Causality, apparent 'superluminality' and reshaping in barrier penetration

Causality, apparent "superluminality," and reshaping in barrier penetration

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We consider tunneling of a nonrelativistic particle across a potential barrier. It is shown that the barrier acts as an effective beam splitter which builds up the transmitted pulse from the copies of the initial envelope shifted in the coordinate space backward relative to the free propagation. Although along each pathway causality is explicitly obeyed, in special cases reshaping can result an overall reduction of the initial envelope, accompanied by an arbitrary coordinate shift. In the case of a high barrier the delay amplitude distribution (DAD) mimics a Dirac δ function, the transmission amplitude is superoscillatory for finite momenta and tunneling leads to an accurate advancement of the (reduced) initial envelope by the barrier width. In the case of a wide barrier, initial envelope is accurately translated into the complex coordinate plane. The complex shift, given by the first moment of the DAD, accounts for both the displacement of the maximum of the transmitted probability density and the increase in its velocity. It is argued that analyzing apparent "superluminality" in terms of spacial displacements helps avoid contradiction associated with time parameters such as the phase time.

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I. INTRODUCTION

In 1932 MacColl was first to notice that a wave packet representing a tunneling particle may emerge from the barrier in a manner that suggests that "there is no appreciable delay in the transmission of the packet through the barrier" [1]. The implication that the particle may have crossed the barrier region with a speed greater than the speed of light c has given the effect the name of "apparent superluminality." A parameter commonly used to estimate the time such a particle spends in the barrier region is the phase time τ_{phase}, essentially the energy derivative of the phase of the transmission amplitude (see, for example, Refs. [2,3]). In accordance with the above, τ_{phase} becomes independent of the barrier width d as $d \to \infty$, a fact often referred to as the Hartman effect [4]. Besides tunneling, a similar behavior was predicted and observed for a variety of systems, including propagation of a photon through a slab of "fast light" material, where it has an even more surprising aspect, since a free photon already moves at the maximal possible speed c (for a recent review, see Ref. [5]). Although it has long been agreed that the the causality is not violated since reshaping [6] destroys causal relationship between the incident and the transmitted peaks, exact mechanism of reshaping, the role of the causality principle and the nature of time parameters used to quantify the effect remain open to further discussion [7]. With this task in mind, we return here to the case of nonrelativistic tunneling across a potential barrier, originally considered in Ref. [1]. In Ref. [8] we analyzed a particular type of a beam splitter in which the transmitted pulse, reshaped through interference, appeared reduced and shifted in the coordinate variable relative to free propagation. Postselection of the particle in a particular spin state allowed one to advance or delay the particle, or to make the shift complex valued. With the initial shape of the pulse preserved, the delay amplitude distribution (DAD), which determines the choice between available pathways, mimicked a Dirac δ-function, while the effective transmission coefficient exhibited supersocillations [9,10] in the momentum range of interest.

The purpose of this article is to demonstrate that a similar mechanism, albeit without the flexibility of choosing the delay at will, is realised in nonrelativistic tunneling across a potential barrier. In Sec. II we change from the momentum to the coordinate representation and show that the causality principle limits the spectrum of delays available to a transmitted particle. In Sec. III we show that, due to the oscillatory nature of the complex valued DAD, causality alone cannot be used to predict the position of the transmitted pulse. In Sec. V we analyze advancement in tunneling across a high rectangular barrier. In Sec. VI we show that in the semiclassical limit of a wide barrier initial envelope experiences a complex coordinate shift. In Sec. VII we link the imaginary part of the shift to the increase in the mean velocity of the transmitted particle. In Sec. VIII we explore the analogy between tunneling and the model of Ref. [8] in order to describe the reshaping mechanism. In Sec. IX we introduce a complex delay time similar to the complex traversal time [11] and briefly discuss the wisdom of such an introduction. Section X contains our conclusions.

