian (see §3). It can be observed that the computations here are significantly easier and more transparent than in the non-graded setting. For a division algebra D finite-dimensional over a henselian valued field F, the valuation on F extends uniquely to D (see Th. 2.1 in [W$_2$], or [W$_1$]), and the filtration on D induced by the valuation yields an associated graded division algebra $\mathrm{gr}(D)$. Previous work on the subject has shown that this transition to graded setting is most "faithful" when the valuation is tame. Indeed, in Section 4, we show that for a tame valued division algebra D over a henselian field, $\mathrm{SK}_1(D)$ coincides with $\mathrm{SK}_1(\mathrm{gr}(D))$ (Th. 4.8). Having established this bridge between the graded setting and non-graded case, we will easily deduce known formulas in the literature for the reduced Whitehead

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group of certain valued division algebras, by passing to the graded setting; this shows the utility of the graded approach (see Cor. 4.10).

In the other direction, if $E = \bigoplus_{\gamma\in\Gamma_E} E_\gamma$ is a graded division algebra whose grade group Γ_E is torsion-free abelian, then E has a quotient division algebra $q(E)$ which has the same index as E. The same question on comparing the reduced Whitehead groups of these objects can also be raised here. It is known that when the grade group is $\mathbb{Z}$, then E has the simple form of a skew Laurent polynomial ring $D[x, x^{-1}, \varphi]$, where D is a division algebra and φ is an automorphism of D. In this setting the quotient division algebra of $D[x, x^{-1}, \varphi]$ is $D(x, \varphi)$. In [PY], Platonov and Yanchevskiĭ compared $\mathrm{SK}_1(D(x, \varphi))$ with $\mathrm{SK}_1(D)$. In particular, they showed that if φ is an inner automorphism then $\mathrm{SK}_1(D(x, \varphi)) \cong \mathrm{SK}_1(D)$. In fact, if φ is inner, then $D[x, x^{-1}, \varphi]$ is an unramified graded division algebra and we prove that $\mathrm{SK}_1(D[x, x^{-1}, \varphi]) \cong \mathrm{SK}_1(D)$ (Cor. 3.6(i)). By combining these, one concludes that the reduced Whitehead group of the graded division algebra $D[x, x^{-1}, \varphi]$, where φ is inner, coincides with SK_1 of its quotient division algebra. In Section 5, we show that this is a very special case of stability of SK_1 for graded division algebras; namely, for any graded division algebra with torsion-free grade group, the reduced Whitehead group coincides with the reduced Whitehead group of its quotient division algebra. This allows us to give a formula for SK_1 for generic abelian crossed product algebras.

The paper is organized as follows: In Section 2, we gather relevant background on the theory of graded division algebras indexed by a totally ordered abelian group and establish several homomorphisms needed in the paper. Section 3 studies the reduced Whitehead group SK_1 of a graded division algebra. We establish analogues to Ershov's linked exact sequences [E] in the graded setting, easily deducing formulas for SK_1 of unramified, totally ramified, and semiramified graded division algebras. In Section 4, we prove that SK_1 of a tame division algebra over a henselian field coincides with SK_1 of its associated graded division algebra. Section 5 is devoted to proving that SK_1 of a graded division algebra is isomorphic to SK_1 of its quotient division algebra. We conclude the paper with two appendices. Appendix A establishes the Wedderburn factorization theorem in the setting of graded division rings, namely that the minimal polynomial of a homogenous element of a graded division ring E splits completely over E (Th. A.1). Appendix B provides a complete proof of the Congruence Theorem for all tame division algebras over henselian valued fields. This theorem was originally proved by Platonov for the case of complete discrete valuations of rank 1, and it was a key tool in his calculations of SK_1 for certain valued division algebras.

2. Graded division algebras

In this section we establish notation and recall some fundamental facts about graded division algebras indexed by a totally ordered abelian group, and about their connections with valued division algebras. In addition, we establish some important homomorphisms relating the group structure of a valued division algebra to the group structure of its associated graded division algebra.

Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a graded ring, i.e., Γ is an abelian group, and R is a unital ring such that each R_γ is a subgroup of $(R, +)$ and $R_\gamma \cdot R_\delta \subseteq R_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$. Set

$$\begin{aligned} \Gamma_R &= \{\gamma \in \Gamma \mid R_\gamma \neq 0\}, \ \text{ the grade set of } R; \\ R^h &= \textstyle\bigcup_{\gamma\in\Gamma_R} R_\gamma, \ \text{ the set of homogeneous elements of } R. \end{aligned}$$

For a homogeneous element of R of degree γ, i.e., an $r \in R_\gamma \setminus 0$, we write $\deg(r) = \gamma$. Recall that R_0 is a subring of R and that for each $\gamma \in \Gamma_R$, the group R_γ is a left and right R_0-module. A subring S of R is a *graded subring* if $S = \bigoplus_{\gamma\in\Gamma_R}(S \cap R_\gamma)$. For example, the center of R, denoted $Z(R)$, is a graded subring of R. If $T = \bigoplus_{\gamma\in\Gamma} T_\gamma$ is another graded ring, a *graded ring homomorphism* is a ring homomorphism $f : R \to T$ with $f(R_\gamma) \subseteq T_\gamma$ for all $\gamma \in \Gamma$. If f is also bijective, it is called a graded ring isomorphism; we then write $R \cong_{\mathrm{gr}} T$.

For a graded ring R, a graded left R-module M is a left R-module with a grading $M = \bigoplus_{\gamma\in\Gamma'} M_\gamma$, where the M_γ are all abelian groups and Γ' is a abelian group containing Γ, such that $R_\gamma \cdot M_\delta \subseteq M_{\gamma+\delta}$ for all $\gamma \in \Gamma_R, \delta \in \Gamma'$. Then, Γ_M and M^h are defined analogously to Γ_R and R^h. We say that M is a *graded free* R-module if it has a base as a free R-module consisting of homogeneous elements.

A graded ring $E = \bigoplus_{\gamma\in\Gamma} E_\gamma$ is called a *graded division ring* if Γ is a torsion-free abelian group and every non-zero homogeneous element of E has a multiplicative inverse. Note that the grade set Γ_E is actually a group. Also, E_0 is a division ring, and E_γ is a 1-dimensional left and right E_0 vector space for every $\gamma \in \Gamma_E$. The requirement that Γ be torsion-free is made because we are interested in graded division rings arising from valuations on division rings, and all the grade groups appearing there are torsion-free. Recall that every torsion-free abelian group Γ admits total orderings compatible with the group structure. (For example, Γ embeds in $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ which can be given a lexicographic total ordering using any base of it as a $\mathbb{Q}$-vector space.) By using any total ordering on Γ_E, it is easy to see that E has no zero divisors and that E^*, the multiplicative group of units of E, coincides with $E^h \setminus \{0\}$ (cf. [HwW$_2$], p. 78). Furthermore, the degree map

$$\deg\colon E^* \to \Gamma_E \tag{2.1}$$

is a group homomorphism with kernel E_0^*.

By an easy adaptation of the ungraded arguments, one can see that every graded module M over a graded division ring E is graded free, and every two homogenous bases have the same cardinality. We thus call M a *graded vector space* over E and write $\dim_E(M)$ for the rank of M as a graded free E-module. Let $S \subseteq E$ be a graded subring which is also a graded division ring. Then, we can view E as a graded left S-vector space, and we write $[E : S]$ for $\dim_S(E)$. It is easy to check the "Fundamental Equality,"

$$[E : S] \;=\; [E_0 : S_0]\,|\Gamma_E : \Gamma_S|, \tag{2.2}$$

where $[E_0 : S_0]$ is the dimension of E_0 as a left vector space over the division ring S_0 and $|\Gamma_E : \Gamma_S|$ denotes the index in the group Γ_E of its subgroup Γ_S.

A *graded field* T is a commutative graded division ring. Such a T is an integral domain, so it has a quotient field, which we denote $q(T)$. It is known, see [HwW$_1$], Cor. 1.3, that T is integrally closed in $q(T)$. An extensive theory of graded algebraic extensions of graded fields has been developed in [HwW$_1$]. For a graded field T, we can define a grading on the polynomial ring $T[x]$ as follows: Let Δ be a totally ordered abelian group with $\Gamma_T \subseteq \Delta$, and fix $\theta \in \Delta$. We have

$$T[x] \;=\; \bigoplus_{\gamma\in\Delta} T[x]_\gamma, \quad \text{where} \quad T[x]_\gamma \;=\; \{\textstyle\sum a_i x^i \mid a_i \in T^h,\ \deg(a_i) + i\theta = \gamma\}. \tag{2.3}$$

This makes $T[x]$ a graded ring, which we denote $T[x]^\theta$. Note that $\Gamma_{T[x]^\theta} = \Gamma_T + \langle\theta\rangle$. A homogeneous polynomial in $T[x]^\theta$ is said to be *θ-homogenizable*. If E is a graded division algebra with center T, and $a \in E^h$ is homogeneous of degree θ, then the evaluation homomorphism $\epsilon_a\colon T[x]^\theta \to T[a]$ given by $f \mapsto f(a)$ is a graded ring homomorphism. Assuming $[T[a] : T] < \infty$, we have $\ker(\epsilon_a)$ is a principal ideal of $T[x]$ whose unique monic generator h_a is called the minimal polynomial of a over T. It is known, see [HwW$_1$], Prop. 2.2, that if $\deg(a) = \theta$, then h_a is θ-homogenizable.

If E is a graded division ring, then its center $Z(E)$ is clearly a graded field. *The graded division rings considered in this paper will always be assumed finite-dimensional over their centers.* The finite-dimensionality assures that E has a quotient division ring $q(E)$ obtained by central localization, i.e., $q(E) = E \otimes_T q(T)$ where $T = Z(E)$. Clearly, $Z(q(E)) = q(T)$ and $\mathrm{ind}(E) = \mathrm{ind}(q(E))$, where the index of E is defined by $\mathrm{ind}(E)^2 = [E : T]$. If S is a graded field which is a graded subring of $Z(E)$ and $[E : S] < \infty$, then E is said to be a *graded division algebra* over S.

A graded division algebra E with center T is said to be *unramified* if $\Gamma_E = \Gamma_T$. From (2.2), it follows then that $[E : S] = [E_0 : T_0]$. At the other extreme, E is said to be *totally ramified* if $E_0 = T_0$. In a case in the middle, E is said to be *semiramified* if E_0 is a field and $[E_0 : T_0] = |\Gamma_E : \Gamma_T| = \mathrm{ind}(E)$. These definitions are motivated by analogous definitions for valued division algebras ([W$_2$]). Indeed, if a valued division algebra is unramified, semiramified, or totally ramfied, then so is its associated graded division algebra (see §4).

A main theme of this paper is to study the correspondence between SK_1 of a valued division algebra and that of its associated graded division algebra. We now recall how to associate a graded division algebra to a valued division algebra.

Let D be a division algebra finite dimensional over its center F, with a valuation $v : D^* \to \Gamma$. So Γ is a totally ordered abelian group, and v satisifies the conditions that for all $a, b \in D^*$,

(1) $v(ab) = v(a) + v(b)$;
(2) $v(a + b) \geq \min\{v(a), v(b)\} \quad (b \neq -a)$.

Let

$$\begin{aligned} V_D &= \{a \in D^* : v(a) \geq 0\} \cup \{0\}, \text{ the valuation ring of } v; \\ M_D &= \{a \in D^* : v(a) > 0\} \cup \{0\}, \text{ the unique maximal left (and right) ideal of } V_D; \\ \overline{D} &= V_D/M_D, \text{ the residue division ring of } v \text{ on } D; \text{ and} \\ \Gamma_D &= \mathrm{im}(v), \text{ the value group of the valuation.} \end{aligned}$$

For background on valued division algebras, see [JW] or the survey paper [W$_2$]. One associates to D a graded division algebra as follows: For each $\gamma \in \Gamma_D$, let

$$\begin{aligned} D^{\geq\gamma} &= \{d \in D^* : v(d) \geq \gamma\} \cup \{0\}, \text{ an additive subgroup of } D; \\ D^{>\gamma} &= \{d \in D^* : v(d) > \gamma\} \cup \{0\}, \text{ a subgroup of } D^{\geq\gamma}; \text{ and} \\ \mathrm{gr}(D)_\gamma &= D^{\geq\gamma}/D^{>\gamma}. \end{aligned}$$

Then define

$$\mathrm{gr}(D) \;=\; \bigoplus_{\gamma\in\Gamma_D} \mathrm{gr}(D)_\gamma.$$

Because $D^{>\gamma}D^{\geq\delta} + D^{\geq\gamma}D^{>\delta} \subseteq D^{>(\gamma+\delta)}$ for all $\gamma, \delta \in \Gamma_D$, the multiplication on $\mathrm{gr}(D)$ induced by multiplication on D is well-defined, giving that $\mathrm{gr}(D)$ is a graded ring, called the *associated graded ring* of D. The multiplicative property (1) of the valuation v implies that $\mathrm{gr}(D)$ is a graded division ring. Clearly, we have $\mathrm{gr}(D)_0 = \overline{D}$ and $\Gamma_{\mathrm{gr}(D)} = \Gamma_D$. For $d \in D^*$, we write $\widetilde{d}$ for the image $d + D^{>v(d)}$ of d in $\mathrm{gr}(D)_{v(d)}$. Thus, the map given by $d \mapsto \widetilde{d}$ is a group epimorphism $D^* \to \mathrm{gr}(D)^*$ with kernel $1 + M_D$.

The restriction $v|_F$ of the valuation on D to its center F is a valuation on F, which induces a corresponding graded field $\mathrm{gr}(F)$. Then it is clear that $\mathrm{gr}(D)$ is a graded $\mathrm{gr}(F)$-algebra, and by (2.2) and the Fundamental Inequality for valued division algebras,

$$[\mathrm{gr}(D) : \mathrm{gr}(F)] \;=\; [\overline{D} : \overline{F}]\,|\Gamma_D : \Gamma_F| \;\leq\; [D : F] \;< \infty.$$

Let F be a field with a henselian valuation v. Recall that a field extension L of F of degree $n < \infty$ is said to be *tamely ramified* or *tame* over F if, with respect to the unique extension of v to L, the residue field $\overline{L}$ is a separable field extension of $\overline{F}$ and $\mathrm{char}(\overline{F}) \nmid n/[\overline{L} : \overline{F}]$. Such an L is necessarily *defectless* over F, i.e., $[L : F] = [\overline{L} : \overline{F}]\,|\Gamma_L : \Gamma_F| = [\mathrm{gr}(L) : \mathrm{gr}(F)]$. Along the same lines, let D be a division algebra with center F (so, by convention, $[D : F] < \infty$); then v on F extends uniquely to a valuation on D. With respect to this valuation, D is said to be *tamely ramified* or *tame* if $Z(\overline{D})$ is separable over $\overline{F}$ and $\mathrm{char}(\overline{F}) \nmid \mathrm{ind}(D)\big/\big(\mathrm{ind}(\overline{D})[Z(\overline{D}) : \overline{F}]\big)$. Recall from [JW], Prop. 1.7 that whenever the field extension $Z(\overline{D})/\overline{F}$ is separable, it is abelian Galois. It is known (cf. Prop. 4.3 in [HwW$_2$]) that D is tame if and only if $[\mathrm{gr}(D) : \mathrm{gr}(F)] = [D : F]$ and $Z(\mathrm{gr}(D)) = \mathrm{gr}(F)$, if and only if D is split by the maximal tamely ramified extension of F, if and only if $\mathrm{char}(\overline{F}) = 0$ or $\mathrm{char}(\overline{F}) = p \neq 0$ and the p-primary component of D is inertially split, i.e., split by the maximal unramified extension of F. We say D is *strongly tame* if

$\mathrm{char}(\overline{F}) \nmid \mathrm{ind}(D)$. Note that strong tameness implies tameness. This is clear from the last characterization of tameness, or from (2.4) below. For a detailed study of the associated graded algebra of a valued division algebra refer to §4 in [HwW$_2$]. Recall also from [Mor], Th. 3, that for a valued division algebra D finite dimensional over its center F (here not necessarily henselian), we have the "Ostrowski theorem"

$$[D:F] \;=\; q^k\,[\overline{D}:\overline{F}]\,|\Gamma_D:\Gamma_F| \tag{2.4}$$

where $q=\mathrm{char}(\overline{D})$ and $k\in\mathbb{Z}$ with $k\geq 0$ (and $q^k=1$ if $\mathrm{char}(\overline{D})=0$). If $q^k=1$ in equation (2.4), then D is said to be *defectless* over F.

Let E be a graded division algebra with, as we always assume, Γ_E a torsion-free abelian group. After fixing some total ordering on Γ_E, define a function

$$\lambda\colon E\setminus\{0\}\to E^* \quad\text{ by }\quad \lambda(\textstyle\sum c_\gamma)=c_\delta,$$

where δ is minimal among the $\gamma\in\Gamma_E$ with $c_\gamma\neq 0$. Note that $\lambda(a)=a$ for $a\in E^*$, and

$$\lambda(ab)=\lambda(a)\lambda(b)\text{ for all }a,b\in E\setminus\{0\}. \tag{2.5}$$

Let $Q=q(E)$. We can extend λ to a map defined on all of Q^* as follows: for $q\in Q^*$, write $q=ac^{-1}$ with $a\in E\setminus\{0\}$, $c\in Z(E)\setminus\{0\}$, and set $\lambda(q)=\lambda(a)\lambda(c)^{-1}$. It follows from (2.5) that $\lambda\colon Q^*\to E^*$ is well-defined and is a group homomorphism. Since the composition $E^*\hookrightarrow Q^*\to E^*$ is the identity, λ is a splitting map for the injection $E^*\hookrightarrow Q^*$. (In Lemma 5.5 below, we will observe that this map induces a monomorphism from $\mathrm{SK}_1(E)$ to $\mathrm{SK}_1(Q)$.)

Now, by composing λ with the degree map of (2.1) we get a map v

$$\begin{tikzcd} Q^* \arrow[r, "\lambda"] \arrow[dr, "v"'] & E^* \arrow[d, "\deg"] \\ & \Gamma_E \end{tikzcd} \tag{2.6}$$

This v is in fact a valuation on Q: for $a,b\in Q^*$, $v(ab)=v(a)+v(b)$ as v is the composition of two group homomorphisms, and it is straightforward to check that $v(a+b)\geq\min(v(a),v(b))$ (check this first for $a,b\in E\setminus\{0\}$). It is easy to see that for the associated graded ring for this valuation on $q(E)$, we have $\mathrm{gr}(q(E))\cong_{\mathrm{gr}} E$; this is a strong indication of the close connection between graded and valued structures.

