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SK_1 of Azumaya algebras over Hensel Pairs

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Abstract Let A be an Azumaya algebra of constant rank n^2 over a Hensel pair (R, I) where R is a semilocal ring with n invertible in R. Then the reduced Whitehead group $\text{SK}_1(A)$ coincides with its reduction $\text{SK}_1(A/IA)$. This generalizes a result of Hazrat (J Algebra 305:687–703, 2006) to non-local Henselian rings.

Keywords Azumaya algebras · Reduced whitehead group · Reduced norm

Let A be an Azumaya algebra over a ring R of constant rank n^2. Then there is an étale faithfully flat commutative ring S over R which splits A, i.e., $A \otimes_R S \cong M_n(S)$. For $a \in A$, considering $a \otimes 1$ as an element of $M_n(S)$, one then defines the reduced characteristic polynomial of a as

$$\text{char}_A(x, a) = \det(x - a \otimes 1) = x^n - \text{Trd}(a)x^{n-1} + \cdots + (-1)^n \text{Nrd}(a).$$

Using descent theory, one can show that $\text{char}_A(x, a)$ is independent of S and the isomorphism above and lies in $R[x]$. Furthermore, the element a is invertible in A if and only if $\text{Nrd}_A(a)$, the reduced norm of a, is invertible in R (see [10, III.1.2], and [14, Theorem 4.3]). Let $\text{SL}(1, A)$ be the set of elements of A with the reduced norm 1. Since the reduced norm map respects the scalar extensions, it defines the smooth group scheme $\text{SL}_{1,A} : T \to \text{SL}(1, A_T)$ where $A_T = A \otimes_R T$ for an R-algebra T. Consider the short exact sequence of smooth group schemes

$$1 \longrightarrow \text{SL}_{1,A} \longrightarrow \text{GL}_{1,A} \xrightarrow{\text{Nrd}} G_m \longrightarrow 1$$

where $\text{GL}_{1,A} : T \to A_T^*$ and $G_m(T) = T^*$ for an R-algebra T and A_T^* and T^* are invertible elements of A_T and T, respectively. This exact sequence induces a long exact sequence

$$1 \longrightarrow \text{SL}(1, A) \longrightarrow A^* \xrightarrow{\text{Nrd}} R^* \longrightarrow H^1_{et}(R, \text{SL}(1, A)) \longrightarrow H^1_{et}(R, \text{GL}(1, A)) \to \cdots \tag{1}$$

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Let A' denote the commutator subgroup of A^*. One defines the reduced Whitehead group of A as $\mathrm{SK}_1(A) = \mathrm{SL}(1, A)/A'$ which is a subgroup of (non-stable) $K_1(A) = A^*/A'$. Let I be an ideal of R. Since the reduced norm is compatible with extensions, it induces the map $\mathrm{SK}_1(A) \to \mathrm{SK}_1(\overline{A})$, where $\overline{A} = A/IA$. A natural question arises here is, under what circumstances and for what ideals I of R, this homomorphism would be injective and/or surjective and thus the reduced Whitehead group of A coincides with its reduction. The following observation shows that even in the case of a split Azumaya algebra, these two groups could differ: consider the split Azumaya algebra $A = M_n(R)$ where R is an arbitrary commutative ring (and $n > 2$). In this case the reduced norm coincides with the ordinary determinant and $\mathrm{SK}_1(A) = \mathrm{SL}_n(R)/[\mathrm{GL}_n(R), \mathrm{GL}_n(R)]$. There are examples such that $\mathrm{SK}_1(A) \neq 1$, in fact not even torsion. But in this setting, obviously $\mathrm{SK}_1(\overline{A}) = 1$ for $\overline{A} = A/mA$ where m is a maximal ideal of R (for some examples see [13, Chap. 2]).

If I is contained in the Jacobson radical $J(R)$, then $IA \subset J(A)$ (see, e.g., [4, Lemma 1.4]) and (non-stable) $K_1(A) \to K_1(\overline{A})$ is surjective, thus its restriction to SK_1 is also surjective.