II. DELAYS AND CAUSALITY IN ONE-DIMENSIONAL SCATTERING

Consider, in a nonrelativistic limit, a one-dimensional wave packet with a mean momentum p_0 incident from the left on a short-range potential $W(x)$. Its transmitted part is given by (we put to unity $\hbar$ and the particle's mass μ)

$$\Psi^T(x,t) = \int T(p)A(p-p_0)\exp(ipx - ip^2t/2)dp, \quad (1)$$

where $A(p-p_0)$ is the momentum distribution of the initial pulse, peaked at $p = p_0$, and $T(p)$ is the transmission amplitude. Consider also the state $\Psi^0(x,t)$ obtained by free ($W = 0$) propagation of the same initial pulse,

$$\Psi^0(x,t) = \int A(p-p_0)\exp(ipx - ip^2t/2)dp. \quad (2)$$

It is convenient to extract the phase factor associated with p_0 thus defining functions $G^T(x,t,p_0)$ and $G^0(x,t,p_0)$ as

$$G^{T,0}(x,t,p_0) = \exp\big(-ip_0x + ip_0^2t/2\big)\Psi^{T,0}(x,t). \qquad (3)$$

Note that $G^0(x,t,p_0)$ represents the envelope of the freely propagating state (2), whereas for $\Psi^T(x,t)$, whose mean momentum may have been changed in transmission, it is not, strictly speaking, so. Following [12] we rewrite the integral (1) as a convolution in the coordinate space, thus obtaining for G^T and G^0 in Eq. (3)

$$G^T(x,t,p_0) = T(p_0)\int_{-\infty}^{\infty} \eta(x',p_0)G^0(x-x',t,p_0)dx'. \qquad (4)$$

In Eq. (4) $\eta(x)$ is the delay amplitude distribution (DAD), related to the Fourier transform of the transmission amplitude $T(p)$

$$\xi(x) = (2\pi)^{-1}\int_{-\infty}^{\infty} T(p)\exp(ipx)dp, \qquad (5)$$

as

$$\eta(x,p_0) = [T(p_0)]^{-1}\exp(-ip_0x)\xi(x), \qquad (6)$$

and normalized to unity

$$\int_{-\infty}^{0} \eta(x,p_0)dx = 1. \qquad (7)$$

Equation (4), which is exact, demonstrates that at any given time t the transmitted pulse $G^T(x,t,p_0)$ builds up from freely propagating envelopes shifted in space by x' (delayed for $x' < 0$ and advanced for $x' > 0$) and weighted by $\eta(x',p_0)$. The support of the $\eta(x,p_0)$ [i.e., all x for which $\eta(x,p_0) \neq 0$] forms a continuum spectrum of available delays.

The causality principle (CP) ensures analyticity of the transmission amplitude in the complex p plane [13] and can be used to obtain information about the spectrum. In particular, for a barrier potential which does not support bound states and, therefore, has no poles in the upper half of the complex p plane, $\xi(x)$ must vanish for $x > 0$, and the spectrum contains no positive shifts (negative delays) [14]. Accordingly, we can write

$$\eta(x,p_0) = \delta(x) + \tilde{\eta}(x,p_0), \quad \tilde{\eta}(x,p_0) \equiv 0, \quad \text{for} \quad x > 0, \qquad (8)$$

where the singular term [which arises because $T(p) \to 1$ for $|x| \to \infty$] corresponds to free propagation, while the smooth part $\tilde{\eta}(x,p_0)$, which describes scattering, vanishes as $W \to 0$. Conversely, the CP ensures that for a barrier the Fourier transform of $T(p)$ contains only plane waves with non-negative frequencies, $x \geqslant 0$,

$$T(p) = \int_0^{\infty} \xi(-x)\exp(ipx)dx. \qquad (9)$$

Finally, for a barrier we can rewrite Eq. (4) in an equivalent form

$$G^T(x,t,p_0) = \int_x^{\infty} \eta(x-x',p_0)G^0(x',t,p_0)dx', \qquad (10)$$

which best serves to demonstrate that the CP prevents transfer of information from the tail of the incident pulse to the front of the transmitted one. Namely should the envelopes of two freely propagating wave packets coincide for $x > x_0$, $G_1^0(x,t,p_0) = G_2^0(x,t,p_0)$, then $G_1^T(x,t,p_0)$ and $G_2^T(x,t,p_0)$ will also coincide in the same range, making it impossible for an observer to distinguish between the two transmitted pulses until their tails arrive at the detector.