3. Reduced norm and reduced Whitehead group of a graded division algebra

Let A be an Azumaya algebra of constant rank n^2 over a commutative ring R. Then there is a commutative ring S faithfully flat over R which splits A, i.e., $A\otimes_R S\cong M_n(S)$. For $a\in A$, considering $a\otimes 1$ as an element of $M_n(S)$, one then defines the *reduced characteristic polynomial*, the *reduced trace*, and the *reduced norm* of a by

$$\mathrm{char}_A(x,a) \;=\; \det(x-(a\otimes 1)) \;=\; x^n-\mathrm{Trd}_A(a)x^{n-1}+\ldots+(-1)^n\mathrm{Nrd}_A(a).$$

Using descent theory, one shows that $\mathrm{char}_A(x,a)$ is independent of S and of the choice of isomorphism $A\otimes_R S\cong M_n(S)$, and that $\mathrm{char}_A(x,a)$ lies in $R[x]$; furthermore, the element a is invertible in A if and only if $\mathrm{Nrd}_A(a)$ is invertible in R (see Knus [K], III.1.2, and Saltman [S$_2$], Th. 4.3). Let $A^{(1)}$ denote the set of elements of A with the reduced norm 1. One then defines the *reduced Whitehead group* of A to be $\mathrm{SK}_1(A)=A^{(1)}/A'$, where A' denotes the commutator subgroup of the group A^* of invertible elements of A. The *reduced norm residue group* of A is defined to be $\mathrm{SH}^0(A)=R^*/\mathrm{Nrd}_A(A^*)$. These groups are related by the exact sequence:

$$1\longrightarrow\mathrm{SK}_1(A)\longrightarrow A^*/A'\xrightarrow{\mathrm{Nrd}}R^*\longrightarrow\mathrm{SH}^0(A)\longrightarrow 1 \tag{3.1}$$

Now let E be a graded division algebra with center T. Since E is an Azumaya algebra over T ([B], Prop. 5.1 or[HwW$_2$], Cor. 1.2), its reduced Whitehead group $\mathrm{SK}_1(E)$ is defined.

Remark 3.1. The reduced norm for an Azumaya algebra is defined using a splitting ring, and in general splitting rings can be difficult to find. But for a graded division algebra E we observe that, analogously to the case of ungraded division rings, any maximal graded subfield L of E splits E. For, the centralizer $C = C_E(L)$ is a graded subring of E containing L, and for any homogeneous $c \in C$, $L[c]$ is a graded subfield of E containing L. Hence, $C = L$, showing that L is a maximal commutative subring of E. Thus, by Lemma 5.1.13(1), p. 141 of [K], as E is Azumaya, $E \otimes_T L \cong \mathrm{End}_L(E) \cong M_n(L)$. Thus, we can compute reduced norms for elements of E by passage to $E \otimes_T L$.

We have other tools as well for computing Nrd_E and Trd_E:

Proposition 3.2. *Let E be a graded division ring with center T. Let $q(T)$ be the quotient field of T, and let $q(E) = E \otimes_T q(T)$, which is the quotient division ring of E. We view $E \subseteq q(E)$. Let $n = \mathrm{ind}(E) = \mathrm{ind}(q(E))$. Then for any $a \in E$,*

(i) $\mathrm{char}_E(x, a) = \mathrm{char}_{q(E)}(x, a)$, *so*
$$\mathrm{Nrd}_E(a) \;=\; \mathrm{Nrd}_{q(E)}(a) \quad \textit{and} \quad \mathrm{Trd}_E(a) \;=\; \mathrm{Trd}_{q(E)}(a). \tag{3.2}$$

(ii) *If K is any graded subfield of E containing T and $a \in K$, then*
$$\mathrm{Nrd}_E(a) \;=\; N_{K/T}(a)^{n/[K:T]} \quad \textit{and} \quad \mathrm{Trd}_E(a) \;=\; \tfrac{n}{[K:T]}\,\mathrm{Tr}_{K/T}(a).$$

(iii) *For $\gamma \in \Gamma_E$, if $a \in E_\gamma$ then $\mathrm{Nrd}_E(a) \in E_{n\gamma}$ and $\mathrm{Trd}(a) \in E_\gamma$. In particular, $E^{(1)} \subseteq E_0$.*

(iv) *Set $\delta = \mathrm{ind}(E)/\big(\mathrm{ind}(E_0)[Z(E_0) : T_0]\big)$. If $a \in E_0$, then,*
$$\mathrm{Nrd}_E(a) \;=\; N_{Z(E_0)/T_0}\mathrm{Nrd}_{E_0}(a)^{\,\delta} \in T_0 \quad \textit{and} \quad \mathrm{Trd}_E(a) \;=\; \delta\,\mathrm{Tr}_{Z(E_0)/T_0}\mathrm{Trd}_{E_0}(a) \in T_0. \tag{3.3}$$

Proof. (i) The construction of reduced characteristic polynonials described above is clearly compatible with scalar extension of the ground ring. Hence, $\mathrm{char}_E(x, a) = \mathrm{char}_{q(E)}(x, a)$ (as we are identifying $a \in E$ with $a \otimes 1$ in $E \otimes_T q(T)$). The formulas in (3.2) follow immediately.

(ii) Let $h_a = x^m + t_{m-1}x^{m-1} + \ldots + t_0 \in q(T)[x]$ be the minimal polynomial of a over $q(T)$. As noted in [HwW_1], Prop. 2.2, since the integral domain T is integrally closed and E is integral over T, we have $h_a \in T[x]$. Let $\ell_a = x^k + s_{k-1}x^{k-1} + \ldots + s_0 \in T[x]$ be the characteristic polynomial of the T-linear function on the free T-module K given by $c \mapsto ac$. By definition, $N_{K/T}(a) = (-1)^k s_0$ and $\mathrm{Tr}_{K/T}(a) = -s_{k-1}$. Since $q(K) = K \otimes_T q(T)$, we have $[q(K) : q(T)] = [K : T] = k$ and ℓ_a is also the characteristic polynomial for the $q(T)$-linear transformation of $q(K)$ given by $q \mapsto aq$. So, $\ell_a = h_a^{k/m}$. Since $\mathrm{char}_{q(E)}(x, a) = h_a^{n/m}$ (see [R], Ex. 1, p. 124), we have $\mathrm{char}_{q(E)}(x, a) = \ell_a^{n/k}$. Therefore, using (i),
$$\mathrm{Nrd}_E(a) \;=\; \mathrm{Nrd}_{q(E)}(a) \;=\; \big[(-1)^k s_0\big]^{n/k} \;=\; N_{K/T}(a)^{n/k}.$$
The formula for $\mathrm{Trd}_E(a)$ in (ii) follows analogously.

(iii) From the equalities $\mathrm{char}_E(x, a) = \mathrm{char}_{q(E)}(x, a) = h_a^{n/m}$ noted in proving (i) and (ii), we have $\mathrm{Nrd}_E(a) = [(-1)^m t_0]^{n/m}$ and $\mathrm{Trd}_E(a) = -\frac{n}{m} t_{m-1}$. As noted in [$\mathrm{HwW}_1$], Prop. 2.2, if $a \in E_\gamma$, then its minimal polynomial h_a is γ-homogenizable in $T[x]$ as in (2.3) above. Hence, $t_0 \in E_{m\gamma}$ and $t_{m-1} \in E_\gamma$. Therefore, $\mathrm{Nrd}_E(a) \in E_{n\gamma}$ and $\mathrm{Trd}(a) \in E_\gamma$. If $a \in E^{(1)}$ then a is homogeneous, since it is a unit of E, and since $1 = \mathrm{Nrd}_E(a) \in E_{n\deg(a)}$, necessarily $\deg(a) = 0$.

(iv) Suppose $a \in E_0$. Then, h_a is 0-homogenizable in $T[x]$, i.e., $h_a \in T_0[x]$. Hence, h_a is the minimal polynomial of a over the field T_0. Therefore, if L is any maximal subfield of E_0 containing a, we have $N_{L/T_0}(a) = [(-1)^m t_0]^{[L:T_0]/m}$. Now,
$$n/m \;=\; \delta\,\mathrm{ind}(E_0)[Z(E_0) : T_0]/m \;=\; \delta\,[L : T_0]/m.$$

Hence,

$$\begin{aligned}\mathrm{Nrd}_E(a) &= \big[(-1)^m t_0\big]^{n/m} = \big[(-1)^m t_0\big]^{\delta[L:T_0]/m} = N_{L/T_0}(a)^\delta \\ &= N_{Z(E_0)/T_0} N_{L/Z(E_0)}(a)^\delta = N_{Z(E_0)/T_0}\mathrm{Nrd}_{E_0}(a)^\delta.\end{aligned}$$

The formula for $\mathrm{Trd}_E(a)$ is proved analogously. □

In the rest of this section we study the reduced Whitehead group SK$_1$ of a graded division algebra. As we mentioned in the introduction, the motif is to show that working in the graded setting is much easier than in the non-graded setting.

The most successful approach to computing SK$_1$ for division algebras over henselian fields is due to Ershov in [E], where three linked exact sequences were constructed involving a division algebra D, its residue division algebra $\overline{D}$, and its group of units U_D (see also [W$_2$], p. 425). From these exact sequences, Ershov recovered Platonov's examples [P$_1$] of division algebras with nontrivial SK$_1$ and many more examples as well. In this section we will easily prove the graded version of Ershov's exact sequences (see diagram (3.5)), yielding formulas for SK$_1$ of unramified, semiramified, and totally ramified graded division algebras. This will be applied in §4, where it will be shown that SK$_1$ of a tame division algebra over a henselian field coincides with SK$_1$ of its associated graded division algebra. We can then readily deduce from the graded results many established formulas in the literature for the reduced Whitehead groups of valued division algebras (see Cor. 4.10). This demonstrates the merit of the graded approach.

If N is a group, we denote by N^n the subgroup of N generated by all n-th powers of elements of N. A homogeneous multiplicative commutator of E, where E is a graded division ring, has the form $aba^{-1}b^{-1}$ where $a, b \in E^* = E^h \setminus \{0\}$. We will use the notation $[a,b] = aba^{-1}b^{-1}$ for $a, b \in E^*$. Since a and b are homogeneous, note that $[a,b] \in E_0$. If H and K are subsets of E^*, then $[H,K]$ denotes the subgroup of E^* generated by $\{[h,k] : h \in H, k \in K\}$. The group $[E^*, E^*]$ will be denoted by E'.

Proposition 3.3. *Let $E = \bigoplus_{\alpha\in\Gamma} E_\alpha$ be a graded division algebra with graded center T, with $\mathrm{ind}(E) = n$. Then,*

(i) *If N is a normal subgroup of E^*, then $N^n \subseteq \mathrm{Nrd}_E(N)[E^*, N]$.*
(ii) *$\mathrm{SK}_1(E)$ is n-torsion.*

Proof. Let $a \in N$ and let $h_a \in q(T)[x]$ be the minimal polynomial of a over $q(T)$, and let $m = \deg(h_a)$. As noted in the proof of Prop. 3.2, $h_a \in T[x]$ and $\mathrm{Nrd}_E(a) = [(-1)^m h_a(0)]^{n/m}$. By the graded Wedderburn Factorization Theorem A.1, we have $h_a = (x - d_1 a d_1^{-1}) \dots (x - d_m a d_m^{-1})$ where each $d_i \in E^* \subseteq E^h$. Note that $[E^*, N]$ is a normal subgroup of E^*, since N is normal in E^*. It follows that

$$\begin{aligned}\mathrm{Nrd}_E(a) &= \big(d_1 a d_1^{-1} \dots d_m a d_m^{-1}\big)^{n/m} = \big([d_1, a]a[d_2, a]a \dots a[d_m, a]a\big)^{n/m} \\ &= a^n d_a \quad \text{where } d_a \in [E^*, N].\end{aligned}$$

Therefore, $a^n = \mathrm{Nrd}_E(a) d_a^{-1} \in \mathrm{Nrd}_E(N)[E^*, N]$, yielding (i). (ii) is immediate from (i) by taking $N = E^{(1)}$. □

The fact that $\mathrm{SK}_1(E)$ is n-torsion is also deducible from the injectivity of the map $\mathrm{SK}_1(E) \to \mathrm{SK}_1(q(E))$ shown in Lemma 5.5 below.

We recall the definition of the group $\widehat{H}^{-1}(G, A)$, which will appear in our description of $\mathrm{SK}_1(E)$. For any finite group G and any G-module A, define the norm map $N_G\colon A \to A$ as follows: for any $a \in A$, let $N_G(a) = \sum_{g\in G} ga$. Consider the G-module $I_G(A)$ generated as an abelian group by $\{a - ga : a \in A \text{ and } g \in G\}$. Clearly, $I_G(A) \subseteq \ker(N_G)$. Then,

$$\widehat{H}^{-1}(G, A) = \ker(N_G)/I_G(A). \tag{3.4}$$

Theorem 3.4. *Let E be any graded division ring finite dimensional over its center T. So, $Z(E_0)$ is Galois over T_0; let $G = \mathcal{G}al(Z(E_0)/T_0)$. Let $\delta = \operatorname{ind}(E)/\big(\operatorname{ind}(E_0)\,[Z(E_0):T_0]\big)$, and let $\mu_\delta(T_0)$ be the group of those δ-th roots of unity lying in T_0. Let $\widetilde{N} = N_{Z(E_0)/T_0} \circ \operatorname{Nrd}_{E_0} \colon E_0^* \to T_0^*$. Then, the rows and column of the following diagram are exact:*

$$\begin{tikzcd}
 & 1 \arrow[d] & & \\
\mathrm{SK}_1(E_0) \arrow[r] & \ker \widetilde{N}/[E_0^*, E^*] \arrow[r, "\operatorname{Nrd}_{E_0}"] \arrow[d] & \widehat{H}^{-1}(G, \operatorname{Nrd}_{E_0}(E_0^*)) \arrow[r] & 1 \\
\Gamma_E/\Gamma_T \wedge \Gamma_E/\Gamma_T \arrow[r, "\alpha"] & E^{(1)}/[E_0^*, E^*] \arrow[r] \arrow[d, "\widetilde{N}"] & \mathrm{SK}_1(E) \arrow[r] & 1 \\
 & \mu_\delta(T_0) \cap \widetilde{N}(E_0^*) \arrow[d] & & \\
 & 1 & &
\end{tikzcd} \tag{3.5}$$

The map α in (3.5) is given as follows: For $\gamma, \delta \in \Gamma_E$, take any nonzero $x_\gamma \in E_\gamma$ and $x_\delta \in E_\delta$. Then, $\alpha\big((\gamma + \Gamma_T) \wedge (\delta + \Gamma_T)\big) = [x_\gamma, x_\delta] \bmod [E_0^*, E^*]$.

Proof. By Prop. 2.3 in [HwW$_2$], $Z(E_0)/T_0$ is a Galois extension and the map $\theta\colon E^* \to \operatorname{Aut}(E_0)$, given by $e \mapsto (a \mapsto eae^{-1})$ for $a \in E_0$, induces an epimorphism $E^* \to G = \mathcal{G}al(Z(E_0)/T_0)$. In the notation for (3.4) with $A = \operatorname{Nrd}_{E_0}(E_0^*)$, we have N_G coincides with $N_{Z(E_0)/T_0}$ on A. Hence,

$$\ker(N_G) \;=\; \operatorname{Nrd}_{E_0}(\ker(\widetilde{N})). \tag{3.6}$$

Take any $e \in E^*$ and let $\sigma = \theta(e) \in \operatorname{Aut}(E_0)$. For any $a \in E_0^*$, let $h_a \in Z(T_0)[x]$ be the minimal polynomial of a over $Z(T_0)$. Then $\sigma(h_a) \in Z(T_0)[x]$ is the minimal polynomial of $\sigma(a)$ over $Z(T_0)$. Hence, $\operatorname{Nrd}_{E_0}(\sigma(a)) = \sigma(\operatorname{Nrd}_{E_0}(a))$. Since $\sigma|_{Z(T_0)} \in G$, this yields

$$\operatorname{Nrd}_{E_0}([a,e]) \;=\; \operatorname{Nrd}_{E_0}(a\sigma(a^{-1})) \;=\; \operatorname{Nrd}_{E_0}(a)\sigma(\operatorname{Nrd}_{E_0}(a))^{-1} \;\in\; I_G(A), \tag{3.7}$$

hence $\widetilde{N}([a,e]) = 1$. Thus, we have $[E_0^*, E^*] \subseteq \ker(\widetilde{N}) \subseteq E^{(1)}$ with the latter inclusion from Prop. 3.2(iv). The formula in Prop. 3.2(iv) also shows that $\widetilde{N}(E^{(1)}) \subseteq \mu_\delta(T_0)$. Thus, the vertical maps in diagram (3.5) are well-defined, and the column in (3.5) is exact. Because Nrd_{E_0} maps $\ker(\widetilde{N})$ onto $\ker(N_G)$ by (3.6) and it maps $[E_0^*, E^*]$ onto $I_G(A)$ by (3.7) (as $\theta(E^*)$ maps onto G), the map labelled Nrd_{E_0} in diagram (3.5) is surjective with kernel $E_0^{(1)}\,[E_0^*, E^*]/[E_0^*, E^*]$. Therefore, the top row of (3.5) is exact. For the lower row, since $[E^*, E^*] \subseteq E_0^*$ and $E^*/\big(E_0^*\,Z(E^*)\big) \cong \Gamma_E/\Gamma_T$, the following lemma yields an epimorphism $\Gamma_E/\Gamma_T \wedge \Gamma_E/\Gamma_T \to [E^*, E^*]/[E_0^*, E^*]$. Given this, the lower row in (3.5) is evidently exact. $\square$

Lemma 3.5. *Let G be a group, and let H be a subgroup of G with $H \supseteq [G,G]$. Let $B = G/\big(H\,Z(G)\big)$. Then, there is an epimorphism $B \wedge B \to [G,G]/[H,G]$.*

Proof. Since $[G,G] \subseteq H$, we have $\big[[G,G],[G,G]\big] \subseteq [H,G]$, so $[G,H]$ is a normal subgroup of $[G,G]$ with abelian factor group. Consider the map $\beta\colon G \times G \to [G,G]/[H,G]$ given by $(a,b) \mapsto aba^{-1}b^{-1}[H,G]$. For any $a,b,c \in G$ we have the commutator identity $[a,bc] = [a,b]\,[b,[a,c]]\,[a,c]$. The middle term $[b,[a,c]]$ lies in $[H,G]$. Thus, β is multiplicative in the second variable; likewise, it is multiplicative in the first variable. As $[H\,Z(G), G] \subseteq [H,G]$, this β induces a well-defined group homomorphism $\beta'\colon B \otimes_{\mathbb{Z}} B \to [G,G]/[H,G]$, which is surjective since $\operatorname{im}(\beta)$ generates $[G,G]/[H,G]$. Since $\beta'(\eta \otimes \eta) = 1$ for all $\eta \in B$, there is an induced epimorphism $B \wedge B \to [G,G]/[H,G]$. $\square$

Corollary 3.6. *Let E be a graded division ring with center T.*

(i) *If E is unramified, then $\mathrm{SK}_1(E) \cong \mathrm{SK}_1(E_0)$.*
(ii) *If E is totally ramified, then $\mathrm{SK}_1(E) \cong \mu_n(T_0)/\mu_e(T_0)$ where $n = \mathrm{ind}(E)$ and e is the exponent of Γ_E/Γ_T.*
(iii) *If E is semiramified, then for $G = \mathcal{G}al(E_0/T_0) \cong \Gamma_E/\Gamma_T$ there is an exact sequence*

$$G \wedge G \ \rightarrow\ \widehat{H}^{-1}(G, E_0^*) \ \rightarrow\ \mathrm{SK}_1(E) \ \rightarrow\ 1. \tag{3.8}$$

(iv) *If E has maximal graded subfields L and K which are respectively unramified and totally ramified over T, then E is semiramified and $\mathrm{SK}_1(E) \cong \widehat{H}^{-1}(\mathcal{G}al(E_0/T_0), E_0^*)$.*

Proof. See §2 for the definitions of unramified, totally ramified, and semiramified graded division algebras.

