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AMENABILITY OF ALGEBRAS OF APPROXIMABLE OPERATORS

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ABSTRACT

We give a necessary and sufficient condition for amenability of the Banach algebra of approximable operators on a Banach space. We further investigate the relationship between amenability of this algebra and factorization of operators, strengthening known results and developing new techniques to determine whether or not a given Banach space carries an amenable algebra of approximable operators. Using these techniques, we are able to show, among other things, the non-amenability of the algebra of approximable operators on Tsirelson's space.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{X}$ be a Banach space which is also an $\mathcal{A}$-bimodule. Then $\mathcal{X}$ is a Banach $\mathcal{A}$-bimodule if there exists a constant M so that $\|a \cdot x\| \leq M\|a\|\|x\|$ and $\|x \cdot a\| \leq M\|a\|\|x\|$ ($a \in \mathcal{A}$, $x \in \mathcal{X}$). A (continuous)

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derivation from $\mathcal{A}$ to $\mathcal{X}$ is a (bounded) linear map $D : \mathcal{A} \to \mathcal{X}$ that satisfies the identity

$$D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in \mathcal{A}).$$

Every map of the form $a \mapsto a \cdot x - x \cdot a$ ($a \in \mathcal{A}$), where $x \in \mathcal{X}$ is fixed, is a continuous derivation. Derivations of this form are called inner derivations.

If $\mathcal{X}$ is a Banach $\mathcal{A}$-bimodule, then its topological dual, $\mathcal{X}^*$, is also a Banach $\mathcal{A}$-bimodule under the actions

$$(a \cdot f)(x) := f(x \cdot a) \quad \text{and} \quad (f \cdot a)(x) := f(a \cdot x) \quad (a \in \mathcal{A},\, x \in \mathcal{X},\, f \in \mathcal{X}^*).$$

The Banach algebra $\mathcal{A}$ is said to be **amenable** if, for every Banach $\mathcal{A}$-bimodule $\mathcal{X}$, every continuous derivation $D : \mathcal{A} \to \mathcal{X}^*$ is inner.

For example, the group algebra, $L^1(G)$, of a locally compact group is amenable if and only if the group G is amenable [13]; a C^*-algebra is amenable if and only if it is nuclear [4, 11]; and a uniform algebra on a compact Hausdorff space Ω is amenable if and only if it is $C(\Omega)$ [25].

In this note we shall be concerned with the amenability of the algebra $\mathcal{A}(X)$ of **approximable operators** on a Banach space X, i.e., the operator norm closure in $\mathcal{B}(X)$ of the ideal $\mathcal{F}(X)$ of continuous finite-rank operators on X, where $\mathcal{B}(X)$ denotes the algebra of all bounded linear operators on X. (When X has the approximation property, $\mathcal{A}(X)$ coincides with the ideal of compact operators on X.) In this setting the main problem is to characterize amenability of $\mathcal{A}(X)$ in terms of properties of X.

The study of amenability of $\mathcal{A}(X)$ goes back to [13], where it is shown that $\mathcal{A}(X)$ is amenable if $X = \ell_p$ for $p \in (1, \infty)$, or $X = C[0, 1]$. Further progress in the study of amenability of this algebra is made in [9]. In this last paper a geometric property, called property $\mathbb{A}$, is introduced, and it is shown that Banach spaces with this property carry amenable algebras of approximable operators. Banach spaces with property $\mathbb{A}$ include all classical Banach spaces, $\mathcal{L}_p$-spaces ($1 \leq p \leq \infty$), spaces with a subsymmetric, shrinking basis, and certain kinds of tensor product of Banach spaces with property $\mathbb{A}$.

In this note we continue the study of amenability of the algebra $\mathcal{A}(X)$. Building upon ideas from [9] we shall develop new techniques that will allow us not only to improve several results from [9] but also to answer some of the questions left open there. In particular, we will show that the algebra of approximable operators on Tsirelson's space is not amenable. An important fact that should become apparent throughout these pages is that a full understanding of amenability of $\mathcal{A}(X)$ will necessarily rely on a good understanding of the finite-dimensional case.