III. COUNTERINTUITIVE ADVANCEMENTS, SUPEROSCILLATIONS AND QUASI-DIRAC DISTRIBUTIONS

Equation (4), which is our main result so far, is worth a brief discussion. While the overall factor $T(p_0)$ represents a reduction in the magnitude of the transmitted pulse, its shape is determined by the DAD $\eta(x,p_0)$ and results from the interference between the subenvelopes $G^0(x-x',t,p_0)$ with different spacial shifts which, because a free wave packet spreads, depend on time. The causality principle restricts the spectrum of available shifts and ensures that in the absence of bound states decomposition (4) does not contain advanced terms. This is a quantum analog of the classical result that a particle is sped up when passing over a region where $W(x) < 0$, e.g., over a potential well, and is delayed compared to free propagation whenever $W(x) > 0$, e.g., when passing over a potential barrier. In the classical limit, $\eta(x,p_0)$ becomes highly oscillatory and has a stationary region around $x = x_{cl}$, corresponding to the classical displacement of a particle crossing $W(x)$ relative to the free one. Thus only one shift x_{cl} and one shape $G^0(x-x_{cl},t,p_0)$ are selected from those available in Eq. (4). Since $\eta(x,p_0)$ must vanish for $x > 0$, one can only have $x_{cl} \leqslant 0$. In this way causality ensures that a classical particle passing over a barrier can only be delayed.

Yet when $\eta(x,p_0)$ has no real stationary points, interference effects play the dominant role and the CP alone cannot predict the final shape or even the location of the transmitted pulse. Indeed, should $T(p)$, for whatever reason, have a simple exponential form,

$$T(p) = B\exp(-i\alpha p), \quad B = \text{const}, \quad \alpha > 0, \qquad (11)$$

equation (6) would yield

$$\eta(x,p_0) = \delta(x-\alpha), \qquad (12)$$

and the transmitted envelope would be a reduced accurate copy of the freely propagating one, advanced by the distance α,

$$G^T(x,t,p_0) = BG^0(x-\alpha,t,p_0). \qquad (13)$$

Naively, one may conclude that this situation cannot be realized for a barrier potential, given that the CP requires, on one hand, that the Fourier spectrum contain no negative frequencies similar to that in Eq. (11) and, on the other hand, that $\eta(x,p_0)$ vanish for all $x > 0$ in contradiction to (12). However, to achieve the advancement in Eq. (13), it is only necessary that Eqs. (11) and (12) be satisfied approximately [8]. Thus, $T(p)$ has to mimic $\exp(-i\alpha p)$ only in a limited region of p containing all initial momenta. Equivalently, $\eta(x,p_0)$ has to mimic $\delta(x-\alpha)$ only for initial wave packets sufficiently broad in the coordinate space. The former is possible, since it is well known [9] that a sum of exponentials, whose frequencies lie within a given interval, can locally reproduce a "superoscillatory" exponential with a frequency outside this interval. For the latter it is sufficient that the DAD $\eta(x,p_0)$ have several of its moments outside its region of support and equal to those of $\delta(x-\alpha)$ [8], $\int_{-\infty}^{0} x^n \eta(x)dx \approx \alpha^n$, $n = 0,1,2\ldots K$. This is possible since the DAD is an alternating distribution rather than a non-negative probabilistic one [15]. If so, the kernel $\eta(x-x')$ termed in Ref. [8] a quasi-Dirac distribution, would act like a spacial shift by a distance α on a polynomial of an order $\leqslant K$ or, more generally, one any function whose Taylor series can be truncated after the first K terms. Next we look for evidence of such a behavior in tunneling across a rectangular barrier.

IV. GAUSSIAN WAVE PACKETS

Although the results of Sec. I apply, in principle, to initial pulses of arbitrary shape, in the following we will consider Gaussian wave packets with positive momenta incident on the barrier from the left. Such a wave packet has a spacial width σ and a mean momentum $p_0 > 0$ and is centered around some $x = 0$ at $t = 0$ so that its momentum distribution $A(p-p_0)$ and the freely propagating envelope in Eq. (4) are given by

$$A(p-p_0) = \sigma^{1/2}/(2\pi)^{3/4} \exp[-(p-p_0)^2\sigma^2/4] \tag{14}$$

and

$$G^0(x,t,p_0) = \left[2\sigma^2/\pi\sigma_t^4\right]^{1/4} \exp\left[-(x-p_0t)^2/\sigma_t^2\right], \tag{15}$$

where $\sigma_t^2 \equiv (\sigma^2 + 2it)$ is a complex valued width which takes into account the effects of spreading. The coordinate probability density for the wave packet in Eq. (15) has a Gaussian shape

$$\rho^0(x,t) \equiv |G^0(x,t,p_0)|^2 = \frac{(2/\pi)^{1/2}}{(\sigma^2+4t^2/\sigma^2)^{1/2}} \times \exp\{-2[x-p_0t]^2/(\sigma^2+4t^2/\sigma^2)\}. \tag{16}$$