(i) Since E is unramified over T, we have E_0 is a central T_0-division algebra, $\mathrm{ind}(E_0) = \mathrm{ind}(E)$, and $E^* = E_0^* T^*$. It follows that $G = \mathcal{G}al(Z(E_0)/T_0)$ is trivial, and thus $\widehat{H}^{-1}(G, \mathrm{Nrd}_{E_0}(E_0))$ is trivial; also, $\delta = 1$, and from (3.3), $\mathrm{Nrd}_{E_0}(a) = \mathrm{Nrd}_E(a)$ for all $a \in E_0$. Furthermore, $[E_0^*, E^*] = [E_0^*, E_0^* T^*] = [E_0^*, E_0^*]$ as T^* is central. Plugging this information into the exact top row of diagram (3.5) and noting that the exact sequence extends to the left by $1 \to [E_0^*, E^*]/[E_0^*, E_0^*] \to \mathrm{SK}_1(E_0)$, part (i) follows.

(ii) When E is totally ramified, $E_0 = T_0$, $\delta = n$, $\widetilde{N}$ is the identity map on T_0, and $[E^*, E_0^*] = [E^*, T_0^*] = 1$. By plugging all this into the exact column of diagram (3.5), it follows that $E^{(1)} \cong \mu_n(T_0)$. Also by [HwW$_2$] Prop. 2.1, $E' \cong \mu_e(T_0)$ where e is the exponent of the torsion abelian group Γ_E/Γ_T. Part (ii) now follows.

(iii) As recalled at the beginning of the proof of Th. 3.4, for any graded division algebra E with center T, we have $Z(E_0)$ is Galois over T_0, and there is an epimorphism $\theta\colon E^* \to \mathcal{G}al(Z(E_0)/T_0)$. Clearly, E_0^* and T^* lie in $\ker(\theta)$, so θ induces an epimorphism $\theta'\colon \Gamma_E/\Gamma_T \to \mathcal{G}al(Z(E_0)/T_0)$. When E is semiramified, by definition $[E_0 : T_0] = |\Gamma_E : \Gamma_T| = \mathrm{ind}(E)$ and E_0 is a field. Let $G = \mathcal{G}al(E_0/T_0)$. Because $|G| = [E_0 : T_0] = |\Gamma_E : \Gamma_T|$, the map θ' must be an isomorphism. In diagram (3.5), since $\mathrm{SK}_1(E_0) = 1$ and clearly $\delta = 1$, the exact top row and column yield $E^{(1)}/[E_0^*, E^*] \cong \widehat{H}^{-1}(G, E_0^*)$. Therefore, the exact row (3.8) follows from the exact second row of diagram (3.5) and the isomorphism $\Gamma_E/\Gamma_T \cong G$ given by θ'.

(iv) Since L and K are maximal subfields of E, we have $\mathrm{ind}(E) = [L : T] = [L_0 : T_0] \le [E_0 : T_0]$ and $\mathrm{ind}(E) = [K : T] = |\Gamma_K : \Gamma_T| \le |\Gamma_E : \Gamma_T|$. It follows from (2.2) that these inequalities are equalities, so $E_0 = L_0$ and $\Gamma_E = \Gamma_K$. Hence, E is semiramified, and (iii) applies. Take any $\eta, \nu \in \Gamma_E/\Gamma_T$, and any inverse images a, b of η, ν in E^*. The left map in (3.8) sends $\eta \wedge \nu$ to $aba^{-1}b^{-1} \bmod I_G(E_0^*)$. Since $\Gamma_E = \Gamma_K$, these a and b can be chosen in K^*, so they commute. Thus, the left map of (3.8) is trivial here, yielding the isomorphism of (iv). □

Remark 3.7. In the setting of Cor. 3.6(iii), there is a further interesting and new formula for $\mathrm{SK}_1(E)$ when $\mathcal{G}al(E_0/T_0)$ is bicyclic, which we describe here without proof. Suppose $G = \mathcal{G}al(E_0/T_0) \cong \langle\sigma\rangle \oplus \langle\tau\rangle$. Let M and P be the fixed fields, $M = E_0^\sigma$ and $P = E_0^\tau$. So, M and P are cyclic Galois over T_0 and $E_0 \cong M \otimes_{T_0} P$. Then, there is an isomorphism

$$\widehat{H}^{-1}(G, E_0^*) \ \cong\ \mathrm{Br}(E_0/T_0) \,/\, \big[\mathrm{Br}(M/T_0) \,+\, \mathrm{Br}(P/T_0)\big]\,, \tag{3.9}$$

where $\mathrm{Br}(T_0)$ is the Brauer group of T_0 and $\mathrm{Br}(E_0/T_0) = \ker\big(\mathrm{Br}(T_0) \to \mathrm{Br}(E_0)\big)$. All the explicit calculations of SK_1 of division algebras in [P$_1$] and [P$_2$] reduce to calculations of relative Brauer groups, using the formula for SK_1 obtained by replacing the relative Brauer group term for the H^{-1} term in the valued version of Cor. 3.6(iv). Now, as our graded division algebra E is semiramified, it is known that E is graded Brauer equivalent to $I \otimes_T N$, where I and N are graded division rings with center T, such that I is "inertial," i.e., $I \cong I_0 \otimes_{T_0} T$, and N is "nicely semiramified," i.e., it has a maximal graded subfield which is unramified over T and another which is totally ramified over T. Furthermore I and N are split by the unramifield graded field extension E_0T of T, so I_0 is split by E_0. One show that in the setting of

Cor. 3.6(iii) with $\mathcal{G}al(E_0/T_0)$ bicyclic (and M, P, and I as above),

$$\mathrm{SK}_1(E) \;\cong\; \mathrm{Br}(E_0/T_0)\big/\big[\,\mathrm{Br}(M/T_0) \,+\, \mathrm{Br}(P/T_0) \,+\, \langle I_0\rangle\big]\,. \tag{3.10}$$

Details will appear in our paper [HW$_2$]. There is an analogous formula in the Henselian valued setting of Cor. 4.10(iii) with $\mathcal{G}al(\overline{D}/\overline{F})$ bicyclic.

For a graded division algebra E with center T, define

$$\mathrm{CK}_1(E) \,=\, E^*/(T^*E'). \tag{3.11}$$

This is the graded analogue to $\mathrm{CK}_1(D)$ for a division algebra D, which is defined as $\mathrm{CK}_1(D) = D^*/(F^*D')$, where $F = Z(D)$. That is, $\mathrm{CK}_1(D)$ is the cokernel of the canonical map $K_1(F) \to K_1(D)$. See [H$_1$] for background on $\mathrm{CK}_1(D)$. Notably, it is known that $\mathrm{CK}_1(D)$ is torsion of bounded exponent $n = \mathrm{ind}(D)$, and CK_1 has functorial properties similar to SK_1. The CK_1 functor was used in [HW$_1$] in showing that for "nearly all" division algebras D, the multiplicative group D^* has a maximal proper subgroup. It is conjectured (see [HW$_1$] and its references) that if $\mathrm{CK}_1(D)$ is trivial, then D is a quaternion division algebra (necessarily over a real Pythagorean field).

Now, for the graded division algebra E with center T, the degree map (2.1) induces a surjective map $E^* \to \Gamma_E/\Gamma_T$ which has kernel $T^*{E_0}^*$. One can then observe that there is an exact sequence

$$1 \;\longrightarrow\; {E_0}^*/{T_0}^*E' \;\longrightarrow\; \mathrm{CK}_1(E) \;\longrightarrow\; \Gamma_E/\Gamma_T \;\longrightarrow\; 1.$$

Thus if E is unramified, $\mathrm{CK}_1(E) \cong {E_0}^*/({T_0}^*E')$ and $E^* \cong T^*{E_0}^*$. It then follows that $E' \cong {E_0}'$, yielding $\mathrm{CK}_1(E) \cong \mathrm{CK}_1(E_0)$. At the other extreme, when E is totally ramified then ${E_0}^*/({T_0}^*E') = 1$, so the exact sequence above yields $\mathrm{CK}_1(E) \cong \Gamma_E/\Gamma_T$.

4. SK_1 of a valued division algebra and its associated graded division algebra

The aim of this section is to study the relation between the reduced Whitehead group (and other related functors) of a valued division algebra with that of its corresponding graded division algebra. We will prove that SK_1 of a tame valued division algebra over a henselian field coincides with SK_1 of its associated graded division algebra. We start by recalling the concept of λ-polynomials introduced in [MW]. We keep the notations introduced in §2.

Let F be a field with valuation v, let $\mathrm{gr}(F)$ be the associated graded field, and F^{alg} the algebraic closure of F. For $a \in F^*$, let $\widetilde{a} \in \mathrm{gr}(F)_{v(a)}$ be the image of a in $\mathrm{gr}(F)$, let $\widetilde{0} = 0_{\mathrm{gr}(F)}$, and for $f = \sum a_i x^i \in F[x]$, let $\widetilde{f} = \sum \widetilde{a}_i x^i \in \mathrm{gr}(F)[x]$.

Definition 4.1. Take any λ in the divisible hull of Γ_F and let $f = a_n x^n + \ldots + a_i x^i + \ldots + a_0 \in F[x]$ with $a_n a_0 \neq 0$. Take any extension of v to F^{alg}. We say that f is a *λ-polynomial* if it satisfies the following equivalent conditions:

(a) Every root of f in F^{alg} has value λ;
(b) $v(a_i) \geq (n-i)\lambda + v(a_n)$ for all i and $v(a_0) = n\lambda + v(a_n)$;
(c) Take any $c \in F^{alg}$ with $v(c) = \lambda$ and let $h = \frac{1}{a_n c^n} f(cx) \in F^{alg}[x]$; then h is monic in $V_{F^{alg}}[x]$ and $h(0) \neq 0$.

If f is a λ-polynomial, let

$$f^{(\lambda)} \;=\; \sum_{i=0}^{n} a_i' x^i \;\in\; \mathrm{gr}(F)[x], \tag{4.1}$$

where a_i' is the image of a_i in $\mathrm{gr}(F)_{(n-i)\lambda+v(a_n)}$ (so $a_0' = \widetilde{a_0}$ and $a_n' = \widetilde{a_n}$, but for $1 \leq i \leq n-1$, $a_i' = 0$ if $v(a_i) > (n-i)\lambda + v(a_n)$). Note that $f^{(\lambda)}$ is a homogenizable polynomial in $\mathrm{gr}(F)[x]$, i.e., $f^{(\lambda)}$ is homogeneous (of degree $v(a_0)$) with respect to the the grading on $\mathrm{gr}(F)[x]$ as in (2.3) with $\theta = \lambda$. Also, $f^{(\lambda)}$ has the same degree as $f$as a polynomial in x.

The λ-polynomials are useful generalizations of polynomials $h \in V_F[x]$ with $h(0) \neq 0$—these are the 0-polynomials. The following proposition collects some basic properties of λ-polynomials over henselian fields, which are analogous to well-known properties for 0-polynomials, and have similar proofs. See, e.g., [EP], Th. 4.1.3, pp. 87–88 for proofs for 0-polynomials, and [MW], Th. 1.9 for proofs for λ-polynomials.

Proposition 4.2. *Suppose the valuation v on F is henselian. Then,*

(i) *If f is a λ-polynomial and $f = gh$ in $F[x]$, then g and h are λ-polynomials and $f^{(\lambda)} = g^{(\lambda)}h^{(\lambda)}$ in $\mathrm{gr}(F)[x]$. So, if $f^{(\lambda)}$ is irreducible in $\mathrm{gr}(F)[x]$, then f is irreducible in $F[x]$.*

(ii) *If $f = \sum_{i=0}^{n} a_i x^i$ is an irreducible polynomial in $F[x]$ with $a_n a_0 \neq 0$, then f is a λ-polynomial for $\lambda = (v(a_0) - v(a_n))/n$. Furthermore, $f^{(\lambda)} = \widetilde{a_n} h^s$ for some irreducible monic λ-homogenizable polynomial $h \in \mathrm{gr}(F)[x]$.*

(iii) *If f is a λ-polynomial in $F[x]$ and if $f^{(\lambda)} = g'h'$ in $\mathrm{gr}(F)[x]$ with $\gcd(g', h') = 1$, then there exist λ-polynomials $g, h \in F[x]$ such that $f = gh$ and $g^{(\lambda)} = g'$ and $h^{(\lambda)} = h'$.*

(iv) *If f is a λ-polynomial in $F[x]$ and if $f^{(\lambda)}$ has a simple root b in $\mathrm{gr}(F)$, then f has a simple root a in F with $\widetilde{a} = b$.*

(v) *Suppose k is a λ-homogenizable polynomial in $gr(F)[x]$ with $k(0) \neq 0$, and suppose $f \in F[x]$ with $\widetilde{f} = k$. Then f is a λ-polynomial and $f^{(\lambda)} = k$.*

Lemma 4.3. *Let $F \subseteq K$ be fields with $[K : F] < \infty$. Let v be a henselian valuation on F such that K is defectless over F. Then, for every $a \in K^*$, with $\widetilde{a}$ its image in $\mathrm{gr}(K)^*$,*

$$\widetilde{N_{K/F}(a)} \;=\; N_{\mathrm{gr}(K)/\mathrm{gr}(F)}(\widetilde{a}).$$

Proof. Let $n = [K : F]$. Note that $[\mathrm{gr}(K) : \mathrm{gr}(F)] = n$ as K is defectless over F. Let $f = x^\ell + c_{\ell-1}x^{\ell-1} + \ldots + c_0 \in F[x]$ be the minimal polynomial of a over F. Then f is irreducible in $F[x]$ and since v is henselian, f is a λ-polynomial, where $\lambda = v(a) = v(c_0)/n$ (see Prop. 4.2(ii)). Let $f^{(\lambda)}$ be the corresponding λ-homogenizable polynomial in $\mathrm{gr}(F)[x]$ as in (4.1). Then $f^{(\lambda)}(\widetilde{a}) = 0$ in $\mathrm{gr}(K)$ (by Prop. 4.2(i) with $g = x - a$), and by Prop. 4.2(ii) $f^{(\lambda)}$ has only one monic irreducible factor in $\mathrm{gr}(F)[x]$, say $f^{(\lambda)} = h^s$, with $\deg(h) = \ell/s$. Since $f^{(\lambda)}(\widetilde{a}) = 0$, h must be the minimal polynomial of $\widetilde{a}$ over $\mathrm{gr}(F)$ and over $q(\mathrm{gr}(F))$. (Recall that since $\mathrm{gr}(F)$ is integrally closed, a monic polynomial in $\mathrm{gr}(F)[x]$ is irreducible in $\mathrm{gr}(F)[x]$ iff it is irreducible in $q(\mathrm{gr}(F))[x]$.) We have $N_{K/F}(a) = (-1)^n c_0^{n/\ell}$. Hence, as $q(\mathrm{gr}(K)) \cong \mathrm{gr}(K) \otimes_{\mathrm{gr}(F)} q(\mathrm{gr}(F))$,

$$\begin{aligned} N_{\mathrm{gr}(K)/\mathrm{gr}(F)}(\widetilde{a}) \;&=\; N_{q(\mathrm{gr}(K))/q(\mathrm{gr}(F))}(\widetilde{a}) \;=\; (-1)^n h(0)^{ns/\ell} \;=\; (-1)^n (h(0)^s)^{n/\ell} \\ &=\; (-1)^n (\widetilde{c_0}^{n/\ell}) \;=\; \widetilde{(-1)^n c_0{}^{n/\ell}} \;=\; \widetilde{N_{K/F}(a)}. \end{aligned}$$
∎

Remark. The preceding lemma is still valid if v on F is not assumed to be henselian, but merely assumed to have a unique and defectless extension to K. This can be proved by scalar extension to the henselization F^h of F. (Since v extends uniquely and defectlessly to K, $K \otimes_F F^h$ is a field, and $\mathrm{gr}(K \otimes_F F^h) \cong_{\mathrm{gr}} \mathrm{gr}(K)$.)

Corollary 4.4. *Let F be a field with henselian valuation v, and let D be a tame F-central division algebra. Then for every $a \in D^*$, $\mathrm{Nrd}_{\mathrm{gr}(D)}(\widetilde{a}) = \widetilde{\mathrm{Nrd}_D(a)}$.*

Proof. Recall from §2 that the assumption D is tame over F means that $[D : F] = [\mathrm{gr}(D) : \mathrm{gr}(F)]$ and $\mathrm{gr}(F) = Z(\mathrm{gr}(D))$. Take any maximal subfield L of D containing a. Then L/F is defectless as D/F is defectless, so $[\mathrm{gr}(L) : \mathrm{gr}(F)] = [L : F] = \mathrm{ind}(D) = \mathrm{ind}(\mathrm{gr}(D))$. Hence, using Lemma 4.3 and Prop. 3.2(ii), we have,

$$\widetilde{\mathrm{Nrd}_D(a)} \;=\; \widetilde{N_{L/F}(a)} \;=\; N_{\mathrm{gr}(L)/\mathrm{gr}(F)}(\widetilde{a}) \;=\; \mathrm{Nrd}_{\mathrm{gr}(D)}(\widetilde{a}).$$
∎

Remarks 4.5. (i) Again, we do not need that v be henselian for Cor. 4.4. It suffices that the valuation v on F extends to D and D is tame over F.

(ii) Analogous results hold for the trace and reduced trace, with analogous proof. In the setting of Lemma 4.3, we have: if $v(\mathrm{Tr}_{K/F}(a)) = v(a)$, then $\mathrm{Tr}_{K/F}(\widetilde{a}) = \widetilde{\mathrm{Tr}_{K/F}(a)}$, but if $v(\mathrm{Tr}_{K/F}(a)) > v(a)$, then $\mathrm{Tr}_{K/F}(\widetilde{a}) = 0$.

(iii) By combining Cor. 4.4 with equation (3.3), for a tame valued division algebra D over henselian field F, we can relate the reduced norm of D with the reduced norm of $\overline{D}$ as follows:

$$\overline{\mathrm{Nrd}_D(a)} \;=\; N_{Z(\overline{D})/\overline{F}}\mathrm{Nrd}_{\overline{D}}(\overline{a})^{\delta}, \tag{4.2}$$

for any $a \in V_D \backslash M_D$ (thus, $\mathrm{Nrd}_D(a) \in V_F \backslash M_F$) and $\delta = \mathrm{ind}(D)/\big(\mathrm{ind}(\overline{D})\,[Z(\overline{D}) : \overline{F}]\big)$ (cf. [E], Cor. 2).

The next proposition will be used several times below. It was proved by Ershov in [E], Prop. 2, who refers to Yanchevskiĭ [Y] for part of the argument. We give a proof here for the convenience of the reader, and also to illustrate the utility of λ-polynomials.