It is observed by Grothendieck [5, Theorem 11.7] that if R is a local Henselian ring with maximal ideal I and G is an affine, smooth group scheme, then $H^1_{et}(R, G) \to H^1_{et}(R/I, G/IG)$ is an isomorphism. This was further extended to Hensel pairs by Strano [15]. Now if further R is a semilocal ring then $H^1_{et}(R, \mathrm{GL}(1, A)) = 0$, and thus from the sequence (1) we have the following commutative diagram:

$$
\begin{array}{ccccccccccc}
 & & & & (1+IA)A'/A' & \longrightarrow & 1+I & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathrm{SK}_1(A) & \longrightarrow & K_1(A) & \xrightarrow{\mathrm{Nrd}} & R^* & \longrightarrow & H^1_{et}(R, \mathrm{SL}(1, A)) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \cong & & \\
1 & \longrightarrow & \mathrm{SK}_1(\overline{A}) & \longrightarrow & K_1(\overline{A}) & \xrightarrow{\mathrm{Nrd}} & \overline{R}^* & \longrightarrow & H^1_{et}(\overline{R}, \mathrm{SL}(1, \overline{A})) & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 1 & & 1 & & & &
\end{array}
\tag{2}
$$

The aim of this note is to prove that for the Hensel pair (R, I) where R is a semilocal ring, the map $\mathrm{SK}_1(A) \to \mathrm{SK}_1(\overline{A})$ is also an isomorphism. This extends a result of [6] to non-local Henselian rings.

Recall that the pair (R, I) where R is a commutative ring and I an ideal of R is called a Hensel pair if for any polynomial $f(x) \in R[x]$, and $b \in R/I$ such that $\overline{f}(b) = 0$ and $\overline{f}'(b)$ is invertible in R/I, then there is $a \in R$ such that $\overline{a} = b$ and $f(a) = 0$ (for other equivalent conditions, see Raynaud [12, Chap. XI]).

In order to prove the statement, we use a result of Vaserstein [17] which establishes the (Dieudonnè) determinant in the setting of semilocal rings. The crucial part is to prove a version of Platonov's congruence theorem [11] in the setting of an Azumaya algebra over a Hensel pair. The approach to do this was motivated by Suslin in [16]. We also need to use the following facts established by Greco in [3,4].

Proposition 1 [4, Prop. 1.6] *Let R be a commutative ring, A be an R-algebra integral over R and finite over its center. Let B be a commutative R-subalgebra of A and I an ideal of R. Then $IA \cap B \subseteq \sqrt{IB}$.*

Corollary 1 [3, Cor. 4.2] *Let (R, I) be a Hensel pair and let $J \subseteq \sqrt{I}$ be an ideal of R. Then (R, J) is a Hensel pair.*

Theorem 1 [3, Th. 4.6] *Let (R, I) be a Hensel pair and let B be a commutative R-algebra integral over R. Then (B, IB) is a Hensel pair.*

We are in a position to prove the main theorem of this note.

Theorem 2 *Let A be an Azumaya algebra of constant rank n^2 over a Hensel pair (R, I) where R is a semilocal ring with n invertible in R. Then $\mathrm{SK}_1(A) \cong \mathrm{SK}_1(\overline{A})$ where $\overline{A} = A/IA$.*

Proof Since for any $a \in A$, $\overline{\mathrm{Nrd}_A(a)} = \mathrm{Nrd}_{\overline{A}}(\overline{a})$, it follows that there is a homomorphism $\phi : \mathrm{SL}(1, A) \to \mathrm{SL}(1, \overline{A})$. We first show that $\ker \phi \subseteq A'$, the commutator subgroup of A^*. In the setting of valued division algebras, this is the Platonov congruence theorem [11]. We shall prove this in several steps. Clearly $\ker \phi = \mathrm{SL}(1, A) \cap 1 + IA$. Note that A is a free R-module (see [1, II, Sect. 5.3, Prop. 5]).

(i) *The group $1 + I$ is uniquely n-divisible and $1 + IA$ is n-divisible.*

Let $a \in 1 + I$. Consider $f(x) = x^n - a \in R[x]$. Since n is invertible in R, $\overline{f}(x) = x^n - 1 \in \overline{R}[x]$ has a simple root. Now this root lifts to a root of $f(x)$ as (R, I) is a Hensel pair. This shows that $1 + I$ is n-divisible. Now if $(1 + a)^n = 1$ where $a \in I$, then $a(a^{n-1} + na^{n-2} + \cdots + n) = 0$. Since the second factor is invertible, $a = 0$, and it follows that $1 + I$ is uniquely n-divisible.

Now let $a \in 1 + IA$. Consider the commutative ring $B = R[a] \subseteq A$. By Theorem 1, (B, IB) is a Hensel pair. On the other hand by Proposition 1, $IA \cap B \subseteq \sqrt{IB}$. Thus by Corollary 1, $(B, IA \cap B)$ is also a Hensel pair. But $a \in 1 + IA \cap B$. Applying the Hensel lemma as in the above, it follows that a has a n-th root and thus $1 + IA$ is n-divisible.