V. TUNNELLING ACROSS A HIGH RECTANGULAR BARRIER

Consider tunneling of a Gaussian wave packet (15) across a rectangular barrier of a width d and a height W, $W(x) = W$ for $x_0 \leqslant x \leqslant x_0 + d$, $x_0 > 0$, and zero otherwise. The transmission amplitude independent of the barrier position x_0 is given by

$$T(p,W) = \frac{4pk\exp(-ipd)}{(k+p)^2\exp(-ikd) - (k-p)^2\exp(ikd)}, \tag{17}$$

where $k = (p^2 - 2W)^{1/2}$. It is readily seen that $T(p)$ is single valued in the complex p plane and has no poles in its upper half. The DAD $\eta(x, p_0 = 0)$ shown in Fig. 1 is real because of the symmetry $T(-p) = T^*(p)$ and vanishes for $x > 0$ as required by causality. It is convenient to rewrite $T(p)$ as a geometric progression

$$T(p,W) = \frac{4pk\exp[-i(p-k)d]}{(p+k)^2} \sum_{n=0}^{\infty} \frac{(p-k)^{2n}}{(p+k)^{2n}} \exp(-i2nkd), \tag{18}$$

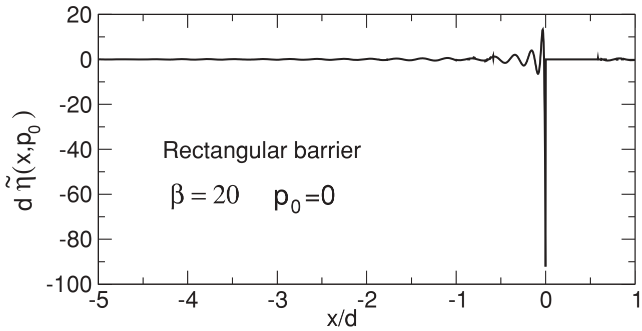

FIG. 1. Regular part of the DAD in Eq. (8) for a rectangular barrier with $\beta \equiv \sqrt{2W}d = 20$ and $p_0 = 0$ obtained by numerical integration of Eq. (5) with $T(p)$ given by Eq. (17).

where we choose the principal branch of the square root $(p^2 - 2W)^{1/2}$, i.e., $k > 0$ for $p^2 > 2W$. Next we fix d and the Gaussian momentum distribution of the incident wave packet $A(p-p_0)$ and increase the barrier height so that

$$W \to \infty, \quad p_0^2/W \to 0. \tag{19}$$

In this limit it is sufficient to retain only the $n = 0$ term in Eq. (18) and expand it to the leading order in W^{-1} to obtain

$$T(p,W) \approx B(W)p\exp(-ipd), \tag{20}$$

with

$$B(W) \equiv -4i(2W)^{-1/2}\exp(-\sqrt{2W}d), \tag{21}$$

which can be made valid for all incident momenta. This is an example of superoscillatory behavior similar to that discussed in Sec. III. Indeed, $T(p,W)/p$ [regular at $p = 0$ according to Eq. (17)], just like $T(p,W)$ itself, has no poles in in the upper half of the complex p plane and its Fourier spectrum cannot contain negative frequencies. Yet according to Eq. (20) in a limited region around $p = 0$ the ratio $T(p)/p$ mimics the behavior of $\exp(-ipd)$. Further, inserting (20) into Eq. (6) shows that that $\eta(x,p_0)$ mimics the behavior of a singular distribution with support at $x = d$,

$$\eta(x,p_0) \approx [\delta(x-d) + i\partial_x\delta(x-d)/p_0], \tag{22}$$

for a class of not-too-narrow wave packets whose momentum distributions probe only the superoscillatory part of $T(p)$. Accordingly, we find the transmitted pulse reduced in magnitude and advanced relative to the free propagation by the barrier width d,

$$G^T(x,t,p_0) \approx T(p_0,W)[G_0(x-d,t,p_0) - i\partial_x G_0(x-d,t,p_0)/p_0], \tag{23}$$

where $T(p_0,W)$ is given by Eq. (20). There is also an additional distortion term proportional to $\partial_x G_0(x-d,t,p_0)$, which becomes negligible for sufficiently fast particles.

Thus, for a given incident Gaussian wave packet one can always find a barrier high enough for the transmitted pulse to be accurately given by Eq. (23).

The price for such an advancement is the reduction of the tunneling probability by a factor $\sim\exp(-2\sqrt{2W}d)$ which makes transmission a very rare event.