Proposition 4.6. *Let $F \subseteq K$ be fields with henselian valuations v such that $[K:F] < \infty$ and K is tamely ramified over F. Then $N_{K/F}(1 + M_K) = 1 + M_F$.*

Proof. If $s \in 1 + M_K$ then $\widetilde{s} = 1$ in $\mathrm{gr}(K)$. So, as K is defectless over F, by Lemma 4.3 $\widetilde{N_{K/F}(s)} = N_{\mathrm{gr}(K)/\mathrm{gr}(F)}(\widetilde{s}) = 1$ in $\mathrm{gr}(F)$, i.e., $N_{K/F}(s) \in 1+M_F$. Thus $N_{K/F}(1+M_K) \subseteq 1+M_F$. To prove that this inclusion is an equality, we can assume $[K:F] > 1$. We have $[\mathrm{gr}(K) : \mathrm{gr}(F)] = [K:F] > 1$, since tamely ramified extensions are defectless. Also, the tame ramification implies that $q(\mathrm{gr}(K))$ is separable over $q(\mathrm{gr}(F))$. For, $q(\mathrm{gr}(F)) \cdot \mathrm{gr}(K)_0$ is separable over $q(\mathrm{gr}(F))$ since $\mathrm{gr}(K)_0 = \overline{K}$ and $\overline{K}$ is separable over $\mathrm{gr}(F)_0 = \overline{F}$. But also, $q(\mathrm{gr}(K))$ is separable over $q(\mathrm{gr}(F)) \cdot \mathrm{gr}(K)_0$ because $[q(\mathrm{gr}(K)) : q(\mathrm{gr}(F)) \cdot \mathrm{gr}(K)_0] = |\Gamma_K : \Gamma_F|$, which is not a multiple of $\mathrm{char}(\overline{F})$. Now, take any homogenous element $b \in \mathrm{gr}(K)$, $b \notin \mathrm{gr}(F)$, and let g be the minimal polynomial of b over $q(\mathrm{gr}(F))$. Then $g \in \mathrm{gr}(F)[x]$, b is a simple root of g, and g is λ-homogenizable where $\lambda = \deg(b)$, by [HwW$_1$], Prop. 2.2. Take any monic λ-polynomial $f \in F[x]$ with $f^{(\lambda)} = g$. Since $f^{(\lambda)}$ has the simple root b in $\mathrm{gr}(K)$ and the valuation on K is henselian, by Prop. 4.2(iv) there is $a \in K$ such that a is a simple root of f and $\widetilde{a} = b$. Let $L = F(a) \subseteq K$. Write $f = x^n + c_{n-1}x^{n-1} + \ldots + c_0$. Take any $t \in 1 + M_F$, and let $h = x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + tc_0 \in F[x]$. Then h is a λ-polynomial (because f is) and $h^{(\lambda)} = f^{(\lambda)} = g$ in $\mathrm{gr}(F)[x]$. Since $h^{(\lambda)}$ has the simple root b in $\mathrm{gr}(L)$, h has a simple root d in L with $\widetilde{d} = b = \widetilde{a}$ by Prop. 4.2(iv). So, $da^{-1} \in 1 + M_L$. The polynomials f and h are irreducible in $F[x]$ by Prop. 4.2(i), as g is irreducible in $\mathrm{gr}(F)[x]$. Since f (resp. h) is the minimal polynomial of a (resp. d) over F, we have $N_{L/F}(a) = (-1)^n c_0$ and $N_{L/F}(d) = (-1)^n c_0 t$. Thus, $N_{L/F}(da^{-1}) = t$, showing that $N_{L/F}(1 + M_L) = 1 + M_F$. If $L = K$, we are done. If not, we have $[K:L] < [K:F]$, and K is tamely ramified over L. So, by induction on $[K:F]$, we have $N_{K/L}(1 + M_K) = 1 + M_L$. Hence,

$$N_{K/F}(1 + M_K) \;=\; N_{L/F}\big(N_{K/L}(1 + M_K)\big) \;=\; N_{L/F}(1 + M_L) \;=\; 1 + M_F. \qquad \square$$

Corollary 4.7. *Let F be a field with henselian valuation v, and let D be an F-central division algebra which is tame with respect to v. Then,* $\mathrm{Nrd}_D(1 + M_D) = 1 + M_F$.

Proof. Take any $a \in 1 + M_D$ and any maximal subfield K of D with $a \in K$. Then, K is defectless over F, since D is defectless over F. So, $a \in 1 + M_K$, and $\mathrm{Nrd}_D(a) = N_{K/F}(a) \in 1 + M_F$ by the first part of the proof of Prop. 4.6, which required only defectlessness, not tameness. Thus, $\mathrm{Nrd}_D(1 + M_D) \subseteq 1 + M_F$. For the reverse inclusion, recall from [HwW$_2$], Prop. 4.3 that as D is tame over F, it has a maximal subfield L with L tamely ramified over F. Then by Prop. 4.6,

$$1 + M_F \;=\; N_{L/F}(1 + M_L) \;=\; \mathrm{Nrd}_D(1 + M_L) \;\subseteq\; \mathrm{Nrd}_D(1 + M_D) \;\subseteq\; 1 + M_F,$$

so equality holds throughout. $\square$

We can now prove the main result of this section:

Theorem 4.8. *Let F be a field with henselian valuation v and let D be a tame F-central division algebra. Then* $\mathrm{SK}_1(D) \cong \mathrm{SK}_1(\mathrm{gr}(D))$.

Proof. Consider the canonical surjective group homomorphism $\rho\colon D^* \to \mathrm{gr}(D)^*$ given by $a \mapsto \widetilde{a}$. Clearly, $\ker(\rho) = 1 + M_D$. If $a \in D^{(1)} \subseteq V_D$ then $\widetilde{a} \in \mathrm{gr}(D)_0$ and by Cor. 4.4,

$$\mathrm{Nrd}_{\mathrm{gr}(D)}(\widetilde{a}) \ = \ \widetilde{\mathrm{Nrd}_D(a)} \ = \ 1.$$

This shows that $\rho(D^{(1)}) \subseteq \mathrm{gr}(D)^{(1)}$. Now consider the diagram

$$\begin{tikzcd}
1 \arrow[r] & (1+M_D)\cap D' \arrow[r] \arrow[d] & D' \arrow[r, "\rho"] \arrow[d] & \mathrm{gr}(D)' \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & (1+M_D)\cap D^{(1)} \arrow[r] & D^{(1)} \arrow[r] & \mathrm{gr}(D)^{(1)} \arrow[r, dotted] & 1
\end{tikzcd} \tag{4.3}$$

The top row of the above diagram is clearly exact. The Congruence Theorem (see Th. B.1 in Appendix B), implies that the left vertical map in the diagram is an isomorphism. Once we prove that $\rho(D^{(1)}) = \mathrm{gr}(D)^{(1)}$, we will have the exactness of the second row of diagram (4.3), and the theorem follows by the exact sequence for cokernels.

To prove the needed surjectivity, take any $b \in \mathrm{gr}(D)^*$ with $\mathrm{Nrd}_{\mathrm{gr}(D)}(b) = 1$. Thus $b \in \mathrm{gr}(D)_0$ by Th. 3.3. Choose $a \in V_D$ such that $\widetilde{a} = b$. Then we have,

$$\overline{\mathrm{Nrd}_D(a)} \ = \ \widetilde{\mathrm{Nrd}_D(a)} \ = \ \mathrm{Nrd}_{\mathrm{gr}(D)}(b) \ = \ 1.$$

Thus $\mathrm{Nrd}_D(a) \in 1 + M_F$. By Cor. 4.7, since $\mathrm{Nrd}_D(1 + M_D) = 1 + M_F$, there is $c \in 1 + M_D$ such that $\mathrm{Nrd}_D(c) = \mathrm{Nrd}(a)^{-1}$. Then, $ac \in D^{(1)}$ and $\rho(ac) = \rho(a) = b$. ☐

Recall from §2 that starting from any graded division algebra E with center T and any choice of total ordering $\leq$ on the torsion-free abelian group Γ_E, there is an induced valuation v on $q(E)$, see (2.6). Let $h(T)$ be the henselization of T with respect to v, and let $h(E) = q(E) \otimes_{q(T)} h(T)$. Then, $h(E)$ is a division ring by Morandi's henselization theorem ([Mor], Th. 2 or see [W₂], Th. 2.3), and with respect to the unique extension of the henselian valuation on $h(T)$ to $h(E)$, $h(E)$ is an immediate extension of $q(E)$, i.e., $\mathrm{gr}(h(E)) \cong_{\mathrm{gr}} \mathrm{gr}(q(E))$. Furthermore, as

$$[h(E) : h(T)] \ = \ [q(E) : q(T)] \ = \ [E : T] \ = \ [\mathrm{gr}(q(E)) : \mathrm{gr}(q(T))] \ = \ [\mathrm{gr}(h(E) : \mathrm{gr}(h(T))]$$

and

$$Z(\mathrm{gr}(h(E))) \ \cong_{\mathrm{gr}} \ Z(\mathrm{gr}(q(E))) \ \cong_{\mathrm{gr}} \ T \ \cong_{\mathrm{gr}} \ \mathrm{gr}(h(T)) \ = \ \mathrm{gr}(Z(h(E))),$$

$h(E)$ is tame (see the characterizations of tameness in §2).

Corollary 4.9. *Let E be a graded division algebra. Then* $\mathrm{SK}_1(h(E)) \cong \mathrm{SK}_1(E)$.

Proof. Since $h(E)$ is a tame valued division algebra, by Th. 4.8, $\mathrm{SK}_1(h(E)) \cong \mathrm{SK}_1(\mathrm{gr}(h(E)))$. But $\mathrm{gr}(h(E)) \cong_{\mathrm{gr}} \mathrm{gr}(q(E)) \cong_{\mathrm{gr}} E$, so the corollary follows. ☐

Having now established that the reduced Whitehead group of a division algebra coincides with that of its associated graded division algebra, we can easily deduce stability of SK_1 for unramified valued division algebra, due originally to Platonov (Cor. 3.13 in [P₁]), and also a formula for SK_1 for a totally ramified division algebra ([LT], p. 363, see also [E], p. 70), and also a formula for SK_1 in the nicely semiramified case ([E], p. 69), as natural consequences of Th. 4.8:

Corollary 4.10. *Let F be a field with Henselian valuation, and let D be a tame division algebra with center F.*

(i) *If D is unramified then* $\mathrm{SK}_1(D) \cong \mathrm{SK}_1(\overline{D})$

(ii) *If D is totally ramified then* $\mathrm{SK}_1(D) \cong \mu_n(\overline{F})/\mu_e(\overline{F})$ *where* $n = \mathrm{ind}(D)$ *and e is the exponent of* Γ_D/Γ_F.

(iii) *If D is semiramified, let* $G = \mathcal{G}al(\overline{D}/\overline{F}) \cong \Gamma_D/\Gamma_F$. *Then, there is an exact sequence*

$$G \wedge G \;\to\; \widehat{H}^{-1}(G, \overline{D}^*) \;\to\; \mathrm{SK}_1(D) \;\to\; 1. \tag{4.4}$$

(iv) *If D is nicely semiramified, then* $\mathrm{SK}_1(D) \cong \widehat{H}^{-1}(\mathcal{G}al(\overline{D}/\overline{F}), \overline{D}^*)$.

See Remark 4.11 below for a description of the maps in (4.4)

Proof. Because D is tame, $Z(\mathrm{gr}(D)) = \mathrm{gr}(F)$ and $\mathrm{ind}(\mathrm{gr}(D)) = \mathrm{ind}(D)$. Therefore, for D in each case (i)–(iv) here, $\mathrm{gr}(D)$ is in the corresponding case of Cor. 3.6. (In case (iii), that D is semiramified means $[\overline{D} : \overline{F}] = |\Gamma_D : \Gamma_F| = \mathrm{ind}(D)$ and $\overline{D}$ is a field. Hence $\mathrm{gr}(D)$ is semiramified. In case (iv), since D is nicely semiramified, by definition (see [JW], p. 149) it contains maximal subfields K and L, with K unramified over F and L totally ramified over F. (In fact, by [M_1], Th. 2.4, D is nicely semiramified if and only if it has such maximal subfields.) Then, $\mathrm{gr}(K)$ and $\mathrm{gr}(L)$ are maximal graded subfields of $\mathrm{gr}(D)$ by dimension count and the graded double centralizer theorem,[HwW_2], Prop. 1.5(b), with $\mathrm{gr}(K)$ unramified over $\mathrm{gr}(F)$ and $\mathrm{gr}(L)$ totally ramified over $\mathrm{gr}(F)$. So, $\mathrm{gr}(D)$ is then in case (iv) of Cor. 3.6.) Thus, in each case Cor. 4.10 for D follows from Cor. 3.6 for $\mathrm{gr}(D)$ together with the isomorphism $\mathrm{SK}_1(D) \cong \mathrm{SK}_1(\mathrm{gr}(D))$ given by Th. 4.8. □

Remark 4.11. By tracing through the isomorphisms used in their construction, one can see that the maps in (4.4) can be described as follows: Let v be the valuation on D. For each $\sigma \in G = \mathcal{G}al(\overline{D}/\overline{F})$ there is by [JW], Prop. 1.7 or [E], Prop. 1 some $d_\sigma \in D^*$ such that $\overline{d_\sigma^{-1} a d_\sigma} = \sigma(\overline{a})$ for all $a \in V_D$. This d_σ is not unique, though its image in Γ_D/Γ_F is uniquely determined. For $\tau \in G$, choose $d_\tau \in D^*$ analogously. Then, $v([d_\sigma, d_\tau]) = 0$ and $N_{\overline{D}/\overline{F}}(\overline{[d_\sigma, d_\tau]}) = 1$ since $\mathrm{Nrd}_D([d_\sigma, d_\tau]) = 1$, by (4.2). The map $G \wedge G \to \widehat{H}^{-1}(G, \overline{D}^*)$ sends $\sigma \wedge \tau$ to the image of $\overline{[d_\sigma, d_\tau]}$ in $\widehat{H}^{-1}(G, \overline{D}^*)$. Now, take any $b \in \overline{D}^*$ with $N_{\overline{D}/\overline{F}}(b) = 1$, and let a be any inverse image of b in V_D. By (4.2) (since here $\delta = 1$ and $Z(\overline{D}) = \overline{D}$), $\mathrm{Nrd}_D(a) \in 1 + M_F$, so by Cor. 4.7 there is $c \in 1 + M_D$ with $\mathrm{Nrd}_D(c) = \mathrm{Nrd}_D(a)$. Then $\overline{ac^{-1}} = b$ and $\mathrm{Nrd}_D(ac^{-1}) = 1$. The map $\widehat{H}^{-1}(G, \overline{D}^*) \to \mathrm{SK}_1(D)$ sends the image of b in $\widehat{H}^{-1}(G, \overline{D}^*)$ to the image of ac^{-1} in $\mathrm{SK}_1(D)$.

Recall that the reduced norm residue group of D is defined as $\mathrm{SH}^0(D) = F^*/\mathrm{Nrd}_D(D^*)$. It is known that $\mathrm{SH}^0(D)$ coincides with the first Galois cohomology group $H^1(F, D^{(1)})$ (see [KMRT], §29). We now show that for a tame division algebra D over a henselian field, $\mathrm{SH}^0(D)$ coincides with SH^0 of its associated graded division algebra.

Theorem 4.12. *Let F be a field with a henselian valuation v and let D be a tame F-central division algebra. Then* $\mathrm{SH}^0(D) \cong \mathrm{SH}^0(\mathrm{gr}(D))$.

Proof. Consider the diagram with exact rows,

$$\begin{tikzcd}
1 \arrow[r] & 1+M_D \arrow[r] \arrow[d] & D^* \arrow[r, "\rho"] \arrow[d, "\mathrm{Nrd}_D"] & \mathrm{gr}(D)^* \arrow[r] \arrow[d, "\mathrm{Nrd}_{\mathrm{gr}(D)}"] & 1 \\
1 \arrow[r] & 1+M_F \arrow[r] & F^* \arrow[r] & \mathrm{gr}(F)^* \arrow[r] & 1
\end{tikzcd} \tag{4.5}$$

where Cor. 4.4 guarantees that the diagram is commutative. By Cor. 4.7, the left vertical map is an epimorphism. The theorem follows by the exact sequence for cokernels. □

Remark. As with SK_1, if D is tame and unramified, then

$$\mathrm{SH}^0(D) \;\cong\; \mathrm{SH}^0(\mathrm{gr}(D)) \;\cong\; \mathrm{SH}^0(\mathrm{gr}(D)_0) \;\cong\; \mathrm{SH}^0(\overline{D}).$$

We conclude this section by establishing a similar result for the CK_1 functor of (3.11) above. Note that here, unlike the situation with SK_1 (Th. 4.8) or with SH^0 (Th. 4.12), we need to assume strong tameness here.

Theorem 4.13. *Let F be a field with henselian valuation v and let D be a strongly tame F-central division algebra. Then* $\mathrm{CK}_1(D) \cong \mathrm{CK}_1(\mathrm{gr}(D))$.

Proof. Consider the canonical epimorphism $\rho\colon D^* \to \mathrm{gr}(D)^*$ given by $a \mapsto \widetilde{a}$, with kernel $1 + M_D$. Since ρ maps D' onto $\mathrm{gr}(D)'$ and F^* onto $\mathrm{gr}(F)^*$, it induces an isomorphism $D^*/\big(F^*D'(1+M_D)\big) \cong \mathrm{gr}(D)^*/\big(\mathrm{gr}(F)^*\mathrm{gr}(D)'\big)$. We have $\mathrm{gr}(F) = Z(\mathrm{gr}(D))$ and by Lemma 2.1 in [H3], as D is strongly tame, $1 + M_D = (1+M_F)[D^*, 1+M_D] \subseteq F^*D'$. Thus, $\mathrm{CK}_1(D) \cong \mathrm{CK}_1(\mathrm{gr}(D))$. □

5. Stability of the reduced Whitehead group

The goal of this section is to prove that if E is a graded division ring (with Γ_E a torsion-free abelian group), then $\mathrm{SK}_1(E) \cong \mathrm{SK}_1(q(E))$, where $q(E)$ is the quotient division ring of E. When $\Gamma_E \cong \mathbb{Z}$, this was essentially proved by Platonov and Yanchevskiĭ in [PY], Th. 1 (see the Introduction). Their argument was based on properties of twisted polynomial rings, and our argument is based on their approach. So, we will first look at twisted polynomial rings. For these, an excellent reference is Ch. 1 in [J].

Let D be a division ring finite dimensional over its center $Z(D)$. Let σ be an automorphism of D whose restriction to $Z(D)$ has finite order, say ℓ. Let $T = D[x,\sigma]$ be the twisted polynomial ring, with multiplication given by $xd = \sigma(d)x$, for all $d \in D$. By Skolem-Noether, there is $w \in D^*$ with $\sigma^\ell = \mathrm{int}(w^{-1})$ (= conjugation by w^{-1}); moreover, w can be chosen so that $\sigma(w) = w$ (by a Hilbert 90 argument, see [J], Th. 1.1.22(iii) or [PY], Lemma 1). Then $Z(T) = K[y]$ (a commutative polynomial ring), where $K = Z(D)^\sigma$, the fixed field of $Z(D)$ under the action of σ, and $y = wx^\ell$. Let $Q = q(T) = D(x,\sigma)$, the division ring of quotients of T. Since T is a finitely-generated $Z(T)$-module, Q is the central localization $T \otimes_{Z(T)} q(Z(T))$ of T. Note that $Z(Q) = q(Z(T)) = K(y)$, and $\mathrm{ind}(Q) = \ell\,\mathrm{ind}(D)$. Observe that within Q we have the twisted Laurent polynomial ring $T[x^{-1}] = D[x,x^{-1},\sigma]$ which is a graded division ring, graded by degree in x, and $T \subseteq T[x^{-1}] \subseteq q(T)$, so that $q(T[x^{-1}]) = Q$. Recall that, since we have left and right division algorithms for T, T is a principal left (and right) ideal domain.

Let $\mathcal{S}$ denote the set of isomorphism classes $[S]$ of simple left T-modules S, and set

$$\mathrm{Div}(T) \;=\; \bigoplus_{[S]\in\mathcal{S}} \mathbb{Z}[S],$$

the free abelian group with base $\mathcal{S}$. For any T-module M satisfying both ACC and DCC, the Jordan-Hölder Theorem yields a well-defined element $jh(M) \in \mathrm{Div}(T)$, given by

$$jh(M) \;=\; \sum_{[S]\in\mathcal{S}} n_{[S]}(M)[S],$$

where $n_{[S]}(M)$ is the number of appearances of simple factor modules isomorphic to S in any composition series of M. Note that for any $f \in T \setminus \{0\}$, the division algorithm shows that $\dim_D(T/Tf) = \deg(f) < \infty$. Hence, T/Tf has ACC and DCC as a T-module. Therefore, we can define a divisor function

$$\delta\colon T\setminus\{0\} \to \mathrm{Div}(T), \quad \text{given by} \quad \delta(f) \;=\; jh(T/Tf).$$

Remark 5.1. Note the following properties of δ:

(i) For any $f, g \in T \setminus \{0\}$, $\delta(fg) = \delta(f) + \delta(g)$. This follows from the isomorphism $Tg/Tfg \cong T/Tf$ (as T has no zero divisors).