The paper has been organized as follows. In the next section, we have gathered some terminology and basic facts we need. In Section 3, we give a necessary and sufficient condition for amenability of $\mathcal{A}(X)$. Unfortunately, practical use of this condition depends on our capability to find good estimates for the projective norm of certain elements called generalized diagonals. In Section 4, we follow a different approach. The results of this section are to a great extent motivated by the notion of approximate primariness introduced in [9]. We explore some of the ideas behind this notion, specially, its connection with factorization properties of operators. Finally, in Section 5, we establish the non-amenability of $\mathcal{A}(X)$ for every Banach space X in a certain family of Tsirelson-like spaces. In doing this we shall rely on results from the previous sections.

2. Preliminaries

In this section we have gathered some notation and basic results that we shall use throughout these pages.

To simplify the statement of the results, we shall denote by ℓ_∞ the linear space, usually denoted by c_0, of all bounded scalar sequences tending to zero. Given a normed space X we denote by X^* its topological dual. If X and Y are isomorphic (resp., isometric) normed spaces, we write this as $X \simeq Y$ (resp., $X \cong Y$), and denote by $d(X, Y)$ the Banach–Mazur distance between them, that is, the infimum of numbers $\|T\|\|T^{-1}\|$, where T is an isomorphism between X and Y.

The adjoint of an operator $T : X \to Y$ is denoted by T^* and we write $\operatorname{rg} T$ (resp., $\operatorname{rk} T$) for the range (resp., rank) of T. The identity operator on a normed space X is denoted by I_X or just I if the space X is clear from context.

By the inversion constant of a surjective linear map, $Q : X \to Y$, between Banach spaces we mean the operator norm of the inverse of the linear isomorphism $\widetilde{Q} : X/\ker Q \to Y$ induced by Q, that is, $\|\widetilde{Q}^{-1}\|$. Given Banach spaces X, Y and Z, a bilinear map $\varphi : X \times Y \to Z$ will be said to be M-open if for every $z \in Z$ there exist $x \in X$ and $y \in Y$ such that $\varphi(x, y) = z$ and $\|x\|\|y\| \leq M\|z\|$.

We write $\|\cdot\|_\wedge$ (resp., $\|\cdot\|$) for the projective (resp., operator) norm. If two norms, $\|\cdot\|_1$ and $\|\cdot\|_2$, on a linear space are equivalent we write this as

$\|\cdot\|_1 \sim \|\cdot\|_2$. Given a set of vectors $\{e_i : i \in I\}$ in a Banach space, we denote by $[e_i]_{i\in I}$ the closure of its linear span.

Let $\mathbf{e} = (e_i)$ be a 1-unconditional basis for the Banach space $(E, \|\cdot\|)$, and let $(X_i, \|\cdot\|_i)$ be a sequence of Banach spaces. We let

$$\Big(\bigoplus_i X_i\Big)_{\mathbf{e}} = \Big\{(x_i) \in \prod_i X_i : \sum_i \|x_i\|_i e_i \text{ converges in } E\Big\},$$

endowed with the norm $\|(x_i)\| := \big\|\sum_i \|x_i\|_i e_i\big\|$. It is well known that $(\bigoplus_i X_i)_{\mathbf{e}}$ is a Banach space. Moreover, if the basis $\mathbf{e}$ is in addition shrinking, then its topological dual can be isometrically identified with the space $(\bigoplus_i X_i^*)_{\mathbf{e}^*}$, where $\mathbf{e}^*$ stands for the 1-unconditional basis of E^* formed by the biorthogonal functionals associated with $\mathbf{e}$. When $\mathbf{e}$ is the unit vector basis of ℓ_p $(1 \le p \le \infty)$ we write $(\bigoplus_i X_i)_p$ instead of $(\bigoplus_i X_i)_{\mathbf{e}}$.

Given a Banach space E with a 1-unconditional basis $\mathbf{e} = (e_i)$ we denote by E^m the space $[e_i]_{i=1}^m$. If X is a Banach space we denote by $E^m(X)$ (resp., $E(X)$) the Banach space $(\bigoplus_i X_i)_{\mathbf{e}}$, where $X_i = X$, $1 \le i \le m$, and $X_i = \{0\}$, $i > m$ (resp., $X_i = X$ for all i). In particular, $\ell_p(X)$ (resp., $\ell_p^m(X)$) denotes the ℓ_p-sum of countably infinitely many (resp., m) copies of X. When appropriate, we may for $n \in \mathbb{N}$ identify $E^n(X)$ with $E^n \otimes X$.