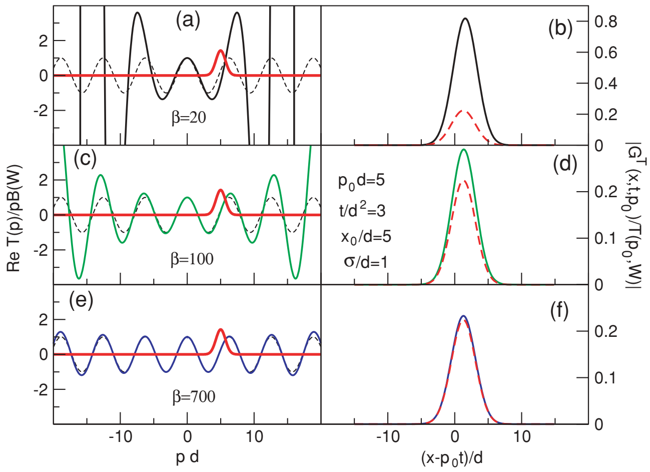

FIG. 2. (Color online) High rectangular barrier: (a) $ReT(p)/pB(W)$ (solid) and $\cos(pd)$ (dashed) for $\beta \equiv \sqrt{2W}d = 20$. Also shown is $|A(p-p_0)|$ scaled to a unit height (thick solid). (b) The shape of the transmitted pulse $|G^T(x,p_0,t)/T(p_0,W)|$: exact (solid) and given by Eq. (23) (dashed); (c and d) same as (a) and (b) but for $\beta = 100$; (e and f) same as (a) and (b) but for $\beta = 700$.

The transmission amplitude $T(p)$ and the transmitted pulse $G^T(x,t,p_0)$ are shown in Fig. 2 for the same Gaussian wave packet and different barrier heights. Figure 2 is similar to Fig. 3 of Ref. [8] with the difference that for a rectangular barrier the superoscillatory band where $T(p)$ can be approximated by Eq. (20) does not have well defined boundaries, whereas for the system studied in Ref. [8] the transmission amplitude exhibited a much more rapid growth marking the edges of the band. Accordingly, the deviations of $G^T(x,t,p_0)$ from the predictions of Eq. (23) at lower barrier heights are less pronounced than the distortion of the shape of the transmitted pulse shown in Fig. 3(b) of Ref. [8].

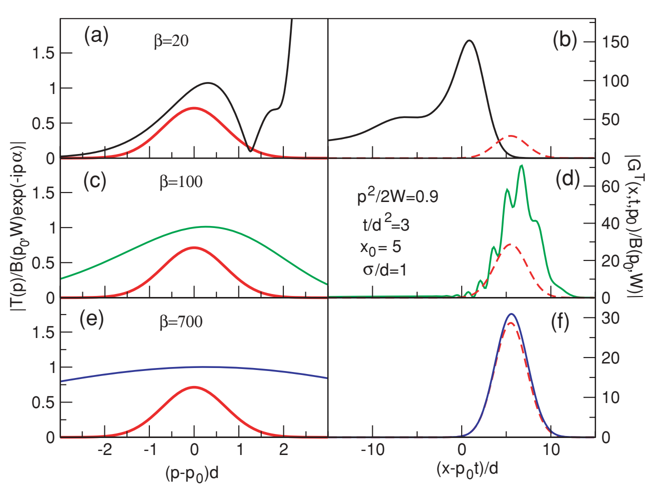

FIG. 3. (Color online) Wide rectangular barrier: (a) ratio between the exact $T(p)$ and its approximation in Eq. (28) for $\beta \equiv \sqrt{2W}d = 20$ (solid). Also shown is $|A(p-p_0)|$ scaled to a unit height (thick solid). (b) The shape of the transmitted pulse $|G^T(x,p_0,t)/T(p_0,W)|$: exact (solid) and given by Eq. (31) (dashed); (c and d) same as (a) and (b) but for $\beta = 100$; (e and f) same as (a) and (b) but for $\beta = 700$.