(ii) $\mathrm{Nrd}_A(1 + IA) = 1 + I$.

From compatibility of the reduced norm, it follows that $\mathrm{Nrd}_A(1 + IA) \subseteq 1 + I$. Now using the fact that $1 + I$ is n-divisible, the equality follows.

(iii) $\mathrm{SK}_1(A)$ *is n^2-torsion.*

We first establish that $N_{A/R}(a) = \mathrm{Nrd}_A(a)^n$. One way to see this is as follows. Since A is an Azumaya algebra of constant rank n^2, $i : A \otimes A^{op} \cong \mathrm{End}_R(A) \cong M_{n^2}(R)$ and there is an étale faithfully flat S algebra such that $j : A \otimes S \cong M_n(S)$. Consider the following diagram

$$\begin{array}{ccccccc}
A \otimes A^{op} \otimes S & \xrightarrow{i\otimes 1} & \mathrm{End}_R(A) \otimes S & \xrightarrow{\cong} & \mathrm{End}_S(A \otimes S) & \xrightarrow{\cong} & M_{n^2}(S) \\
\downarrow & & & & & & \vdots\ \psi \\
A^{op} \otimes A \otimes S & \xrightarrow{1\otimes j} & A^{op} \otimes M_n(S) & \xrightarrow{\cong} & M_n(A^{op} \otimes S) & \xrightarrow{\cong} & M_{n^2}(S)
\end{array}$$

where the automorphism ψ is the compositions of isomorphisms in the diagram. By a theorem of Artin (see, e.g., [10, Sect. III, Lemma 1.2.1]), one can find an ètale faithfully flat S algebra T such that $\psi \otimes 1 : M_{n^2}(T) \to M_{n^2}(T)$ is an inner automorphism. Now the determinant of the element $a \otimes 1 \otimes 1$ in the first row is $N_{A/R}(a)$ and in the second row is $\mathrm{Nrd}_A(a)^n$ and since $\psi \otimes 1$ is inner, thus they coincide.

Therefore if $a \in \mathrm{SL}(1, A)$, then $N_{A/R}(a) = 1$. We will show that $a^{n^2} \in A'$. Consider the sequence of R-algebra homomorphism

$$f : A \to A \otimes A^{op} \to \mathrm{End}_R(A) \cong M_{n^2}(R) \hookrightarrow M_{n^2}(A)$$

and the R-algebra homomorphism $i : A \to M_{n^2}(A)$ where a maps to aI_{n^2}, where I_{n^2} is the identity matrix of $M_{n^2}(A)$. Since R is a semilocal ring, the Skolem–Noether theorem is present in this setting (see [10, Prop. 5.2.3]) and thus there is $g \in \mathrm{GL}_{n^2}(A)$ such that $f(a) = gi(a)g^{-1}$. Also, since A is a finite algebra over R, A is a semilocal ring. Since n is invertible in R, by Vaserstein's result [17], the Dieudonnè determinant extends to the setting of $M_{n^2}(A)$. Taking the determinant from $f(a)$ and $gi(a)g^{-1}$, it follows that $1 = N_{A/R}(a) = a^{n^2}c_a$ where $c_a \in A'$. This shows that $\mathrm{SK}_1(A)$ is n^2-torsion.

(iv) *Platonov's Congruence Theorem:* $\mathrm{SL}(1, A) \cap (1 + IA) \subseteq A'$.

Let $a \in \mathrm{SL}(1, A) \cap (1 + IA)$. By part (i), there is $b \in 1 + IA$ such that $b^{n^2} = a$. Then $\mathrm{Nrd}_A(a) = \mathrm{Nrd}_A(b)^{n^2} = 1$. By part (ii), $\mathrm{Nrd}_A(b) \in 1 + I$ and since $1 + I$ is uniquely n-divisible, $\mathrm{Nrd}_A(b) = 1$, so $b \in \mathrm{SL}(1, A)$. By part (iii), $b^{n^2} \in A'$, so $a \in A'$. Thus $\ker\phi \subseteq A'$ where $\phi : \mathrm{SL}(1, A) \to \mathrm{SL}(1, \overline{A})$.