Given Banach spaces X and Y we write $\mathcal{A}(X,Y)$ (resp., $\mathcal{F}(X,Y)$) for the Banach (resp., normed) space of approximable (resp., finite-rank) operators from X to Y. When appropriate we shall identify $\mathcal{F}(X,Y)$ with $X^* \otimes Y$, so that for $x^* \in X^*$, $y \in Y$ the rank-1 operator $x \mapsto x^*(x)y$ is denoted $x^* \otimes y$. When $X = Y$ we simply write $\mathcal{A}(X)$ (resp., $\mathcal{F}(X)$). Likewise, we shall use tensor notation for operators $E^m(X) \mapsto E^n(X)$ for a Banach space E with a 1-unconditional basis and an arbitrary Banach space X so that, for $m, n \in \mathbb{N}$ we identify $\mathcal{A}(E^m(X), E^n(X))$ with $\mathcal{A}(E^m, E^n) \otimes \mathcal{A}(X)$.

For any Banach space X and positive integers $n > m$, there is a natural isometric embedding $E^m(X) \hookrightarrow E^n(X)$ which in turn induces an isometric Banach algebra homomorphism $\mathcal{A}(E^m(X)) \hookrightarrow \mathcal{A}(E^n(X))$. Letting m and n vary we obtain a direct system of Banach algebras and isometric Banach algebra homomorphisms. Its inductive limit is also a Banach algebra that we denote by $\mathcal{A}_0(E(X))$. Note that $\mathcal{A}_0(\ell_p(X)) = \mathcal{A}(\ell_p(X))$, $1 < p \le \infty$.

Recall that a Banach space X is said to have the λ-bounded approximation property, λ-BAP in short, if there is a net $(T_\alpha) \subset \mathcal{F}(X)$ of bound λ converging strongly to the identity operator on X. We write this as $T_\alpha \xrightarrow{s} I_X$. If, in addition, the T_α's can be chosen to be projections then X is called a π_λ-space. A Banach space is said to have the bounded approximation property, BAP in short, if it has the λ-BAP for some λ, and is said to be a π-space if it is a π_λ-space for some λ.

Recall that a bounded net (e_α) in a normed algebra $\mathcal{A}$ is called a bounded approximate identity, BAI in short, for $\mathcal{A}$ if $\lim_\alpha e_\alpha a = \lim_\alpha a e_\alpha = a$ $(a \in \mathcal{A})$. A normed $\mathcal{A}$-bimodule, $\mathcal{X}$ is **essential**, if $\mathcal{A} \cdot \mathcal{X} \cdot \mathcal{A}$ is dense in $\mathcal{X}$. Clearly, if $\mathcal{A}$ has a BAI, then this BAI is also a BAI for any essential $\mathcal{A}$-bimodule. It is well-known that the algebra of approximable operators on a Banach space X has a BAI of bound λ if and only if X^* has the λ-BAP [10, Theorem 3.3], [24].

Lastly, there is an intrinsic characterization of amenability that is particularly useful in this setting. Precisely, a Banach algebra $\mathcal{A}$ is amenable if and only if it has an approximate diagonal, i.e., a bounded net (d_α) in $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $\pi(d_\alpha)a \to a$ and $ad_\alpha - d_\alpha a \to 0$ $(a \in \mathcal{A})$, where $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$, $a \otimes b \mapsto ab$ [14, Lemma 1.2 and Theorem 1.3]. The Banach algebra $\mathcal{A}$ is said to be K-amenable if it has an approximate diagonal of bound K. The smallest such K is called the amenability constant of $\mathcal{A}$.

Other definitions and results shall be given as they are needed.

3. Property $\mathbb{A}$ revised

Recall from [9] that a Banach space X is said to have property $\mathbb{A}$ if there exist a constant $K > 0$ and a bounded net of projections $(P_\alpha) \subset \mathcal{A}(X)$ such that

i) $P_\alpha \xrightarrow{s} I_X$;
ii) $P_\alpha^* \xrightarrow{s} I_{X^*}$;
iii) for each α there is a finite group $G_\alpha \subset \mathcal{F}(X_\alpha)$ whose linear span is $\mathcal{F}(X_\alpha)$ and such that $\max_{T\in G_\alpha} \|T\| \leq K$ (where $X_\alpha = \operatorname{rg} P_\alpha$).