VI. TUNNELLING ACROSS A WIDE RECTANGULAR BARRIER

Next we consider the case of tunneling across a rectangular barrier whose width increases while its height and the mean kinetic energy of the particle are kept constant,

$$d \to \infty, \quad p_0 = \text{const.} \tag{24}$$

We will also assume that the width of the incident wave packet increases proportionally to the barrier width,

$$\sigma/d \equiv \gamma = \text{const.}, \tag{25}$$

so that its momentum space width σ_p decreases with d,

$$\sigma_p = 2/\sigma = 2/\gamma d. \tag{26}$$

It is easy to show that under these conditions the transmitted pulse will have the shape of the initial envelope not just advanced relative to free propagation but also shifted into the complex coordinate plane. Indeed, retaining only the $n=0$ term in Eq. (18) and expanding the exponent in a Taylor series around p_0 we may write ($k_0 \equiv \sqrt{p_0^2 - 2W}$)

$$-id(p-k) = -id\sum_{n=0}^{\infty} \partial_p^n (p-k)|_{p=p_0}(p-p_0)^n/n!$$
$$\approx -id(p_0-k_0) - id\left[1 + \frac{ip_0}{\sqrt{2W-p_0}}\right](p-p_0) \tag{27}$$

for all initial momenta $|p-p_0| \lesssim \sigma_p \sim 1/d$. We may also replace p with p_0 everywhere in the pre-exponential factor to finally obtain

$$T(p,W) \approx B(p_0,W)\exp(-ip\alpha) \tag{28}$$

with

$$\alpha \equiv d + ip_0 d/\sqrt{2W - p_0} \tag{29}$$

and

$$B(p_0,W) = \frac{4p_0k_0\exp[-id(p_0-k_0) + i\alpha p_0]}{(p_0+k_0)^2}. \tag{30}$$

Thus, in the range of interest, $p_0 - \sigma_p \lesssim p \lesssim p_0 + \sigma_p$, $T(p)$ exhibits a kind of a superoscillatory behavior with a complex valued frequency α, $\text{Re}\alpha > 0$, similar to that studied in Ref. [8]. As a result, for the transmitted pulse we find

$$G^T(x,t,p_0) = B(p_0,W)G^0(x - \text{Re}\alpha - i\text{Im}\alpha, t, p_0). \tag{31}$$

With $T(p,W)$ given by Eq. (28) we expect [8] at least several moments of the DAD $\eta(x,p_0)$ to equal α^n, $\bar{x^n} \equiv \int x^n \eta(x,p_0)dx = \alpha^n$, $n = 0,1,\ldots$. Using the identity

$$\bar{x^n} = i^n \partial_p^n T(p)/T(p)|_{p=p_0}, \quad n = 0,1,\ldots, \tag{32}$$

and noting that as $d \to \infty$ the main contribution to $\bar{x^n}$ comes from differentiating n times the exponential of the first term in the expansion (18), we obtain

$$\lim_{d\to\infty}\bar{x^n}/d^n = (\alpha/d)^n + O(1/d). \tag{33}$$

We can now confirm the result (31) by repeating the calculation in the coordinate space. Consider, for simplicity, a Gaussian function which can be expanded in a Taylor series

$$\exp(-x^2/\gamma^2 d^2) \approx \sum_n (-1)^n (x/\gamma d)^{2n}/n!. \tag{34}$$

Inserting (34) into Eq. (10) and using Eq. (33) we obtain ($C_k^n = n!/k!(n-k)!$ is the binomial coefficient)

$$\begin{aligned}\int \eta(p_0,x')\exp[-(x-x')^2/\gamma^2 d^2]dx' \\ \approx \sum_n \frac{(-1)^n}{n!(\gamma d)^{2n}} \sum_{k=0}^{2n} C_k^{2n} x^k \bar{x}^{2n-k} \\ \approx \exp[-(x-\alpha)^2/\gamma^2 d^2] + O(1/d).\end{aligned} \tag{35}$$

Thus, for a given ratio σ/d one can always find a barrier wide enough for the transmitted pulse to be accurately given by Eq. (31).

The price for an accurate translation of the freely propagating envelope into the complex x plane is the reduction of the tunneling probability by a factor $|B(p_0,W)|^2$ which makes the transmission a very rare event.

The ratio between the exact transmission amplitude $T(p)$ and the one given by Eq. (28) as well as the transmitted envelope $G^T(x,t,p_0)$ are shown in Fig. 3 for the same ratio σ/d and different barrier width. Figure 3 is similar to Figs. 3(c) and 3(f) of Ref. [8] with the difference that for a rectangular barrier the complex superoscillatory band where $T(p)$ can be approximated by Eq. (28) does not have well defined boundaries.