(ii) We can extend δ to a map $\delta\colon Q^* \to \mathrm{Div}(T)$, where $Q = q(T)$, by $\delta(fh^{-1}) = \delta(f) - \delta(h)$ for any $f \in T\setminus\{0\}$, $h \in Z(T)\setminus\{0\}$. It follows from (i) that δ is well-defined and is a group homomorphism on Q^*. Clearly, δ is surjective, as every simple T-module is cyclic.

(iii) For all $q, s \in Q^*$, $\delta(sqs^{-1}) = \delta(q)$. This is clear, as δ is a homomorphism into an abelian group.
(iv) For all $q \in Q^*$, $\delta(\mathrm{Nrd}_Q(q)) = n\,\delta(q)$, where $n = \mathrm{ind}(Q)$. This follows from (iii), since Wedderburn's factorization theorem applied to the minimal polynomial of q over $Z(Q)$ shows that $\mathrm{Nrd}_Q(q) = \prod_{i=1}^n s_i q {s_i}^{-1}$ for some $s_i \in Q^*$.
(v) If $\mathrm{Nrd}_Q(q) = 1$, then $\delta(q) = 0$. This is immediate from (iv), as $\mathrm{Div}(T)$ is torsion-free.

Lemma 5.2. *Take any $f, g \in T \setminus \{0\}$ with $T/Tf \cong T/Tg$, so $\deg(f) = \deg(g)$. If $\deg(f) \geq 1$, there exist $s, t \in T \setminus \{0\}$ with $\deg(s) = \deg(t) < \deg(f)$ such that $fs = tg$.*

Proof. (cf. [J], Prop. 1.2.8) We have $\deg(f) = \dim_D(T/Tf) = \dim_D(T/Tg) = \deg(g)$. Let $\alpha\colon T/Tf \to T/Tg$ be a T-module isomorphism, and let $\alpha(1+Tf) = s+Tg$. By the division algorithm, s can be chosen with $\deg(s) < \deg(g)$. We have

$$fs + Tg \;=\; f(s+Tg) \;=\; f\alpha(1+Tf) \;=\; \alpha(f+Tf) \;=\; \alpha(0) \;=\; 0 \quad \text{in } T/Tg.$$

Hence, $fs = tg$ for some $t \in T$. Since $\deg(f) = \deg(g)$, we have

$$\deg(t) \;=\; \deg(s) \;<\; \deg(g) \;=\; \deg(f). \qquad \square$$

Proposition 5.3. *Consider the group homomorphism $\delta\colon Q^* \to \mathrm{Div}(T)$ defined in Remark 5.1(ii) above. Then $\ker(\delta) = D^*Q'$.*

Proof. (cf. [PY], proof of Lemma 5) Clearly, $D^* \subseteq \ker(\delta)$ and $Q' \subseteq \ker(\delta)$, so $D^*Q' \subseteq \ker(\delta)$. For the reverse inclusion take $h \in \ker(\delta)$ and write $h = fg^{-1}$ with $f, g \in T \setminus \{0\}$. (As Q is a central localization of T, g may be chosen in $Z(T)$, but we do not need this.) Since $\delta(fg^{-1}) = 0$, we have $\delta(f) = \delta(g)$, so $\deg(f) = \deg(g)$. If $\deg(f) = 0$, then $h \in D^*$, and we're done. So, assume $\deg(f) > 1$. Write $f = pf_1$ with p irreducible in T. Then, T/Tp is one of the simple composition factors of T/Tf. If $g = q_1q_2\ldots q_k$ with each q_i irreducible in T, then the composition factors of T/Tg are (up to isomorphism) $T/Tq_1, \ldots, T/Tq_k$. Because $\delta(f) = \delta(g)$, i.e. $jh(T/Tf) = jh(T/Tg)$, we must have $T/Tp \cong T/Tq_j$ for some j. Write $g = g_1qg_2$ where $q = q_j$. By Lemma 5.2, there exist $s, t \in T\setminus\{0\}$ with $\deg(s) = \deg(t) < \deg(p) = \deg(q)$ and $ps = tq$. Then, working modulo Q', we have

$$h \;=\; fg^{-1} \;=\; (pf_1)(g_1qg_2)^{-1} \;\equiv\; f_1(pq^{-1})(g_1g_2)^{-1} \;\equiv\; f_1(ts^{-1})(g_1g_2)^{-1} \;\equiv\; (f_1t)(g_1g_2s)^{-1}.$$

Let $h' = (f_1t)(g_1g_2s)^{-1}$. Since $h' \equiv h \pmod{Q'}$, we have $\delta(h') = \delta(h) = 0$, while $\deg(f_1t) < \deg(f)$. By iterating this process we can repeatedly lower the degree of numerator and denominator to obtain $h'' \in D^*$ with $h'' \equiv h' \equiv h \pmod{Q'}$. Hence, $h \in D^*Q'$, as desired. $\square$

Remark. Since $K_1(Q) = Q^*/Q'$, Prop. 5.3 can be stated as saying that there is an exact sequence

$$\mathrm{K}_1(D) \longrightarrow \mathrm{K}_1(Q) \xrightarrow{\;\delta\;} \mathrm{Div}(T) \longrightarrow 0. \tag{5.1}$$

This can be viewed as part of an exact localization sequence in K-Theory. We prefer the explicit description of $\mathrm{Div}(T)$ and δ given here, as it helps to understand the maps associated with $\mathrm{Div}(T)$.

Let $R = Z(T) = K[y]$. So, $q(R) = Z(Q)$. We define $\mathrm{Div}(R)$ just as we defined $\mathrm{Div}(T)$ above. Note that this $\mathrm{Div}(R)$ coincides canonically with the usual divisor group of fractional ideals of the PID R, since for $a \in R \setminus \{0\}$, the simple composition factors of R/Ra are the simple modules R/P as P ranges over the prime ideal factors of the ideal Ra.

Proposition 5.4. *For $R = Z(T) = K[y]$, there is a map $\mathrm{Nrd}\colon \mathrm{Div}(T) \to \mathrm{Div}(R)$ such that the following diagram commutes:*

$$\begin{tikzcd} D^* \arrow[r] \arrow[d, "\mathrm{Nrd}_D"'] & Q^* \arrow[r, "\delta_T"] \arrow[dd, "\mathrm{Nrd}_Q"] & \mathrm{Div}(T) \arrow[dd, "\mathrm{Nrd}"] \\ Z(D)^* \arrow[d, "N_{Z(D)/K}"'] & & \\ K^* \arrow[r] & q(R)^* \arrow[r, "\delta_R"] & \mathrm{Div}(R) \end{tikzcd} \tag{5.2}$$

Moreover, Nrd *is injective.*

Proof. Let $E = T[x^{-1}] = D[x, x^{-1}, \sigma]$, which with its grading by degree in x is a graded division ring with $E_0 = D$ and $q(E) = Q$. Since $\mathrm{ind}(Q) = \mathrm{ind}(D)\,[Z(D) : K]$, by (3.3), for $d \in D^* = E_0^*$, $\mathrm{Nrd}_Q(d) = N_{Z(D)/K}(\mathrm{Nrd}_D(d))$. This gives the commutativity of the left rectangle in the diagram.

For the right vertical map in diagram (5.2), note that there is a canonical map, call it $N\colon \mathrm{Div}(T) \to \mathrm{Div}(R)$ given by taking a T-module M (with ACC and DCC) and viewing it as an R-module; that is $N(jh_T(M)) = jh_R(M)$. But, this is not the map $\mathrm{Nrd}\colon \mathrm{Div}(T) \to \mathrm{Div}(R)$ we need here! (Consider N a norm map, while our desired Nrd is a *reduced* norm map.)

The map $\delta_R\colon q(R)^* \to \mathrm{Div}(R)$ is defined the same way as δ_T. Let $\psi = \delta_R \circ \mathrm{Nrd}_Q\colon Q^* \to \mathrm{Div}(R)$. Then, $Q' \subseteq \ker(\psi)$ as $q(R)^*$ is abelian, and $D^* \subseteq \ker(\psi)$ by the commutative left rectangle of (5.2). Prop. 5.3 thus yields $\ker(\delta_T) \subseteq \ker(\psi)$. Since δ_T is surjective, there is an induced homomorphism $\mathrm{Nrd}\colon \mathrm{Div}(T) \to \mathrm{Div}(R)$ such that $\mathrm{Nrd} \circ \delta_T = \psi = \delta_R \circ \mathrm{Nrd}_Q$. This Nrd is the desired map.

We have a scalar extension map from R-modules to T-modules given by $M \to T \otimes_R M$. This induces a map $\rho\colon \mathrm{Div}(R) \to \mathrm{Div}(T)$ given by $\rho(jh_R(M)) = jh_T(T \otimes_R M)$. For any $r \in R$, we have $T \otimes_R (R/Rr) \cong T/Tr$. Thus for any $g \in T \setminus \{0\}$,

$$\begin{aligned} \rho(\mathrm{Nrd}(\delta_T(g))) &= \rho(\delta_R(\mathrm{Nrd}_Q(g))) = \rho(jh_R(R/R\mathrm{Nrd}_Q(g))) \\ &= jh_T(T/T\mathrm{Nrd}_Q(g)) = \delta_T(\mathrm{Nrd}_Q(g)) = n\,\delta_T(g), \end{aligned}$$

using Remark 5.1(iv). This shows that $\rho \circ \mathrm{Nrd}\colon \mathrm{Div}(T) \to \mathrm{Div}(T)$ is multiplication by n, which is an injection, as $\mathrm{Div}(T)$ is a torsion-free abelian group. Hence Nrd must be injective. □

Remark. Here is a description of how the maps $\mathrm{Nrd}\colon \mathrm{Div}(T) \to \mathrm{Div}(R)$ and $N\colon \mathrm{Div}(T) \to \mathrm{Div}(R)$ and $\rho\colon \mathrm{Div}(R) \to \mathrm{Div}(T)$ are related, and a formula for Nrd on generators of $\mathrm{Div}(T)$. Proofs are omitted. We have

$$\rho \circ \mathrm{Nrd} = n\,\mathrm{id}_{\mathrm{Div}(T)}; \tag{5.3}$$

and

$$N = n \cdot \mathrm{Nrd}. \tag{5.4}$$

Let S be any simple left T-module, and $[S]$ the corresponding basic generator of $\mathrm{Div}(T)$. Let $M = \mathrm{ann}_T(S)$, and let $P = \mathrm{ann}_R(S)$, which is a maximal ideal of R. Let k = matrix of size of $T/M = \dim_\Delta(S)$, where $\Delta = \mathrm{End}_T(S)$, so $T/M \cong M_k(\Delta)$. Then,

$$\mathrm{Nrd}([S]) = n_S[R/P], \quad \text{where} \quad n_S = \tfrac{1}{nk}\dim_{R/P}(T/M) = \mathrm{ind}(T/M). \tag{5.5}$$

We now consider an arbitrary graded division ring E. As usual, we assume throughout that Γ_E is a torsion-free abelian group and $[E : Z(E)] < \infty$.

Lemma 5.5. *Let E be a graded division ring, and let $Q = q(E)$. Then, the canonical map $\mathrm{SK}_1(E) \to \mathrm{SK}_1(Q)$ is injective.*

Proof. Recall from Prop. 3.2(i) that $\mathrm{Nrd}_E(a) = \mathrm{Nrd}_Q(a)$ for all $a \in E$, so the inclusion $E^* \hookrightarrow Q^*$ yields a map $\mathrm{SK}_1(E) = E^{(1)}/E' \to Q^{(1)}/Q' = \mathrm{SK}_1(Q)$. Also recall the homomorphism $\lambda\colon Q^* \to E^*$ of (2.6), which maps Q' to E'. Since the composition $E^* \hookrightarrow Q^* \xrightarrow{\lambda} E^*$ is the identity map, for any $a \in E^{(1)} \cap Q'$, we have $a = \lambda(a) \in E'$. Thus, the map $\mathrm{SK}_1(E) \to \mathrm{SK}_1(Q)$ is injective. □

Proposition 5.6. *Let E be a graded division ring, and let $Q = q(E)$. Then,*

$$Q^{(1)} \;=\; (Q^{(1)} \cap E_0)Q'.$$

Once this proposition is proved, it will quickly yield the main theorem of this section:

Theorem 5.7. *Let E be a graded division ring. Then,* $\mathrm{SK}_1(E) \cong \mathrm{SK}_1(q(E))$.

Proof. Set $Q = q(E)$. Since the reduced norm respects scalar extensions, $Q^{(1)} \cap E_0 \subseteq E^{(1)}$. The image of the map $\xi\colon \mathrm{SK}_1(E) \to \mathrm{SK}_1(Q)$ is $E^{(1)}Q'/Q'$, which thus contains $(Q^{(1)} \cap E_0)Q'/Q' = Q^{(1)}/Q' = \mathrm{SK}_1(Q)$ (using Prop. 5.6). Thus ξ is surjective, as well as being injective by Lemma 5.5, proving the theorem. □

Proof of Prop. 5.6. We first treat the case where Γ_E is finitely generated.

Case I. *Suppose $\Gamma_E = \mathbb{Z}^n$ for some $n \in \mathbb{N}$.*

Let $F = Z(E)$, a graded field, and let $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ (1 in the i-th position), so $\Gamma_E = \mathbb{Z}\varepsilon_1 \oplus \ldots \oplus \mathbb{Z}\varepsilon_n$. For $1 \le i \le n$, let $\Delta_i = \mathbb{Z}\varepsilon_1 \oplus \ldots \oplus \mathbb{Z}\varepsilon_i \subseteq \Gamma_E$; and let $S_i = E_{\Delta_i} = \bigoplus_{\gamma \in \Delta_i} E_\gamma$, which is a graded sub-division ring of E. Let $Q_i = q(S_i)$, the quotient division ring of S_i; so $Q_n = Q$ as $S_n = E$. Set $R_0 = Q_0 = E_0$. Note that $[S_i : (S_i \cap F)] < \infty$, so Q_i is obtainable from S_i by inverting the nonzero elements of $S_i \cap F$. This makes it clear that $Q_i \subseteq Q_{i+1}$, for each i.

For each j, $1 \le j \le n$, choose and fix a nonzero element $x_j \in E_{\varepsilon_j}$. Let $\varphi_j = \mathrm{int}(x_j) \in \mathrm{Aut}(E)$ (i.e., φ_j is conjugation by x_j). Since φ_j is a degree-preserving automorphism of E, φ_j maps each S_i to itself. Hence, φ_j extends uniquely to an automorphism to Q_i, also denoted φ_j. Since each Γ_E/Γ_F is a torsion abelian group, there is $\ell_j \in \mathbb{N}$ such that $\ell_j\varepsilon_j \in \Gamma_F$. Then, if we choose any nonzero $z_j \in F_{\ell_j\varepsilon_j}$, we have $x_j^{\ell_j} \in E_{\ell_j\varepsilon_j} = E_0 z_j$. So, $x_j^{\ell_j} = c_j z_j$ for some $c_j \in E_0^*$, and $z_j \in F = Z(E)$. Then $\varphi_j^{\ell_j} = \mathrm{int}({x_j}^{l_j}) = \mathrm{int}(c_j z_j) = \mathrm{int}(c_j)$. Thus, $\varphi_j^{\ell_j}|_{S_i}$ is an inner automorphism of S_i for each i, as $c_j \in E_0^* \subseteq S_i$.

Now, fix i with $1 \le i \le n$. We will prove:

$$Q_i^* \cap Q^{(1)} \;\subseteq\; (Q_{i-1}^* \cap Q^{(1)})[Q_i^*, Q^*]. \tag{5.6}$$

We have $S_i = S_{i-1}[x_i, x_i^{-1}] \cong S_{i-1}[x_i, x_i^{-1}, \varphi_i]$ (twisted Laurent polynomial ring). Likewise, within Q_i we have $Q_{i-1}[x_i] \cong Q_{i-1}[x_i, \varphi_i]$ (twisted polynomial ring), with $\varphi_i^{\ell_i}$ an inner automorphism of Q_{i-1}. In order to invoke Prop. 5.4, let

$$T \;=\; Q_{i-1}[x_i] \;\cong\; Q_{i-1}[x_i, \varphi_i] \quad \text{and let} \quad R = Z(T).$$

Since $S_{i-1}[x_i] \subseteq T \subseteq Q_i = q(S_{i-1}[x_i])$, we have $q(T) = Q_i$. Let $\mathcal{G} \subseteq \mathrm{Aut}(Q_i)$ be the subgroup of automorphisms of Q_i generated by $\varphi_{i+1}, \ldots, \varphi_n$, and let $G = \mathcal{G}/(\mathcal{G} \cap \mathrm{Inn}(Q_i))$, where $\mathrm{Inn}(Q_i)$ is the group of inner automorphisms of Q_i. Since Skolem-Noether shows that $\mathrm{Inn}(Q_i)$ is the kernel of the restriction map $\mathrm{Aut}(Q_i) \to \mathrm{Aut}(Z(Q_i))$, this G maps injectively into $\mathrm{Aut}(Z(Q_i))$. For $\sigma \in G$, we write $\sigma|_{Z(Q_i)}$ for the automorphism of $Z(Q_i)$ determined by σ. Note that G is a finite abelian group, since the images of the φ_i have finite order in G and commute pairwise. (For, we have $x_j x_k = c_{jk} x_k x_j$ for some $c_{jk} \in E_0^*$. Hence $\varphi_j\varphi_k = \mathrm{int}(c_{jk})\varphi_k\varphi_j$ and $\mathrm{int}(c_{jk}) \in \mathrm{Inn}(Q_i)$, as $c_{jk} \in E_0^* \subseteq Q_i^*$). Every element of $\mathcal{G}$ is an automorphism of $S_{i-1}[x_i]$ preseerving degree in x_i, so an automorphism of T, since this is true of each φ_j. Therefore we have a group action of $\mathcal{G}$ on T by ring automorphisms, and an induced action of $\mathcal{G}$ on $\mathrm{Div}(T)$. Note that as any $\psi \in \mathcal{G}$ permutes the maximal left ideals of T, the action of ψ on $\mathrm{Div}(T)$ arises from an action on the base of $\mathrm{Div}(T)$ consisting of isomorphism classes of simple T-modules. That is, $\mathrm{Div}(T)$ is a permutation

$\mathcal{G}$-module. $\mathcal{G}$ also acts on $R = Z(T)$ by ring automorphisms, and on $\mathrm{Div}(R)$, and all the maps in the commutative diagram below (see Prop. 5.4) are $\mathcal{G}$-module homomorphisms.

$$\begin{tikzcd} Q_i^* \arrow[r, "\delta_T"] \arrow[d, "\mathrm{Nrd}_{Q_i}"'] & \mathrm{Div}(T) \arrow[d, "\mathrm{Nrd}"] \\ Z(Q_i)^* \arrow[r, "\delta_R"] & \mathrm{Div}(R) \end{tikzcd} \tag{5.7}$$

Since inner automorphisms of Q_i act trivially on $\mathrm{Div}(T)$ (see Remark 5.1(iii)), and on $Z(Q_i)$ and $\mathrm{Div}(R)$, these $\mathcal{G}$-modules are actually G-modules. Let

$$\mathfrak{N} \;=\; \mathrm{Nrd}(\mathrm{Div}(T)) \;\subseteq\; \mathrm{Div}(R).$$

Because $\mathrm{Nrd}\colon \mathrm{Div}(T) \to \mathrm{Div}(R)$ is injective (see Prop. 5.4), $\mathfrak{N}$ is a G-module isomorphic to $\mathrm{Div}(T)$, so $\mathfrak{N}$ is a permutation G-module. Within $\mathfrak{N}$ we have two distinguished $\mathcal{G}$-submodules,

$$\begin{aligned} \mathfrak{N}_0 &= \ker(N_G), \text{where } N_G\colon \mathfrak{N} \to \mathfrak{N} \text{ is the norm, given by } N_G(b) = \textstyle\sum_{\sigma\in G} \sigma(b); \text{ and} \\ I_G(\mathfrak{N}) &= \big\langle \{\beta - \sigma(\beta) \mid \beta \in \mathfrak{N},\, \sigma \in G\}\big\rangle \subseteq \mathfrak{N}_0. \end{aligned}$$

By definition, $\widehat{H}^{-1}(G, \mathfrak{N}) = \mathfrak{N}_0/I_G(\mathfrak{N})$. But, because $\mathfrak{N}$ is a permutation G-module, $\widehat{H}^{-1}(G, \mathfrak{N}) = 0$. (This is well known, and is an easy calculation, as $\mathfrak{N}$ is a direct sum of G-modules of the form $\mathbb{Z}[G/H]$ for subgroups H of G.) That is, $\mathfrak{N}_0 = I_G(\mathfrak{N})$.