Property $\mathbb{A}$ was introduced in [9] in an attempt to explain amenability of $\mathcal{A}(X)$ as a consequence of some sort of approximation property. Indeed, Banach spaces with this property must carry amenable algebras of approximable operators [9, Theorem 4.2]. Though we believe the converse is unlikely to be true, we do not know of an example of a Banach space X without property $\mathbb{A}$ and that $\mathcal{A}(X)$ is amenable. The main result of this section, Corollary 3.3 below, is a characterization of amenability of the algebra of approximable operators in terms of a property analogous to property $\mathbb{A}$.

We start with the following.

Proposition 3.1: *Let X be a Banach space such that $\mathcal{A}(X)$ is K-amenable. Suppose in addition that $\mathcal{A}(X)$ contains a bounded net of projections, $(P_\alpha)_{\alpha\in A}$, such that $P_\alpha \xrightarrow{s} I_X$ and $P_\alpha^* \xrightarrow{s} I_{X^*}$. Then $\mathcal{A}(X)$ has an approximate diagonal $(\delta_\alpha)_{\alpha\in A}$ with the following properties:*

a) $\limsup_\alpha \|\delta_\alpha\|_\wedge \leq \lambda K$, *where* $\lambda = \limsup_\alpha \|P_\alpha\|$;
b) $\pi(\delta_\alpha) = P_\alpha$ $(\alpha \in A)$;
c) $W \cdot \delta_\alpha = \delta_\alpha \cdot W$ *for every* $W \in P_\alpha \mathcal{A}(X) P_\alpha$ $(\alpha \in A)$; *and*
d) *For every* $\alpha \in A$ *there exists* $\beta = \beta(\alpha) \in A$ *such that*

$$\delta_\alpha \in \mathcal{A}(X)P_\beta \otimes P_\beta \mathcal{A}(X).$$

Proof. Let $(d_i)_{i\in I}$ be an approximate diagonal for $\mathcal{A}(X)$ bounded by K. Since $P_\alpha \xrightarrow{s} I_X$ and $P_\alpha^* \xrightarrow{s} I_{X^*}$, we can assume, without loss of generality, that for every $i \in I$ there exists $\beta_i \in A$ such that $d_i \in \mathcal{A}(X)P_{\beta_i} \otimes P_{\beta_i}\mathcal{A}(X)$. Let $x \in X$ and $x^* \in X^*$ be fixed vectors such that $x^*(x) = 1$, and let (ε_α) be a net of positive numbers converging to zero ($\varepsilon_\alpha = 1/\operatorname{rk} P_\alpha$, $\alpha \in A$, will do).

Given $i \in I$, let $\Phi_i : X^* \otimes X \to \mathcal{F}(X) \otimes \mathcal{F}(X)$ be the linear map which is defined on elementary tensors by $\Phi_i(\xi^* \otimes \xi) := x^* \otimes \xi \cdot d_i \cdot \xi^* \otimes x$. It is readily seen that Φ_i is an $\mathcal{F}(X)$-bimodule morphism.

For each $\alpha \in A$ choose $i(\alpha) \in I$ 'big enough' so that

$$\left|1 - x^*\left(\pi\left(d_{i(\alpha)}\right)x\right)\right| \leq \varepsilon_\alpha, \tag{1}$$

and

$$\left\|\Phi_{i(\alpha)}(P_\alpha) - P_\alpha \cdot d_{i(\alpha)}\right\|_\wedge \leq \varepsilon_\alpha. \tag{2}$$

Since $(\pi(d_i))$ is a bounded approximate identity for $\mathcal{A}(X)$, it is clear that (1) holds for every $i \in I$ 'big enough'. To see that the same is true about (2) note that for every $i \in I$ we have

$$\begin{aligned}\Phi_i(\xi^* \otimes \xi) - \xi^* \otimes \xi \cdot d_i &= x^* \otimes \xi \cdot d_i \cdot \xi^* \otimes x - \xi^* \otimes \xi \cdot d_i \\ &= x^* \otimes \xi \cdot \left(d_i \cdot \xi^* \otimes x - \xi^* \otimes x \cdot d_i\right) \quad (\xi \in X,\, \xi^* \in X^*).\end{aligned}$$

This last equality, combined in the obvious way with the facts that P_α is finite-rank and that (d_i) is an approximate diagonal, gives the desired conclusion.