VII. MOMENTUM FILTERING

Equation (31) describes, in a compact form, two effects related to the transmission of a Gaussian wave packet. One is a constant shift in the position of the transmitted envelope; the other is an increase of its average velocity due to suppression of lower momenta contained in the initial distribution. Inserting Eq. (28) and (14) into (1) and completing the square in the exponent we have

$$\begin{aligned}-(p-p_0)^2\sigma^2/4 - i\alpha p = -(p - p_0 - 2\mathrm{Im}\alpha/\sigma^2)^2\sigma^2/4 \\ - ip\mathrm{Re}\alpha + p_0\mathrm{Im}\alpha + (\mathrm{Im}\alpha)^2/\sigma^2,\end{aligned} \tag{36}$$

which shows that after the transmission the mean momentum has increased by

$$\Delta p_0 = 2p_0 d/\left[\sigma^2\left(2W - p_0^2\right)^{1/2}\right]. \tag{37}$$

Note that Δp_0 vanishes for a wave packet very broad in the coordinate (narrow in the momentum) space. Accordingly, for observable probability density with the help of Eqs. (29) and (15) we find

$$\begin{aligned}\rho^T(x,t) \equiv |G^T(x,t,p_0|^2 \\ = C\exp\{-2[x-(p_0+\Delta p_0)t-d]^2/(\sigma^2+4t^2/\sigma^2)\},\end{aligned} \tag{38}$$

where $C = [2/\pi]^{1/2}\sigma|\sigma_t|^{-2}|B(p_0,W)|^2$ Thus, the transmitted probability density has a Gaussian shape which broadens with time and whose maximum propagates along the trajectory

$$x = (p_0 + \Delta p_0)t + d. \tag{39}$$

The maximum arrives at a detector earlier than that of a freely propagating pulse [cf. Eq. (16)], first, because of the increase in the mean velocity and, second, because of additional advancement by a distance d the pulse has received on traversing the barrier. This advancement, if interpreted incorrectly, gives rise to the notion of "superluminality."

VIII. "SUPERLUMINALITY" AND HARTMAN EFFECT VS. RESHAPING

One might try the following classical reasoning: the particle emerges from the barrier with its mean velocity slightly increased and with an additional advancement. The advancement is due to a shorter duration τ spent inside the barrier, $\tau < d/p_0$. Neglecting $\Delta p_0 t$, for the separation Δx between the maxima of the free and the tunnelled pulses one has $\Delta x = p_0(d/p_0 - \tau)$. With $\Delta x = \mathrm{Re}\alpha = d + O(1)$ [cf. Eq. (33)] we have $\tau \sim O(1/p_0)$ and not $O(d/p_0)$ as one might expect. The fact that τ defined in this manner becomes independent of d in the limit of large barrier widths is known as the Hartman effect (see Ref. [3] and references therein). It is readily seen that for a wide barrier d/τ can be greater than the speed of light c, hence the term "superluminality" in the title of this section. It is well known (see, for example, Refs. [3,10]) that relating the advancement d to the duration τ spent in the barrier is incorrect, since there is no causal relationship between the incident and transmitted peaks.

With the help of Eq. (4) we can analyze reshaping mechanism responsible for destroying this realtionship. A barrier acts as a beam splitter with an infinite (continuum) number of arms. On exit from each arm there is an initial pulse shifted backward (delayed) by a distance x' and the probability amplitude for passing through the arm is $\eta(x',p_0)$. The shifted shapes are then recombined to produce the tunnelled pulse which, although in none of the arms causality is violated, has an apparently "superluminal" aspect. Resulting wave packet is invariably deformed, yet it is possible to limit deformation to overall reduction accompanied by a coordinate shift, which is what happens in the to cases considered in Secs. V and VI. Standard quantum mechanics states that if two or more different shifts contribute to the sum, no definite shift (delay) can be assigned to the product of their interference. Accordingly, the separation between the free and the transmitted maxima is obtained as the first moment $\bar{x}$ of an alternating complex valued DAD $\eta(x,p_0)$, for which neither $\mathrm{Re}\bar{x}$ nor $\mathrm{Im}\bar{x}$ are restricted to lie within its region of support [15]. Averaging with an amplitude rather than a probability distribution destroys any direct link between the causal spectrum of delays in the arms of a beam splitter and apparently noncausal advancement of the transmitted peak.

IX. COMPLEX DELAYS AND THE PHASE TIME

In the case studied in Sec. IV the observable time parameter of interest is the delay with which the peak of the transmitted probability density arrives at a detector located at some x_d.