Take any generator $\beta - \sigma(\beta)$ of $I_G(\mathfrak{N})$, where $\sigma \in G$ and $\beta \in \mathfrak{N}$, say $\beta = \mathrm{Nrd}(\eta)$, where $\eta \in \mathrm{Div}(T)$. Take any $b \in Q_i^*$ with $\delta_T(b) = \eta$, and choose $u \in E^*$ which is some product of the φ_j $(i+1 \le j \le n)$, such that $\mathrm{int}(u)|_{Z(Q_i)} = \sigma|_{Z(Q_i)}$. Then, $\delta_R(\mathrm{Nrd}_{Q_i}(b)) = \mathrm{Nrd}(\delta_T(b)) = \beta$ (see (5.7)). Also, because $\mathrm{int}(u)|_{Q_i}$ is an automorphism of Q_i, we have $\mathrm{Nrd}_{Q_i}(ub^{-1}u^{-1}) = u\,\mathrm{Nrd}_{Q_i}(b^{-1})u^{-1}$. Thus, $bub^{-1}u^{-1} \in [Q_i^*, Q^*] \cap Q_i$ and

$$\begin{aligned} \mathrm{Nrd}_{Q_i}(bub^{-1}u^{-1}) &= \mathrm{Nrd}_{Q_i}(b)\,\mathrm{Nrd}_{Q_i}(ub^{-1}u^{-1}) \;=\; \mathrm{Nrd}_{Q_i}(b)\,u\,\mathrm{Nrd}_{Q_i}(b^{-1})u^{-1} \\ &= \mathrm{Nrd}_{Q_i}(b)/\sigma(\mathrm{Nrd}_{Q_i}(b)). \end{aligned}$$

Hence, in $\mathrm{Div}(R)$,

$$\delta_R\big(\mathrm{Nrd}_{Q_i}(bub^{-1}u^{-1})\big) \;=\; \delta_R\big(\mathrm{Nrd}_{Q_i}(b)/\sigma \mathrm{Nrd}_{Q_i}(b)\big) \;=\; \beta - \sigma(\beta).$$

Since such $\beta - \sigma(\beta)$ generate $I_G(\mathfrak{N})$, it follows that for any $\gamma \in I_G(\mathfrak{N})$, there is $c \in [Q_i^*, Q^*] \cap Q_i$, with $\gamma = \delta_R(\mathrm{Nrd}_{Q_i}(c)) = \mathrm{Nrd}(\delta_T(c))$ (see(5.7)).

To prove (5.6), we need a formula for Nrd_Q for an element of Q_i. For this, note that $E = S_i[x_{i+1}, x_{i+1}^{-1}, \ldots, x_n, x_n^{-1}]$ which can be considered a graded ring over S_i. Now, let $C = Q_i[x_{i+1}, x_{i+1}^{-1}, \ldots, x_n, x_n^{-1}] \subseteq Q$. This C is a graded division ring with $C_0 = Q_i$ and $\Gamma_C = \mathbb{Z}\varepsilon_{i+1} \oplus \ldots \oplus \mathbb{Z}\varepsilon_n$. Since $E \subseteq C \subseteq Q = q(E)$, we have $q(C) = Q$. For the graded field $Z(C)$ we have $Z(C)_0$ consists of those elements of $Z(C_0) = Z(Q_i)$ centralized by $x_{i+1}, \ldots, x_n$, i.e., $Z(C)_0$ is the fixed field $Z(Q_i)^{\mathcal{G}} = Z(Q_i)^G$. Since, as noted earlier, G injects into $\mathrm{Aut}(Z(Q_i))$, we have $G \cong \mathcal{G}al(Z(Q_i)/Z(C)_0)$. Thus, for any $q \in Q_i = C_0$, by Prop. 3.2(i) and (iv),

$$\begin{aligned} \mathrm{Nrd}_Q(q) &= \mathrm{Nrd}_{q(C)}(q) \;=\; \mathrm{Nrd}_C(q) \;=\; N_{Z(C_0)/Z(C_0)^G}(\mathrm{Nrd}_{C_0}(q))^m \\ &= N_{Z(Q_i)/Z(Q_i)^G}(\mathrm{Nrd}_{Q_i}(q))^m, \end{aligned}$$

where $m = \mathrm{ind}(Q)/\,\mathrm{ind}(Q_i)[Z(Q_i) : Z(Q_i)^G]$.

To verify (5.6), take any $a \in Q_i^* \cap Q^{(1)}$. Thus,

$$1 \;=\; \mathrm{Nrd}_Q(a) \;=\; N_{Z(Q_i)/Z(Q_i)^G}(\mathrm{Nrd}_{Q_i}(a))^m.$$

Hence, for $\alpha = \delta_T(a) \in \operatorname{Div}(T)$, using the identification of G with $\mathcal{G}al(Z(Q_i)/Z(C)_0)$ and the commutative diagram (5.7),

$$\begin{aligned} 0 \;&=\; \delta_R(\operatorname{Nrd}_Q(a)) \;=\; \delta_R\big(N_{Z(Q_i)/Z(Q_i)^G}(\operatorname{Nrd}_{Q_i}(a))^m\big) \;=\; \sum_{\sigma\in G} \sigma\big(\delta_R(\operatorname{Nrd}_{Q_i}(a)^m)\big) \\ &=\; N_G\big(\delta_R(\operatorname{Nrd}_{Q_i}(a))^m\big) \;=\; m\, N_G(\operatorname{Nrd}(\delta_T(a))) \;=\; m\, N_G(\operatorname{Nrd}(\alpha)). \end{aligned}$$

Since $\operatorname{Div}(R)$ is torsion-free, we have $N_G(\operatorname{Nrd}(\alpha)) = 0$, i.e., $\operatorname{Nrd}(\alpha) \in \ker(N_G) = \mathfrak{N}_0 = I_G(\mathfrak{N})$. Therefore, as we saw above, there is $c \in [Q_i^*, Q^*] \cap Q_i^*$ with $\operatorname{Nrd}(\alpha) = \operatorname{Nrd}(\delta_T(c))$. Let $a' = a/c \in Q_i^*$. Then,

$$\operatorname{Nrd}(\delta_T(a')) \;=\; \operatorname{Nrd}(\delta_T(a)) - \operatorname{Nrd}(\delta_T(c)) \;=\; \operatorname{Nrd}(\alpha) - \operatorname{Nrd}(\alpha) \;=\; 0.$$

Because $\operatorname{Nrd}\colon \operatorname{Div}(T) \to \operatorname{Div}(R)$ is injective (see Prop. 5.4), it follows that $\delta_T(a') = 0$ in $\operatorname{Div}(T)$. Therefore, as $T = Q_{i-1}[x,\varphi_i]$ and $q(T) = Q_i$, by Prop. 5.3 there is $a'' \in Q_{i-1}$ with $a'' \equiv a' \pmod{Q_i'}$. So, $a'' \equiv a \pmod{[Q_i^*, Q^*]}$, and hence $\operatorname{Nrd}_Q(a'') = \operatorname{Nrd}_Q(a) = 1$, i.e., $a'' \in Q_{i-1}^* \cap Q^{(1)}$. Thus, $a \in (Q_{i-1}^* \cap Q^{(1)})[Q_i^*, Q^*]$, proving (5.6).

The inclusion (5.6) shows that for any i, $1 \le i \le n$ and any $a \in Q^{(1)} \cap Q_i$ there is $b \in Q^{(1)} \cap Q_{i-1}$ with $b \equiv a \pmod{Q'}$. Hence, by downward induction on i, for any $q \in Q^{(1)} = Q^{(1)} \cap Q_n$ there is $d \in Q_0 \cap Q^{(1)} = E_0 \cap Q^{(1)}$ with $d \equiv q$ mod $Q')$. So, $Q^{(1)} \subseteq (Q^{(1)} \cap E_0)Q'$. The reverse inclusion is clear, completing the proof of Case I.

Case II. *Suppose Γ_E is not a finitely generated abelian group.*

The basic point is that E is a direct limit of sub-graded division algebras with finitely generated grade group, so we can reduce to Case I. But we need to be careful about the choice of the sub-division algebras to assure that they have the same index as E, so that the reduced norms are compatible.

Let $F = Z(E)$. Since $|\Gamma_E/\Gamma_F| < \infty$, there is a finite subset, say $\{\gamma_1, \ldots, \gamma_k\}$ of Γ_E whose images in Γ_E/Γ_F generate this group. Let Δ_0 be any finitely generated subgroup of Γ_E, and let Δ be the subgroup of Γ_E generated by Δ_0 and $\gamma_1, \ldots, \gamma_k$. Then, Δ is also a finitely generated subgroup of Γ_E, but with the added property that $\Delta + \Gamma_F = \Gamma_E$. Let

$$E_\Delta \;=\; \bigoplus_{\delta\in\Delta} E_\delta,$$

which is a graded sub-division ring of E, with $E_{\Delta,0} = E_0$ and $\Gamma_{E_\Delta} = \Delta$. Since $\Delta + \Gamma_F = \Gamma_E$, we have $E_\Delta F = E$. (For, take any $\gamma \in \Gamma_E$ and write $\gamma = \delta + \eta$ with $\delta \in \Delta$ and $\eta \in \Gamma_F$, and any nonzero $d \in E_{\Delta,\delta}$ and $c \in F_\eta$. Then, $E_\gamma = dcE_0 \subseteq E_\Delta F$.) Because $E_\Delta F = E$, we have $Z(E_\Delta) = F \cap E_\Delta = F_{\Delta\cap\Gamma_F}$. Note that

$$\begin{aligned} [E_\Delta : Z(E_\Delta)] \;&=\; [E_{\Delta,0} : F_{\Delta\cap\Gamma_F,0}]\,|\Gamma_\Delta : (\Gamma_\Delta \cap \Gamma_F)| \;=\; [E_0 : F_0]\,|(\Gamma_\Delta + \Gamma_F) : \Gamma_F| \\ &=\; [E_0 : F_0]\,|\Gamma_E : \Gamma_F| \;=\; [E : F]. \end{aligned}$$

The graded homomorphism $E_\Delta \otimes_{Z(E_\Delta)} F \to E$ is onto as $E_\Delta F = E$, and is then also injective by dimension count (or by the graded simplicity of $E_\Delta \otimes_{Z(E_\Delta)} F$). Thus, $E_\Delta \otimes_{Z(E_\Delta)} F \cong E$. It follows that $q(E_\Delta) \otimes_{q(Z(E_\Delta))} q(F) \cong q(E)$. Specifically,

$$\begin{aligned} q(E_\Delta) \otimes_{q(Z(E_\Delta))} q(F) \;&\cong\; (E_\Delta \otimes_{Z(E_\Delta)} q(Z(E_\Delta))) \otimes_{q(Z(E_\Delta))} q(F) \;\cong\; E_\Delta \otimes_{Z(E_\Delta)} q(F) \\ &\cong\; (E_\Delta \otimes_{Z(E_\Delta)} F) \otimes_F q(F) \;\cong\; E \otimes_F q(F) \;\cong\; q(E). \end{aligned}$$

Therefore, for any $a \in q(E_\Delta)$, $\operatorname{Nrd}_{q(E_\Delta)}(a) = \operatorname{Nrd}_{q(E)}(a)$.

Now, if we take any $a \in Q^{(1)}$ where $Q = q(E)$, there is a subgroup $\Delta \subseteq \Gamma_E$ with Δ finitely generated and $\Delta + \Gamma_F = \Gamma_E$ and $a \in E_\Delta$. Since $\operatorname{Nrd}_{q(E_\Delta)}(a) = \operatorname{Nrd}_Q(a) = 1$, we have, by Case I applied to E_Δ, $a \in \big(q(E_\Delta)^{(1)} \cap E_0\big)q(E_\Delta)' \subseteq (Q^{(1)} \cap E_0)Q'$, completing the proof for Case II. □

Remark. (i) Prop. 5.6 for those E with $\Gamma_E \cong \mathbb{Z}$ was proved in [PY], and our proof of this is essentially the same as theirs, expressed in a somewhat different language. Platonov and Yanchevskiĭ also in effect assert Prop. 5.6 for E with Γ_E finitely generated, expressed as a result for iterated quotient division rings of twisted polynomial rings. (See [PY], Lemma 8.) By way of proof of [PY], Lemma 8, the authors say nothing more than that it follows by induction from the rank 1 case. It is not clear whether the proof given here coincides with their unstated proof, since the transition from rank 1 to finite rank is not transparent.

(ii) So far the functor CK_1 has manifested properties similar to SK_1. However, the similarity does not hold here, since the functor CK_1 is not (homotopy) stable. In fact, for a division algebra D over its center F of index n, one has the following split exact sequence,

$$1 \to \mathrm{CK}_1(D) \to \mathrm{CK}_1(D(x)) \to \bigoplus_p \mathbb{Z}/(n/n_p)\mathbb{Z} \to 1$$

where p runs over irreducible monic polynomials of $F[x]$ and n_p is the index of central simple algebra $D \otimes_F \big(F[x]/(p)\big)$ (see Th. 2.10 in [H$_1$]). This is provable by mapping the exact sequence (5.1) with $T = F[x]$ to the sequence for $T = D[x]$ and taking cokernels.

Example 5.8. Let E be a semiramified graded division ring with $\Gamma_E \cong \mathbb{Z}^n$, and let $T = Z(E)$. Since Γ_E/Γ_T is a torsion group, there are a base $\{\gamma_1, \ldots, \gamma_n\}$ of the free abelian group Γ_E and some $r_1, \ldots, r_n \in \mathbb{N}$ such that $\{r_1\gamma_1, \ldots, r_n\gamma_n\}$ is a base of Γ_T. Choose any nonzero $z_i \in E_{\gamma_i}$ and $x_i \in T_{r_i\gamma_i}$, $1 \le i \le n$. Let $F = T_0$ and $M = E_0$, and let $G = \mathcal{G}al(M/F)$. Because E is semiramified, M is Galois over F with $[M : F] = |\Gamma_E : \Gamma_T| = \mathrm{ind}(E) = r_1 \ldots r_n$, and $G \cong \Gamma_E/\Gamma_T$. Since $z_i^{r_i} \in E_{r_i\gamma_i} = E_0 x_i$, there is $b_i \in M$ with $z_i^{r_i} = b_i x_i$. Let $u_{ij} = z_i z_j z_i^{-1} z_j^{-1} \in M$. Let $\sigma_i \in G$ be the automorphism of M determined by conjugation by z_i. From the isomorphism $G \cong \Gamma_E/\Gamma_T$, each σ_i has order r_i in G and $G \cong \langle \sigma_1 \rangle \times \ldots \times \langle \sigma_n \rangle$. Clearly, $T = F[x_1, x_1^{-1}, \ldots x_n, x_n^{-1}]$, an iterated Laurent polynomial ring, and $E = M[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]$, an iterated twisted Laurent polynomial ring whose multiplication is completely determined by the $b_i \in M$, the $u_{ij} \in M$, and the action of the σ_i on M.

Let $D = q(E)$, which is a division ring with center $q(T) = F(x_1, \ldots, x_n)$, a rational function field over F. Then, D is the *generic abelian crossed product* determined by M/F, the base $\{\sigma_1, \ldots, \sigma_n\}$ of G, the b_i and the u_{ij}, as defined in [AS]. As was pointed out in [BM], all generic abelian crossed products arise this way as rings of quotients of semiramified graded division algebras. Generic abelian crossed products were used in [AS] to give the first examples of noncyclic p-algebras, and in [S$_1$] to prove the existence of noncrossed product p-algebras. It is known by [T], Prop. 2.1 that D is determined up to F-isomorphism by M and the u_{ij}. By Cor. 3.6(iii) and Th. 5.7, there is an exact sequence

$$G \wedge G \;\to\; \widehat{H}^{-1}(G, M^*) \;\to\; \mathrm{SK}_1(D) \;\to\; 1, \tag{5.8}$$

where the left map is determined by sending $\sigma_i \wedge \sigma_j$ to u_{ij} mod $I_G(M^*)$. An important condition introduced by Amitsur and Saltman in [AS] was *nondegeneracy* of $\{u_{ij}\}$. This condition was essential for the noncyclicity results in [AS], and is also key to the results on noncyclicity and indecomposability of generic abelian crossed products in recent work of McKinnie in [Mc$_1$], [Mc$_2$] and Mounirh [M$_2$]. The original definition of nondegeneracy in [AS] was somewhat mysterious. A cogent characterization was given recently in [Mc$_3$], Lemma 5.1: A family $\{u_{ij}\}$ in M^* (meeting the conditions to appear in a generic abelian crossed product) is nondegenerate iff for every rank 2 subgroup H of G, the map $H \wedge H \to \widehat{H}^{-1}(H, M^*)$ appearing in the complex (5.8) for the generic abelian crossed product $C_D(M^H)$ is nonzero. In the first nontrivial case, where $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$ with p a prime number, we have $\{u_{ij}\}$ is nondegenerate iff the map $G \wedge G \to \widehat{H}^{-1}(G, M^*)$ is nonzero, iff the epimorphism $\widehat{H}^{-1}(G, M^*) \to \mathrm{SK}_1(D)$ is not injective. Thus, the nondegeneracy is encoded in $\mathrm{SK}_1(D)$, and it occurs just when $\mathrm{SK}_1(D)$ is not "as large as possible."

Appendix A. The Wedderburn factorization theorem

Let D be a noncommutative division ring with center F, and let $a \in D$ with minimal polynomial f in $F[x]$. Any conjugate of a is also a root of this polynomial. Since the number of conjugates of a is infinite ([L], 13.26), this suggests that f might split completely in $D[x]$. In fact, this is the case, and it is called the Wedderburn factorization theorem. We now carry over this theorem to the setting of graded division algebra. (This is used in proving Th. 3.3).