Now define a new net $(\delta_\alpha) \in \mathcal{A}(X)\widehat{\otimes}\mathcal{A}(X)$ by

$$\delta_\alpha := \varkappa_\alpha \Phi_{i(\alpha)}(P_\alpha) \quad (\alpha \in A),$$

where $\varkappa_\alpha = 1/x^*(\pi(d_{i(\alpha)})x)$. We show next that (δ_α) has all required properties. First note that

$$\begin{aligned}\|\delta_\alpha\|_\wedge = \varkappa_\alpha\|\Phi_{i(\alpha)}(P_\alpha)\|_\wedge \ &\le\ \varkappa_\alpha\|P_\alpha\cdot d_{i(\alpha)}\|_\wedge + \varkappa_\alpha\|\Phi_{i(\alpha)}(P_\alpha) - P_\alpha\cdot d_{i(\alpha)}\|_\wedge \\ &\le \varkappa_\alpha\|P_\alpha\|K + \varkappa_\alpha\varepsilon_\alpha,\end{aligned}$$

and so, $\limsup_\alpha \|\delta_\alpha\|_\wedge \le \lambda K$, that is, (a) is satisfied.

That (δ_α) satisfies (b) follows immediately from its definition above and the definition of Φ_i. As for (c), just recall that $\Phi_{i(\alpha)}$ is an $\mathcal{F}(X)$-bimodule morphism so $W\cdot\delta_\alpha = \delta_\alpha\cdot W$ whenever $WP_\alpha = P_\alpha W$. By our assumption about (d_i), at the beginning of the proof, it is clear that (d) is satisfied too.

Finally, since $P_\alpha \xrightarrow{s} I_X$ and $P_\alpha^* \xrightarrow{s} I_{X^*}$, we have that

$$W\cdot\delta_\alpha - \delta_\alpha\cdot W = (W - P_\alpha W P_\alpha)\cdot\delta_\alpha + \delta_\alpha\cdot(P_\alpha W P_\alpha - W) \to 0 \quad (W\in\mathcal{A}(X)).$$

Obviously, $(\pi(\delta_\alpha))$ is a BAI for $\mathcal{A}(X)$, so, (δ_α) is an approximate diagonal for $\mathcal{A}(X)$. ∎

Thus, if $\mathcal{A}(X)$ is amenable and has a net of projections as in the lemma, then it has an approximate diagonal whose elements behave themselves like diagonals in a sense that we make more precise in the next definition.

Definition 3.2: Let X and Y be finite-dimensional Banach spaces, and let $\mathcal{A}$ be a subalgebra of $\mathcal{F}(X)$. We call an element $\Delta \in \mathcal{F}(Y,X)\widehat{\otimes}\mathcal{F}(X,Y)$ a generalized diagonal (g.d. in short) for $\mathcal{A}$, if

i) $W\Delta = \Delta W$ $(W\in\mathcal{A})$; and
ii) $\pi(\Delta)W = W$ $(W\in\mathcal{A})$.

It is easily seen that when $\mathcal{A} = \mathcal{F}(X)$, an element $\Delta\in\mathcal{F}(Y,X)\widehat{\otimes}\mathcal{F}(X,Y)$ is a generalized diagonal for $\mathcal{A}$ if and only if there exists an $\mathcal{A}$-bimodule morphism $\rho:\mathcal{A}\to\mathcal{F}(Y,X)\widehat{\otimes}\mathcal{F}(X,Y)$ so that $\pi\circ\rho = I_{\mathcal{A}}$ and $\rho(I_X) = \Delta$. Furthermore, if $(x_k)_{k=1}^m$ and $(y_i)_{i=1}^n$ are bases of X and Y, respectively, then it follows from this last observation, that Δ can be written as

$$\Delta = \sum_{i,j} a_{i,j} \sum_k (y_j^*\otimes x_k)\otimes(x_k^*\otimes y_i), \tag{3}$$

for some scalars $a_{i,j}$ satisfying $\sum_i a_{i,i} = 1$, where, as is customary, the y_j^*'s (resp., the x_k^*'s) denote the biorthogonal functionals associated with the basis $(y_i)_{i=1}^n$ (resp., $(x_k)_{k=1}^m$). Conversely, it can be easily verified that every element of the form (3) is a generalized diagonal for $\mathcal{F}(X)$.