It is easy to see that ϕ is surjective. In fact, if $\overline{a} \in \mathrm{SL}(1, \overline{A})$ then $1 = \mathrm{Nrd}_{\overline{A}}(\overline{a}) = \overline{\mathrm{Nrd}_A(a)}$ thus, $\mathrm{Nrd}_A(a) \in 1+I$. By part (i), there is $r \in 1+I$ such that $\mathrm{Nrd}_A(ar^{-1}) = 1$ and $\overline{ar^{-1}} = \overline{a}$. Thus ϕ is an epimorphism. Consider the induced map $\overline{\phi} : \mathrm{SL}(1, A) \to \mathrm{SL}(1, \overline{A})/\overline{A}'$. Since $I \subseteq J(R)$, and by part (iii), $\ker\phi \subseteq A'$ it follows that $\ker\overline{\phi} = A'$ and thus $\overline{\phi} : \mathrm{SK}_1(A) \cong \mathrm{SK}_1(\overline{A})$. □

Let R be a semilocal ring and $(R, J(R))$ a Hensel pair. Let A be an Azumaya algebra over R of constant rank n^2 and n invertible in R. Then by Theorem 2, $\mathrm{SK}_1(A) \cong \mathrm{SK}_1(\overline{A})$ where $\overline{A} = A/J(R)A$. But $J(A) = J(R)A$, so $\overline{A} = M_{k_1}(D_1) \times \cdots \times M_{k_r}(D_r)$ where D_i are division algebras. Thus $\mathrm{SK}_1(A) \cong \mathrm{SK}_1(\overline{A}) = \mathrm{SK}_1(D_1) \times \cdots \times \mathrm{SK}_1(D_r)$.

Using a result of Goldman [2], one can remove the condition of Azumaya algebra having a constant rank from the Theorem.

Corollary 2 *Let A be an Azumaya algebra over a Hensel pair (R, I) where R is semilocal and the least common multiple of local ranks of A over R is invertible in R. Then* $\mathrm{SK}_1(A) \cong \mathrm{SK}_1(\overline{A})$ *where* $\overline{A} = A/IA$.

Proof One can decompose R uniquely as $R_1 \oplus \cdots \oplus R_t$ such that $A_i = R_i \otimes_R A$ have constant ranks over R_i which coincide with local ranks of A over R (see [2, Sect. 2 and Theorem 3.1]). Since (R_i, IR_i) are Hensel pairs, the result follows by using Theorem 2. □

Remark Let D be a tame unramified division algebra over a Henselian field F, i.e., the value group of D coincides with value group of F and $\mathrm{char}(\overline{F})$ does not divide the index of D (see [18] for a nice survey on valued division algebras). Let V_D be the valuation ring of D and $U_D = V_D^*$. Jacob and Wadsworth observed that V_D is an Azumaya algebra over its center V_F (Theorem 3.2 in [18] and Example 2.4 in [8]). Since $D^* = F^*U_D$ and $V_D \otimes_{V_F} F \simeq D$, it can be seen that $\mathrm{SK}_1(D) = \mathrm{SK}_1(V_D)$. On the other hand our main Theorem states that $\mathrm{SK}_1(V_D) \simeq \mathrm{SK}_1(\overline{D})$. Comparing these, we conclude the stability of SK_1 under reduction, namely $\mathrm{SK}_1(D) \simeq \mathrm{SK}_1(\overline{D})$ (compare this with the original proof, Corollary 3.13 in [11]).

Now consider the group $\mathrm{CK}_1(A) = A^*/R^*A'$ for the Azumaya algebra A over the Hensel pair (R, I). A proof similar to Theorem 3.10 in [6], shows that $\mathrm{CK}_1(A) \cong \mathrm{CK}_1(\overline{A})$. Thus in the case of tame unramified division algebra D, one can observe that $\mathrm{CK}_1(D) \cong \mathrm{CK}_1(\overline{D})$.

For an Azumaya algebra A over a semilocal ring R, by the exact sequence (1), one has

$$R^*/\mathrm{Nrd}_A(A^*) \cong H^1_{\text{èt}}(R, \mathrm{SL}(1, A)).$$

If (R, I) is also a Hensel pair, then by the Grothendieck-Strano result,

$$R^*/\mathrm{Nrd}_A(A^*) \cong H^1_{\text{èt}}(R, \mathrm{SL}(1, A)) \cong H^1_{\text{èt}}(\overline{R}, \mathrm{SL}(1, \overline{A})) \cong \overline{R}^*/\mathrm{Nrd}_{\overline{A}}(\overline{A}^*).$$

However specializing to a tame unramified division algebra D, the stability does not follow in this case. In fact for a tame and unramified division algebra D over a Henselian field F with the valued group Γ_F and index n one has the following exact sequence (see [7, Theorem 1]):

$$1 \longrightarrow H^1(\overline{F}, \mathrm{SL}(1, \overline{D})) \longrightarrow H^1(F, \mathrm{SL}(1, D)) \longrightarrow \Gamma_F / n\Gamma_F \longrightarrow 1.$$

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