With the help of Eq. (38) we can express this delay in terms of a complex valued coordinate shift $\alpha \approx \bar{x}$ which initial Gaussian pulse experiences upon traversing the barrier, so that there is no need to introduce any additional time parameters. If, against our own advice, we attribute the coordinate shift $\bar{x}$ to the difference between the durations τ and d/p_0 spent in the barrier in tunneling and free motion, for τ we obtain

$$\tau = (d - \bar{x})/p_0 = d/p_0 - i\partial_p \ln T(p_0)/p_0. \qquad (40)$$

Equation (40) defines a complex time parameter, whose real part is the phase time [2,3] often used to quantify advancement of the transmitted pulse,

$$\tau_{\text{phase}} \equiv d/p_0 + \partial_p \Phi(p_0)/p_0 = \mathrm{Re}\tau. \qquad (41)$$

We note, however, that little is gained by introducing the time parameters (40) and (41) as "superluminal" tunneling is readily analyzed in terms of spacial shifts. It can also be shown that the envelope plays the role of a pointer in a highly inaccurate (weak) quantum measurement of such a shift (see Ref. [12] and references therein). Both τ and τ_{phase} are artifacts of a naive extrapolation of particle-like behavior to a wavelike situation where, just like in Ref. [6], the initial peak is first destroyed and then recreated in a different place by an explicitly causal reshaping mechanism.

X. CONCLUSIONS AND DISCUSSION

In summary, transformation to the coordinate representation in Eq. (4) helps one analyze the reshaping mechanism of quantum tunneling as well as the role played by the causality principle. Like any system characterized by a transmission amplitude $T(p)$, a potential barrier can be seen as an effective beam splitter with a continuum of arms (pathways). On exit from each arm there is a copy of the initial envelope (subenvelope) shifted relative to free propagation. All subenvelopes recombine to shape the transmitted wave packet. The probability amplitude for traveling along a particular pathway is given by the delay amplitude distribution (6), essentially a nonanalytic Fourier transform of $T(p)$ with an additional phase determined by the particle's mean momentum. Causality principle ensures that along neither pathway causality is violated. Thus, for a barrier, none of the subenvelopes are advanced, and the Fourier spectrum of a barrier transmission amplitude contains only non-negative frequencies.

Restrictions imposed by the CP cannot, however, prevent the reduced tunnelled pulse to be advanced even though all its constituent parts are delayed relative to free propagation. For example, an accurate advancement by a distance α is achieved if a sufficient number of moments of the complex oscillatory distribution $\eta(x,p_0)$ equal α^n, $n = 0,1,\ldots,$ where α lies outside the spectrum of available shifts. Equivalently, in a limited region of momenta, $T(p)$ mimics the exponential $\exp(-i\alpha p)$ with a frequency outside its Fourier spectrum. The width of this superoscillatory band imposes the limit on the minimal coordinate width of a wave packet which can be advanced without distorting the shape of the envelope.

For a rectangular barrier of the width d, one can find at least two regimes where a situation similar to the one just described is realized. Well above the barrier a single shift is selected from the spectrum and one can speak, in a classical sense, of a duration spent in the barrier region. Whenever more than two subenvelopes envelopes interfere, no such duration can be assigned to the distorted (reshaped) transmitted pulse. In the case of a high barrier considered in Sec. V, this distortion takes the form of an overall reduction in size accompanied by forward shift by the barrier width d. In the special case of tunneling across a wide barrier considered in Sec. VI the distortion takes the form of an overall reduction accompanied by a complex valued coordinate shift α. The shift accounts for the shift of the maximum of the transmitted probability density as well as for the increase in its velocity. Since the free Hamiltonian commutes with a coordinate shift, whether real or complex, the above remains true at any time, once the transmission is completed. This analysis can be compared with the description of the effect in terms of the phase time (41): were the transmitted pulse (31) to represent (which it does not) a classical particle crossing the barrier region, such a particle would have to cross a wide barrier infinitely fast. Arguably, the latter statement raises more questions then provides answers and contributes to the extended discussion of the subject which continues in the literature [2].

The origin of α is of some interest. The shift $\alpha \approx \bar{x}$ is the complex-valued first moment of the alternating delay amplitude distribution $\eta(x,p_0)$. It has been shown in Ref. [15] that such nonprobabilistic averages arise whenever one attempts to answer the "which way?" (in our case, "which shift?") question without destroying interference between different pathways. Standard quantum mechanics cannot give (and, according to Ref. [16], best avoids trying to give) a consistent answer to this question, and the overinterpretation of the "weak value" α leads to a false notion of "superluminarity" as discussed above.

Finally, our analysis applies to a wave packet of an arbitrary shape with a sufficiently narrow momentum distribution. The Gaussian wave packets considered above have an additional advantage of being sufficiently well localised in both the coordinate and the momentum spaces and, for this reason, provide a good illustration of the quantum speed up effect.

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