Now the main result of this section is merely a restatement of Proposition 3.1 in terms of generalized diagonals.

COROLLARY 3.3: *Let X be a Banach space. Suppose $\mathcal{A}(X)$ contains a bounded net of projections, $(P_\alpha)_{\alpha\in A}$, such that $P_\alpha \xrightarrow{s} I_X$ and $P_\alpha^* \xrightarrow{s} I_{X^*}$. Set $X_\alpha = \operatorname{rg} P_\alpha$ $(\alpha \in A)$. Then $\mathcal{A}(X)$ is amenable if and only if there is a constant $K > 0$ such that for every $\alpha \in A$ there exists $\beta = \beta(\alpha) \in A$ such that $\mathcal{F}(X_\beta, X_\alpha)\widehat{\otimes}\mathcal{F}(X_\alpha, X_\beta)$ contains a generalized diagonal for $\mathcal{F}(X_\alpha)$ of norm no greater than K.*

Proof. First suppose $\mathcal{A}(X)$ is amenable. By Proposition 3.1, $\mathcal{A}(X)$ has an approximate diagonal, $(\delta_\alpha)_{\alpha\in A}$, satisfying (a)–(d) of the same proposition. For each $\alpha \in A$, let $\beta = \beta(\alpha) \in A$ be as in (d). Let P_α^c (resp., P_β^c) denote the corestriction of P_α (resp., P_β) to its range, and let $\imath_\alpha : X_\alpha \to X$ (resp., $\imath_\beta : X_\beta \to X$) denote the canonical embedding of X_α (resp., X_β) into X. Then define $\Delta_\alpha \in \mathcal{F}(X_\beta, X_\alpha)\widehat{\otimes}\mathcal{F}(X_\alpha, X_\beta)$ by $\Delta_\alpha := \Phi_\alpha(\delta_\alpha)$, where $\Phi_\alpha : \mathcal{A}(X)\widehat{\otimes}\mathcal{A}(X) \to \mathcal{F}(X_\beta, X_\alpha)\widehat{\otimes}\mathcal{F}(X_\alpha, X_\beta)$ is the linear map defined on elementary tensors by $\Phi_\alpha(R\otimes S) := P_\alpha^c R\,\imath_\beta \otimes P_\beta^c S\,\imath_\alpha$ $(R, S \in \mathcal{A}(X))$. It is easy to verify that Δ_α is a g.d. for $\mathcal{F}(X_\alpha)$ $(\alpha \in A)$. The desired conclusion now follows on noting that the family $(\Phi_\alpha)_{\alpha\in A}$ is uniformly bounded.

Conversely, for each $\alpha \in A$, let $\Delta_\alpha \in \mathcal{F}(X_\beta, X_\alpha)\widehat{\otimes}\mathcal{F}(X_\alpha, X_\beta)$ be a g.d. for $\mathcal{F}(X_\alpha)$ of norm $\le K$, and let $\Psi_\alpha : \mathcal{F}(X_\beta, X_\alpha)\widehat{\otimes}\mathcal{F}(X_\alpha, X_\beta) \to \mathcal{A}(X)\widehat{\otimes}\mathcal{A}(X)$ be the linear map defined on elementary tensors by $\Psi_\alpha(U\otimes V) := \imath_\alpha U P_\beta^c \otimes \imath_\beta V P_\alpha^c$ $(U \in \mathcal{F}(X_\beta, X_\alpha),\ V \in \mathcal{F}(X_\alpha, X_\beta))$. Then $(\Psi_\alpha(\Delta_\alpha))$ is an approximate diagonal for $\mathcal{A}(X)$. ∎