Theorem A.1 (Wedderburn Factorization Theorem)**.** *Let E be a graded division ring with center T (with Γ_E torsion-free abelian). Let a be a homogenous element of E which is algebraic over T with minimal polynomial $h_a \in T[x]$. Then, h_a splits completely in E. Furthermore, there exist n conjugates $a_1, \ldots, a_n$ of a such that $h_a = (x - a_n)(x - a_{n-1}) \ldots (x - a_1)$ in $E[x]$.*

Proof. The proof is similar to Wedderburn's original proof for a division ring ([We], see also [L] for a nice account of the proof). We sketch the proof for the convenience of the reader. For $f = \sum c_i x^i \in E[x]$ and $a \in E$, our convention is that $f(a)$ means $\sum c_i a^i$. Since Γ_E is torsion-free, we have $E^* = E^h \setminus \{0\}$.

I: Let $f \in E[x]$ with factorization $f = gk$ in $E[x]$. If $a \in E$ satisfies $k(a) \in T \cdot E^*$, then $f(a) = g(a')k(a)$, for some conjugate a' of a. (Here E could be any ring with $T \subseteq Z(E)$.)

Proof. Let $g = \sum b_i x^i$. Then, $f = \sum b_i k x^i$, so $f(a) = \sum b_i k(a) a^i$. But, $k(a) = te$, where $t \in T$ and $e \in E^*$. Thus, $f(a) = \sum b_i t e a^i = \sum b_i e a^i e^{-1} t e = \sum b_i (eae^{-1})^i te = g(eae^{-1})k(a)$. □

II: Let $f \in E[x]$ be a non-zero polynomial. Then $r \in E$ is a root of f if and only if $x - r$ is a right divisor of f in $E[x]$. (Here, E could be any ring.)

Proof. We have $x^i - r^i = (x^{i-1} + x^{i-2}r + \ldots + r^{i-1})(x - r)$ for any $i \geq 1$. Hence,

$$f - f(r) = g \cdot (x - r) \tag{A.1}$$

for some $g \in E[x]$. So, if $f(r) = 0$, then $f = g \cdot (x - r)$. Conversely, if $x - r$ is a right divisor of f, then equation (A.1) shows that $x - r$ is a right divisor of the constant $f(r)$. Since $x - r$ is monic, this implies that $f(r) = 0$. □

III: If a non-zero monic polynomial $f \in E[x]$ vanishes identically on the conjugacy class A of a (i.e., $f(b) = 0$ for all $b \in A$), then $\deg(f) \geq \deg(h_a)$.

Proof. Consider $f = x^m + d_1 x^{m-1} + \ldots + d_m \in E[x]$ such that $f(A) = 0$ and $m < \deg(h_a)$ with m as small as possible. Suppose $a \in E_\gamma$, so $A \subseteq E_\gamma$, as the units of E are all homogeneous. Since the $E_{m\gamma}$-component of $f(b)$ is 0 for each $b \in A$, we may assume that each $d_i \in E_{i\gamma}$. Because $f \notin T[x]$, some $d_i \notin T$. Choose j minimal with $d_j \notin T$, and some $e \in E^*$ such that $ed_j \neq d_j e$. For any $c \in E$, write $c' := ece^{-1}$. Thus $d_j' \neq d_j$ but $d_\ell' = d_\ell$ for $\ell < j$. Let $f' = x^m + d_1' x^{m-1} + \ldots + d_m' \in E[x]$. Now, for all $b \in A$, we have $f'(b') = [f(b)]' = 0' = 0$. Since $eAe^{-1} = A$, this shows that $f'(A) = 0$. Let $g = f - f'$, which has degree $j < m$ with leading coefficient $d_j - d_j'$. Then, $g(A) = 0$. But, $d_j - d_j' \in E_{j\gamma} \setminus \{0\} \subseteq E^*$. Thus, $(d_j - d_j')^{-1} g$ is monic of degree $j < m$ in $E[x]$, and it vanishes on A. This contradicts the choice of f; hence, $m \geq \deg(h_a)$. □

We now prove the theorem. Since $h_a(a) = 0$, by (II), $h_a \in E[x] \cdot (x - a)$. Take a factorization

$$h_a = g \cdot (x - a_r) \ldots (x - a_1),$$

where $g \in E[x]$, $a_1, \ldots, a_r \in A$ and r is as large as possible. Let $k = (x - a_r) \ldots (x - a_1) \in E[x]$. We claim that $k(A) = 0$, where A is the conjugacy class of a. For, suppose there exists $b \in A$ such that $k(b) \neq 0$. Since $k(b)$ is homogenous, we have $k(b) \in E^*$. But, $h_a = gk$, and $h_a(b) = 0$, as $b \in A$; hence, (I) implies that $g(b') = 0$ for some conjugate b' of b. We can then write $g = g_1 \cdot (x - b')$, by (II). So h_a has a right factor $(x - b')k = (x - b')(x - a_r) \ldots (x - a_1)$, contradicting our choice of r. Thus $k(A) = 0$, and using (III), we have $r \geq \deg(h_a)$, which says that $h_a = (x - a_r) \ldots (x - a_1)$. □

Remark (Dickson Theorem)*.* One can also see that, with the same assumptions as in Th. A.1, if $a, b \in E$ have the same minimal polynomial $h \in T[x]$, then a and b are conjugates. For, $h = (x - b)k$ where $k \in T[b][x]$. But then by (III), there exists a conjugate of a, say a', such that $k(a') \neq 0$. Since $h(a') = 0$, by (I) some conjugate of a' is a root of $x - b$. (This is also deducible using the graded version of the Skolem-Noether theorem, see [HwW$_2$], Prop. 1.6.)

Appendix B. The Congruence theorem for tame division algebras

For a valued division algebra D, the congruence theorem provides a bridge for relating the reduced Whitehead group of D to the reduced Whitehead group of its residue division algebra. This was used by Platonov [P$_1$] to produce non-trivial examples of $\mathrm{SK}_1(D)$, by carefully choosing D with a suitable residue division algebra. Keeping the notations of Section 2, Platonov's congruence theorem states that for a division algebra D with a complete discrete valuation of rank 1, such that $Z(\overline{D})$ is separable over $\overline{F}$, $(1 + M_D) \cap D^{(1)} \subseteq D'$. This crucial theorem was established with a lengthy and rather complicated proof in [P$_1$]. In [E], Ershov states that the "same" proof will go through for tame valued division algebras over henselian fields. However, this seems highly problematical, as Platonov's original proof used properties of maximal orders over discrete valuation rings which have no satisfactory analogues for more general valuation rings. For the case of strongly tame division algebras, i.e., $\mathrm{char}(\overline{F}) \nmid [D : F]$, a short proof of the congruence theorem was given in [H$_2$] and another (in the case of discrete rank 1 valuations) in [Sus]. In this appendix, we provide a complete proof for the general situation of a tame valued division algebra.

Theorem B.1 (Congruence Theorem)**.** *Let F be a field with a henselian valuation v, and let D be a tame F-central division algebra. Then $(1 + M_D) \cap D^{(1)} \subseteq D'$.*

Tameness is meant here as described in §2, which is the weaker sense used in [JW] and [E]. Among the several characterizations of tameness mentioned in §2, the ones we use here are that D is tame if and only if D is split by the maximal tamely ramified extension of F, if and only if $\mathrm{char}(\overline{F}) = 0$ or $\mathrm{char}(\overline{F}) = \overline{p} \neq 0$ and the $\overline{p}$- primary component of D is inertially split, i.e., split by the maximal unramified extension of F.

The proof of the theorem will use the following well-known lemma:

Lemma B.2. *Let D be a division ring with center F and let L be a field extension of F with $[L : F] = \ell$. If $a \in D$ and $a \otimes 1 \in (D \otimes_F L)'$, then $a^\ell \in D'$.*

Proof. The regular representation $L \to M_\ell(F)$ yields a ring monomorphism $D \otimes_F L \to M_\ell(D)$. Therefore, we have a composition of group homomorphisms

$$(D \otimes_F L)^* \to \mathrm{GL}_\ell(D) \to D^*/D', \quad a \mapsto \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a \end{pmatrix}_{\ell \times \ell} \mapsto a^\ell D',$$

where the second map is the Dieudonné determinant. (See [D], §20 for properties of the Dieudonné determinant.) The lemma follows at once, since the image of the composition is abelian, so its kernel contains $(D \otimes_F L)'$. ☐

Note that in the preceding lemma, there is no valuation present, and D could be of infinite dimension over F.

Proof of Theorem B.1. The proof is carried out in four steps.

Step 1. The theorem is true if D is strongly tame over F, i.e., $\mathrm{char}(\overline{F}) \nmid [D : F]$. This has a short proof given in [H$_2$] and another (in the case of discrete valuation of rank 1) in [Sus], Lemma 1.6. For the convenience of the reader, we recall the argument from [H$_2$]:

Let $n = \operatorname{ind}(D)$, so $\operatorname{char}(\overline{F}) \nmid n$. Take any $s \in D^*$, and let $f = x^k + c_{k-1}x^{k-1} + \ldots + c_0 \in F[x]$ be the minimal polynomial of s over F. By applying the Wedderburn factorization theorem to f (see [L], (16.9), pp. 251–252, or Appendix A above), we see that there exist $d_1, \ldots, d_k \in D^*$ with $(-1)^k c_0 = (d_1 s d_1^{-1}) \ldots (d_k s d_k^{-1})$. Hence, as D^*/D' is abelian,

$$\operatorname{Nrd}_D(s) \;=\; [(-1)^k c_0]^{n/k} \;\equiv\; \big[s^k(d_1 s d_1^{-1} s^{-1}) \ldots (d_k s d_k^{-1} s^{-1})\big]^{n/k} \;\equiv\; s^n \pmod{D'}. \tag{B.1}$$

Now, take any $a \in 1 + M_D$ with $\operatorname{Nrd}_D(a) = 1$. Since $\operatorname{char}(\overline{F}) \nmid n$, Hensel's Lemma applied over $F(a)$ shows that there is $s \in 1 + M_{F(a)} \subseteq 1 + M_D$ with $s^n = a$. Then, $\operatorname{Nrd}_D(s) = 1 + m \in 1 + M_F$ by Cor. 4.7. But,

$$(1+m)^n \;=\; \operatorname{Nrd}_D(s^n) \;=\; \operatorname{Nrd}_D(a) \;=\; 1.$$

If $m \neq 0$, then we have $1 = (1+m)^n = 1 + n\,m + r$ with $v(r) \geq 2v(m)$, which would imply that $v(n\,m) = v(r) > v(m)$. This cannot occur since $\operatorname{char}(\overline{F}) \nmid n$; hence, $m = 0$. Thus, by (B.1)

$$a \;=\; s^n \;\equiv\; \operatorname{Nrd}_D(s) \;=\; 1 + m \;=\; 1 \pmod{D'},$$

i.e., $a \in D'$. This completes Step 1.

Step 2. We prove the theorem if D is inertially split of prime power degree over F. This is a direct adaptation of Platonov's argument in [P_1] for discrete (rank 1) valuations. (When v is discrete, every tame division algebra is inertially split.)

Suppose $\operatorname{ind}(D) = p^k$, p prime and D is inertially split. Then, D has a maximal subfield K which is unramified over F (cf. [JW], Lemma 5.1, or [W_2], Th. 3.4) Take any $a \in (1 + M_D) \cap D^{(1)}$. We first push a into K. Since $\overline{K}$ is separable over $\overline{F}$, there is $y \in \overline{K}$ with $\overline{K} = \overline{F}(y)$. Choose any $z \in V_K$ with $\overline{z} = y$. So $K = F(z)$, by dimension count, as $\overline{F(z)} \supseteq \overline{F}(y)$. Note that $\overline{az} = \overline{z}$ in $\overline{D}$. If f is the minimal polynomial of az over F, then $f \in V_F[x]$ as $az \in V_D$, and $\overline{z} = \overline{az}$ is a root of the image $\overline{f}$ of f in $\overline{F}[x]$. We have $\deg(\overline{f}) = \deg(f) = [F(az) : F] \leq [K : F] = [\overline{F}(\overline{z}) : \overline{F}]$. Hence, $\overline{f}$ is the minimal polynomial of $\overline{z}$ over $\overline{F}$, so $\overline{z}$ is a simple root of $\overline{f}$. By Hensel's lemma applied over K, K contains a root b of f with $\overline{b} = \overline{z}$. Since b and az have the same minimal polynomial f over F, by Skolem–Noether there is $t \in D^*$ with $b = tazt^{-1}$. So $az = t^{-1}bt$. Then,

$$a \;=\; t^{-1}btz^{-1} \;=\; (t^{-1}btb^{-1})(bz^{-1}).$$

We have $bz^{-1} \in K$, as $b, z \in K$, and $bz^{-1} \equiv a \pmod{D'}$; so, $\operatorname{Nrd}_D(bz^{-1}) = \operatorname{Nrd}_D(a) = 1$, and $bz^{-1} \in 1 + M_D$, as $\overline{b} = \overline{z}$. Therefore, we may replace a by bz^{-1}, so we may assume $a \in K$.

Let N be the normal closure of K over F, and let $G = \mathcal{G}al(N/F)$. Since K is unramified over F and the maximal unramified extension F_{nr} of F is Galois over F (cf. [EP], Th. 5.2.7, Th. 5.2.9, pp. 124–126), $N \subseteq F_{\mathrm{nr}}$; so N is also unramified over F. Let P be a p-Sylow subgroup of G and let $L = N^P$, the fixed field of P. Thus, $[L : F] = |G : P|$, which is prime to p, and N is Galois over L with $\mathcal{G}al(N/L) = P$. Since $\gcd\big([L : F], \operatorname{ind}(D)\big) = 1$, $D_1 = L \otimes_F D$ is a division ring and $K_1 = L \otimes_F K$ is a field with $K_1 \cong L \cdot K \subseteq N$. So, K_1 is unramified over F and hence over L. We have $\operatorname{Nrd}_{D_1}(1 \otimes a) = \operatorname{Nrd}_D(a) = 1$ and $1 \otimes a \in 1 + M_D$, so if we knew the result for D_1, we would have $1 \otimes a \in D_1'$. But then by Lemma B.2, $a^{[L:F]} \in D'$. But we also have $a^{\operatorname{ind}(D)} \in D'$, since $\mathrm{SK}_1(D)$ is $\operatorname{ind}(D)$-torsion (by [D], p. 157, Lemma 2 or Lemma B.2 above with L a maximal subfield of D). Since $\gcd\big([L : F], \operatorname{ind}(D)\big) = 1$, it would follow that $a \in D'$, as desired. Thus, it suffices to prove the result for D_1.

To simplify notation, replace D_1 by D, K_1 by K, $1 \otimes a$ by a, and L by F. Because $F \subseteq K \subseteq N$ with N Galois over F, any subfield T of K minimal over F corresponds to a maximal subgroup of $\mathcal{G}al(N/F)$ containing $\mathcal{G}al(N/K)$. Since $[N : F]$ is a power of p, by p-group theory such a maximal subgroup is normal in $\mathcal{G}al(N/F)$ and of index p. Thus, T is Galois over F and $[T : F] = p$. So $\mathcal{G}al(T/F)$ is a cyclic group , say $\mathcal{G}al(T/F) = \langle \sigma \rangle$. Let $E = C_D(T)$, the centralizer of T in D; so $F \subseteq T \subseteq K \subseteq E \subseteq D$. Note that K is a maximal subfield of E, since it is a maximal subfield of D.

Let $c = N_{K/T}(a) = \operatorname{Nrd}_E(a)$. Because K is unramified over T and $a \in V_K$, we have $c \in V_T$ and $\overline{c} = N_{\overline{K}/\overline{T}}(\overline{a}) = N_{\overline{K}/\overline{T}}(\overline{1}) = \overline{1}$, so $c \in 1 + M_T$. We have,

$$N_{T/F}(c) \;=\; N_{T/F}(N_{K/T}(a)) \;=\; N_{K/F}(a) \;=\; \operatorname{Nrd}_D(a) \;=\; 1\,.$$

By Hilbert 90, $c = b/\sigma(b)$ for some $b \in T$. This equation still holds if we replace b in it by any F^*-multiple of b. Thus, as $\Gamma_T = \Gamma_F$ since T is unramified over F, we may assume that $v(b) = 0$. But further, since T is unramified and cyclic Galois over F, its residue field $\overline{T}$ is cyclic Galois of degree p over $\overline{F}$, with $\mathcal{G}al(\overline{T}/\overline{F}) = \langle\overline{\sigma}\rangle$ where $\overline{\sigma}$ is the automorphism of $\overline{T}$ induced by σ on T. In $\overline{T}$ we have $\overline{b}/\overline{\sigma}(\overline{b}) = \overline{b/\sigma(b)} = \overline{c} = \overline{1}$. Therefore, $\overline{b}$ lies in the fixed field of $\overline{\sigma}$ in $\overline{T}$, which is $\overline{F}$. Hence, there is $\eta \in V_F$ with $\overline{\eta} = \overline{b}$ in $\overline{T}$. By replacing b by $b\eta^{-1}$, we can assume $\overline{b} = \overline{1}$, i.e., $b \in 1 + M_T$.

Since K is unramified and hence tame over T, Prop. 4.6 shows $N_{K/T}(1 + M_K) = 1 + M_T$. So, there is $s \in 1 + M_K$ with $N_{K/F}(s) = b$. Now, by Skolem–Noether, there is an inner automorphism φ of D such that $\varphi(T) = T$ and $\varphi|_T = \sigma$. Since $E = C_D(T)$, we have φ is a (non-inner) automorphism of E, and $\varphi(K)$ is a maximal subfield of E (since K is a maximal subfield of E). We have $\operatorname{Nrd}_E(\varphi(s)) = N_{\varphi(K)/\varphi(T)}(\varphi(s)) = \varphi\big(N_{K/T}(s)\big) = \sigma(b)$. Thus,

$$\operatorname{Nrd}_E(s/\varphi(s)) \;=\; b/\sigma(b) \;=\; c.$$

Now, there is $u \in D^*$ with $\varphi(s) = usu^{-1}$. So, $\varphi(s) \in 1 + M_D$. Let $a' = a/(s/\varphi(s)) = a/(sus^{-1}u^{-1}) \in E$. So $a' \equiv a \pmod{D'}$. But further, $a' \in E \cap (1 + M_D) = 1 + M_E$ (as $a, s, \varphi(s) \in (1 + M_D) \cap E$). Also,

$$\operatorname{Nrd}_E(a') \;=\; \operatorname{Nrd}_E(a)\big/\operatorname{Nrd}_E(s/\varphi(s)) \;=\; N_{K/T}(a)/c \;=\; 1.$$

Since $[E : T] < [D : F]$ and E is inertially split over T (since it is split by its maximal subfield K which is unramified over T), by induction on index the theorem holds for T over E. Hence, $a' \in E'$. Since $a \equiv a' \pmod{D'}$, we thus have $a \in D'$, as desired. This completes the proof of Step 2.

Step 3. Suppose $D = P \otimes_F Q$, where $\gcd(\operatorname{ind}(P), \operatorname{ind}(Q)) = 1$, and suppose the theorem is true for $L \otimes_F Q$ and $K \otimes_F P$ for some maximal subfield L of P and K of Q. Then we show using Prop. B.3 below that the theorem is true for D.

Let $C = C_D(L)$. Then, $C = C_{P\otimes_F Q}(L \otimes_F F) \cong C_L(P) \otimes_F C_Q(F) = L \otimes_F Q$. Also,

$$L \otimes_F D \;\cong\; (L \otimes_F P) \otimes_L (L \otimes_F Q) \;\cong\; M_\ell(L) \otimes_L C \;\cong\; M_\ell(C),$$

where $\ell = [L : F] = \operatorname{ind}(P)$. Take any $a \in (1 + M_D) \cap D^{(1)}$. For $1 \otimes a \in L \otimes D = M_\ell(C)$, Prop. B.3 shows that there is $c \in 1 + M_C$ with $\operatorname{ddet}(a) \equiv c \pmod{C'}$, where ddet denotes the Dieudonné determinant. Then,

$$1 \;=\; \operatorname{Nrd}_D(a) \;=\; \operatorname{Nrd}_{M_\ell(C)}(1 \otimes a) \;=\; \operatorname{Nrd}_C(\operatorname{ddet}(1 \otimes a)) \;=\; \operatorname{Nrd}_C(c).$$

Hence, $c \in (1+M_C)\cap C^{(1)}$ which lies in C' by hypothesis as $C \cong L\otimes_F Q$. That is, $\operatorname{ddet}(1 \otimes a) = 1 \in C^*/C'$. Hence, $1 \otimes a \in \ker(\operatorname{ddet}) = (L \otimes_F D)'$. Therefore, by Lemma B.2, $a^\ell \in D'$. Likewise, by looking at $1 \otimes a \in K \otimes_F D$, we obtain $a^k \in D'$ where $k = [K : F] = \operatorname{ind}(Q)$. Since $\gcd(\ell, k) = 1$, it follows that $a \in D'$, completing Step 3.

Step 4. We now prove the theorem in full. Let F be a henselian field, and let D be a tame F-central division algebra. If $\operatorname{char}(\overline{F}) = 0$, then D is strongly tame over F, so the theorem holds for D by Step 1. If $\operatorname{char}(\overline{F}) = \overline{p} \neq 0$ we have $D \cong P \otimes_F Q$ where P is the $\overline{p}$-primary component of D and Q is the tensor product of all the other primary components of D. So, $\gcd(\operatorname{ind}(P), \operatorname{ind}(Q)) = 1$. For any maximal subfield L of P, $L \otimes_F Q$ is a division algebra tame over L with $\operatorname{ind}(L \otimes_F Q) = \operatorname{ind}(Q)$, which is prime to $\overline{p}$. So, $L \otimes_F Q$ is strongly tame over L, and the theorem holds for $L \otimes_F Q$ by Step 1. On the other hand, for any maximal subfield K of Q, we have $K \otimes_F P$ is tame over K and $\operatorname{ind}(K \otimes_F P) = \operatorname{ind}(P)$, which is a power of $\overline{p}$; hence, $K \otimes_F P$ is inertially split, as noted in §2. Hence, by Step 2 the theorem holds for $K \otimes_F P$. Thus, by Step 3 the theorem holds for D. $\square$

The following proposition will complete the proof of the Congruence Theorem.

Proposition B.3. *Let F be a henselian valued field, and let D be an F-central division algebra which is defectless over F. Let L be a field, $F \subseteq L \subseteq D$, and let $C = C_D(L)$, so $L \otimes_F D \cong M_\ell(C)$ where $\ell = [L : F]$. Take any $a \in 1 + M_D$. Then, for $1 \otimes a \in L \otimes_F D \cong M_\ell(C)$,*

$$\operatorname{ddet}(1 \otimes a) \in 1 + M_C \pmod{C'},$$

where ddet *denotes the Dieudonné determinant.*

Proof. D is an L-D bimodule via multiplication in D. Hence (as L is commutative) D is a right $L \otimes_F D$-module, with module action given by $a(\sum \ell_i \otimes d_i) = \sum \ell_i a d_i$. In fact, D is simple as a right $L \otimes_F D$-module, since it is already a simple right D-module. Hence, by Wedderburn's Theorem, $L \otimes_F D \cong \operatorname{End}_\Delta(D)$, where $\Delta = \operatorname{End}_{L \otimes_F D}(D)$ (acting on D on the left). Since (for D acting on D on the right) $\operatorname{End}_D(D) \cong D$ (elements of D acting on D by left multiplication) $\operatorname{End}_{L \otimes_F D}(D)$ consists of left multiplication by elements of D which commute with the left action of L on D, i.e., $\Delta \cong C_D(L) = C$. So,

$$L \otimes_F D \ \cong\ \operatorname{End}_\Delta(D) \ \cong\ \operatorname{End}_C(D) \ \cong\ M_\ell(C)\,,$$

where $\ell = [D : C] = [L : F]$. The last isomorphism is obtained by choosing a base $\{b_1, \ldots, b_\ell\}$ of D as a left C-vector space ($D = Cb_1 \oplus \ldots \oplus Cb_\ell$) and writing the matrix for an element of $L \otimes_F D$ acting C-linearly on D (on the right) relative to this base, with matrix entries in C.

Because D is defectless over F, D is also defectless over C, i.e., $[D : C] = [\operatorname{gr}(D) : \operatorname{gr}(C)]$; thus, the valuation w on D extending v on F is a $w|_C$-norm by [RTW], Cor. 2.3. This means that we can choose our base $\{b_1, \ldots, b_\ell\}$ to be a splitting base for w over $w|_C$, i.e., satisfying, for all $c_1, \ldots, c_\ell \in C$,

$$w\big(\sum_{i=1}^{\ell} c_i b_i\big) \ = \ \min_{1 \le i \le \ell} \big(w(c_i) + w(b_i)\big). \tag{B.2}$$

Let $\gamma_i = w(b_i)$ for $1 \le i \le \ell$.

Let

$$\begin{aligned} R \ &= \ \{A = (a_{ij}) \in M_\ell(C) : w(a_{ij}) \ge \gamma_i - \gamma_j \text{ for all } i, j\};\\ J \ &= \ \{A = (a_{ij}) \in M_\ell(C) : w(a_{ij}) > \gamma_i - \gamma_j \text{ for all } i, j\};\\ 1 + J \ &= \ \{I_\ell + A : A \in J\}, \quad \text{where } I_\ell \in M_\ell(c) \text{ is the identity matrix.} \end{aligned}$$

Because w is a valuation, it is easy to check that R is a subring of $M_\ell(C)$ and J is an ideal of R. Therefore, $1 + J$ is closed under multiplication. Take any $f \in \operatorname{End}_C(D)$ (which acts on D on the right), and let $A = (a_{ij})$ be the matrix of f relative to the C-base $\{b_1, \ldots b_\ell\}$ of D, i.e., $b_i f = \sum_{j=1}^{\ell} a_{ij} b_j$ for all i. So,

$$w(b_i f) \ = \ w\big(\sum_{j=1}^{\ell} a_{ij} b_j\big) \ = \ \min_{1 \le j \le \ell} \big(w(a_{ij}) + \gamma_j\big).$$

Thus, $w(b_i f) \ge w(b_i) = \gamma_i$ iff $w(a_{ij}) \ge \gamma_i - \gamma_j$ for $1 \le j \le \ell$. From this it is clear that $A = (a_{ij}) \in R$ iff $w(b_i f) \ge w(b_i)$ for all i. Analogously, $A \in J$ iff $w(b_i f) > w(b_i)$ for all i.

Now, take any $u \in 1 + M_D$, say $u = 1 + m$ with $m \in M_D$. Then, $1 \otimes m \in L \otimes_F D$ corresponds to $\rho_m \in \operatorname{End}_C(D)$, where $d\rho_m = dm$ for all $d \in D$. Let $S \in M_\ell(C)$ be the matrix for ρ_m. Since $w(m) > 0$, we have

$$w(b_i \rho_m) \ = \ w(b_i m) \ = \ w(b_i) + w(m) \ > \ w(b_i) \quad \text{for all } i.$$

Hence, $S \in J$, by the preceding paragraph.

Claim. For any matrix $T \in 1 + J$, we have $\operatorname{ddet}(T) \in 1 + M_C \pmod{C'}$.

The Proposition follows at once from this claim, since the matrix for $1 \otimes (1 + m)$ is $I_\ell + S \in 1 + J$.

Proof of Claim. Take $T \in 1 + J$. The idea is that the process of bringing T to upper triangular form by row operations is carried out entirely within $1 + J$. Write $T = I_\ell + Z$ with $Z = (z_{ij}) \in J$. So, $w(z_{ii}) > \gamma_i - \gamma_i = 0$ for all i, i.e., $z_{ii} \in M_C$. Thus, for all i, j, we have

$$t_{ii} \ = \ 1 + z_{ii} \in 1 + M_C \qquad \text{and} \qquad t_{ij} \ = \ z_{ij}, \quad \text{so} \quad w(t_{ij}) \ > \ \gamma_i - \gamma_j \quad \text{when} \quad i \neq j.$$

Fix k with $1 \le k \le \ell - 1$. Since $t_{kk} \in 1 + M_C$, $w(t_{kk}) = 0$, so $t_{kk} \neq 0$. Let $Y = (y_{ij}) \in M_\ell(C)$ be the matrix for the row operations to bring 0's to all entries in the k-th column of T below the main diagonal, i.e., the i-th row of YT is: (the i-th row of T) $-$ ($t_{ik}t_{kk}^{-1}\cdot$ the k-th row of T) for $k < i \le \ell$ (with the first k rows unchanged). So, $y_{ii} = 1$ for all i; $y_{ik} = -t_{ik}t_{kk}^{-1}$ for our fixed k and all i with $k < i \le \ell$; and $y_{ij} = 0$ otherwise. For $i > k$,we have

$$w(y_{ik}) \ = \ w(t_{ik}) - w(t_{kk}) \ > \ \gamma_i - \gamma_k.$$

Hence, $Y \in 1 + J$ and Y is a unipotent lower triangular matrix. Since $1 + J$ is closed under multiplication, we have $YT \in 1 + J$. To bring T to upper triangular form we apply the row operations successively for columns 1 to $\ell - 1$. We end up with an upper triangular matrix $T' = Y_{\ell-1}Y_{\ell-2}\dots Y_2Y_1T \in 1 + J$, where each $Y_k \in 1 + J$ is the matrix for zeroing the k-th column as described above, but applied to the matrix $Y_{k-1}\dots Y_1T \in 1 + J$ (not to T). Say $T' = (t'_{ij})$. Each Y_k is unipotent and lower triangular, so $\operatorname{ddet}(Y_k) = 1 \in C^*/C'$, So, $\operatorname{ddet}(T') = \operatorname{ddet}(Y_{k-1})\dots\operatorname{ddet}(Y_1)\operatorname{ddet}(T) = \operatorname{ddet}(T)$ in C^*/C'. Since T' is upper triangular with each $t'_{ii} \in 1 + M_C$, we have

$$\operatorname{ddet}(T) \ = \ \operatorname{ddet}(T') \ = \ t'_{11}\dots t'_{\ell\ell} \in 1 + M_C \text{ (equality modulo } C'),$$

proving the Claim. □

References

[AS] S. A. Amitsur, D. J. Saltman, *Generic Abelian crossed products and p-algebras*, J. Algebra, **51** (1978), 76–87. 21

[B] M. Boulagouaz, *Le gradué d'une algèbre à division valuée*, Comm. Algebra, **23** (1995), 4275–4300. 1, 5

[BM] M. Boulagouaz, K. Mounirh, *Generic abelian crossed products and graded division algebras*, pp. 33–47 in *Algebra and Number Theory*, eds. M. Boulagouaz and J.-P. Tignol, Lecture Notes in Pure and Appl. Math., Vol. 208, Dekker, New York, 2000. 21

[D] P. Draxl, *Skew Fields*, London Math. Soc. Lecture Note Series, Vol. 81, Cambridge Univ. Press, Cambridge, 1983. 23, 24

[EP] A. J. Engler, A. Prestel, *Valued Fields*, Springer-Verlag, Berlin, 2005. 11, 24

[E] Yu. Ershov, *Henselian valuations of division rings and the group SK_1*, Mat. Sb. (N.S.), **117** (1982), 60–68 (in Russian); English trans., Math. USSR-Sb., **45** (1983), 63–71. 2, 7, 12, 13, 14, 23

[G] P. Gille, Le probléme de Kneser-Tits, exposé Bourbaki, No. 983, to appear in Astérisque; preprint available at: `http://www.dma.ens.fr/~gille/` . 1

[H$_1$] R. Hazrat, SK$_1$*-like functors for division algebras*, J. Algebra, **239** (2001), 573–588. 10, 21

[H$_2$] R. Hazrat, *Wedderburn's factorization theorem, application to reduced K-theory*, Proc. Amer. Math. Soc., **130** (2002), 311–314. 23

[H$_3$] R. Hazrat, *On central series of the multiplicative group of division rings*, Algebra Colloq., **9** (2002), 99–106. 15

[H$_4$] R. Hazrat, SK$_1$ *of Azumaya algebras over Hensel pairs*, to appear in Math. Z.; preprint available (No. 282) at: `http://www.math.uni-bielefeld.de/LAG/` . 1

[HW$_1$] R. Hazrat, A. R. Wadsworth, *On maximal subgroups of the multiplicative group of a division algebra*, to appear in J. Algebra; preprint available (No. 260) at: `http://www.math.uni-bielefeld.de/LAG/` . 10

[HW_2] R. Hazrat, A. R. Wadsworth, *Unitary* SK_1 *for graded and valued division algebras*, preprint in preparation. 10

[HwW_1] Y.-S. Hwang, A. R. Wadsworth, *Algebraic extensions of graded and valued fields*, Comm. Algebra, **27** (1999), 821–840. 1, 3, 6, 12

[HwW_2] Y.-S. Hwang, A. R. Wadsworth, *Correspondences between valued division algebras and graded division algebras*, J. Algebra, **220** (1999), 73–114. 1, 3, 4, 5, 8, 9, 12, 14, 23

[JW] B. Jacob, A. R. Wadsworth, *Division algebras over Henselian fields*, J. Algebra, **128** (1990), 126–179. 4, 14, 23, 24

[J] N. Jacobson, *Finite-Dimensional Division Algebras over Fields*, Springer-Verlag, Berlin, 1996. 15, 16

[K] M. -A. Knus, *Quadratic and Hermitian Forms over Rings*, Springer-Verlag, Berlin, 1991. 5, 6

[KMRT] M. -A. Knus, A. Merkurjev, M. Rost, J.-P. Tignol, *The Book of Involutions*, AMS Coll. Pub., Vol. 44, Amer. Math. Soc., Providence, RI, 1998. 14

[L] T.-Y. Lam, *A First Course in Noncommutative Rings*, Graduate Texts in Math., Vol. 131, Springer-Verlag, New York, 1991. 21, 22, 24

[LT] D. W. Lewis, J.-P. Tignol, *Square class groups and Witt rings of central simple algebras*, J. Algebra, **154** (1993), 360–376. 13

[Mc_1] K. McKinnie, *Prime to p extensions of the generic abelian crossed product*, J. Algebra, **317** (2007), 813–832. 21

[Mc_2] K. McKinnie, *Indecomposable p-algebras and Galois subfields in generic abelian crossed products*, J. Algebra, **320** (2008), 1887–1907. 21

[Mc_3] K. McKinnie, *Degeneracy and decomposability in abelian crossed products*, preprint, arXiv: 0809.1395. 21

[Mer] A. S. Merkurjev, *K-theory of simple algebras*, pp. 65–83 in *K-theory and Algebraic Geometry: connections with quadratic forms and division algebras*, eds. B. Jacob and A. Rosenberg, Proc. Sympos. Pure Math., Vol. 58, Part 1, Amer. Math. Soc., Providence, RI, 1995. 1

[Mor] P. Morandi, *The Henselization of a valued division algebra*, J. Algebra, **122** (1989), 232–243. 5, 13

[M_1] K. Mounirh, *Nicely semiramified division algebras over Henselian fields*, Int. J. Math. Math. Sci., **2005** (2005), 571–577. 14

[M_2] K. Mounirh, *Nondegenerate semiramified valued and graded division algebras*, Comm. Algebra, **36** (2008), 4386–4406. 21

[MW] K. Mounirh, A. R. Wadsworth, *Subfields of nondegenerate tame semiramified division algebras*, preprint, arXiv:0905:3694. 10, 11

[PS] I. A. Panin, A. A. Suslin, *On a conjecture of Grothendieck concerning Azumaya algebras*, St. Petersburg Math. J., **9** (1998), 851–858. 1

[P_1] V. P. Platonov, *The Tannaka-Artin problem and reduced K-theory*, Izv. Akad. Nauk SSSR Ser. Mat., **40** (1976), 227–261 (in Russian); English trans., Math. USSR-Izv., **10** (1976), 211–243. 1, 7, 9, 13, 23, 24

[P_2] V. P. Platonov, *Infiniteness of the reduced Whitehead group in the Tannaka-Artin problem*, Mat. Sb. (N.S.) **100(142)** (1976), 191–200, 335 (in Russian); English trans., Math. USSR-Sb., **29** (1976), 167–176. 9

[P_3] V. P. Platonov, *Algebraic groups and reduced K-theory*, pp. 311–317 in *Proceedings of the ICM (Helsinki 1978)*, ed. O. Lehto, Acad. Sci. Fennica, Helsinki. 1

[PY] V. P. Platonov, V. I. Yanchevskiĭ, SK_1 *for division rings of noncommutative rational functions.* Dokl. Akad. Nauk SSSR, **249** (1979), 1064–1068 (in Russian); English trans., Soviet Math. Doklady, **20** (1979), 1393–1397. 2, 15, 16, 20, 21

[R] I. Reiner, *Maximal Orders*, Academic Press, New York, 1975. 6

[RTW] J.-F. Renard, J.-P. Tignol., A. R. Wadsworth, *Graded Hermitian forms and Springer's theorem*, Indag. Math., N.S., **18** (2007), 97–134. 26

[S_1] D. J. Saltman, *Noncrossed product p-algebras and Galois p-extensions*, J. Algebra, **52** (1978), 302–314. 21

[S_2] D. J. Saltman, *Lectures on Division Algebras*, Reg. Conf. Series in Math., no. 94, AMS, Providence, 1999. 5

[St] C. Stuth, *A generalization of the Cartan-Brauer-Hua theorem*, Proc. Amer. Math Soc., **15** (1964), 211–217.

[Sus] A. A. Suslin, SK_1 *of division algebras and Galois cohomology*, pp. 75–99 in *Algebraic K-theory*, ed. A. A. Suslin, Adv. Soviet Math., Vol. 4, Amer. Math. Soc., Providence, RI, 1991. 23

[T] J.-P. Tignol, *Produits croisés abéliens*, J. Algebra, **70** (1981), 420–436. 21

[TW] J.-P. Tignol, A. R. Wadsworth, *Value functions and associated graded rings for semisimple algebras*, to appear in Trans. Amer. Math. Soc.; preprint available (No. 247) at: `http://www.math.uni-bielefeld.de/LAG/` . 1

[W_1] A. R. Wadsworth, *Extending valuations to finite dimensional division algebras*, Proc. Amer. Math. Soc., **98** (1986), 20–22. 1

[W_2] A. R. Wadsworth, *Valuation theory on finite dimensional division algebras*, pp. 385–449 in Valuation Theory and Its Applications, Vol. I, eds. F.-V. Kuhlmann et al., Fields Institute Commun., Vol. 32, Amer. Math. Soc., Providence, RI, 2002. 1, 4, 7, 13, 24

[We] J. H. M. Wedderburn, *On division algebras*, Trans. Amer. Math. Soc., **15** (1921), 162–166. 22

[Y] V. I. Yanchevskiĭ, *Reduced unitary K-Theory and division rings over discretely valued Hensel fields*, Izv. Akad. Nauk SSSR Ser. Mat., **42** (1978), 879–918 (in Russian); English trans., Math. USSR Izvestiya, **13** (1979), 175–213. 12

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