Remark 3.4: If X is not a π-space but still X^* has the BAP, as must be the case if $\mathcal{A}(X)$ is amenable [9], then we can argue as follows. First, we choose a net of projections (P_α) in $\mathcal{F}(X)$ such that $P_\alpha \xrightarrow{s} I_X$ and $P_\alpha^* \xrightarrow{s} I_{X^*}$. Such a net, of course, would be necessarily unbounded. Then we choose a bounded net (T_α) in $\mathcal{F}(X)$ such that $P_\alpha T_\alpha = P_\alpha = T_\alpha P_\alpha$ for every α, and set $X_\alpha := \operatorname{rg} T_\alpha$. It can be shown that $\mathcal{A}(X)$ is amenable if and only if there is a constant $K > 0$ such that for every $\alpha \in A$ there exists $\beta = \beta(\alpha) \in A$ such that $\mathcal{F}(X_\beta, X_\alpha)\widehat{\otimes}\mathcal{F}(X_\alpha, X_\beta)$ contains a generalized diagonal for $\mathcal{A}_\alpha = P_\alpha|^{X_\alpha}\mathcal{A}(X)P_\alpha|_{X_\alpha} (\subseteq \mathcal{F}(X_\alpha))$ of norm no greater than K (here $P_\alpha|_{X_\alpha}$ and $P_\alpha|^{X_\alpha}$ denote the restriction and corestriction, respectively, of P_α to X_α).

Remark 3.5: Note that (iii) of the definition of property $\mathbb{A}$ guarantees the existence of a diagonal (and hence a generalized diagonal) for $\mathcal{F}(X_\alpha)$ in $\mathcal{F}(X_\alpha)\widehat{\otimes}\mathcal{F}(X_\alpha)$ whose norm does not exceed K, namely, $\frac{1}{|G_\alpha|}\sum_{T\in G_\alpha} T\otimes T^{-1}$.

Example 3.6: Let (n_k) be an unbounded sequence of positive integers, and let $1\le p\neq q\le\infty$. It is shown in [9, Theorem 6.5] that the algebra $\mathcal{A}\big((\bigoplus_k \ell_p^{n_k})_q\big)$ is amenable. It seems to be unknown whether or not this algebra has property $\mathbb{A}$. However, it is relatively easy to show that this algebra satisfies the condition of Corollary 3.3. Indeed, fix $i\in\mathbb{N}$ and let $m=\max\{n_1,\dots,n_i,i\}$. The algebra $\mathcal{F}\big(\ell_q^m(\ell_p^m)\big)$ has a diagonal Δ_m of norm 1 (see the discussion below). (Furthermore, note that Δ_m can be given explicitly.) As (n_k) is unbounded, there are positive integers $k_1<k_2<\dots<k_m$ so that $m\le\min\{n_{k_j}:1\le j\le m\}$. Clearly, we can think of $(\bigoplus_{j=1}^i \ell_p^{n_j})_q =: X_i$ (resp., $\ell_q^m(\ell_p^m)$) as a 1-complemented subspace of $\ell_q^m(\ell_p^m)$ (resp., $(\bigoplus_{j=1}^{k_m}\ell_p^{n_j})_q =: X_{k_m}$). Let $P_1:\ell_q^m(\ell_p^m)\to X_i$ and $P_2: X_{k_m}\to\ell_q^m(\ell_p^m)$ be the natural projections, and let $\imath_1: X_i\to\ell_q^m(\ell_p^m)$ and $\imath_2:\ell_q^m(\ell_p^m)\to X_{k_m}$ be the corresponding inclusion maps. It is easy to see that the image of Δ_m by the linear map $\mathcal{F}(\ell_q^m(\ell_p^m))\widehat{\otimes}\mathcal{F}(\ell_q^m(\ell_p^m))\to\mathcal{F}(X_{k_m},X_i)\widehat{\otimes}\mathcal{F}(X_i,X_{k_m})$, $R\otimes S\mapsto P_1RP_2\otimes\imath_2S\imath_1$, is a generalized diagonal for $\mathcal{F}(X_i)$ in $\mathcal{F}(X_{k_m},X_i)\widehat{\otimes}\mathcal{F}(X_i,X_{k_m})$ of norm at most 1. The rest is clear.

It can be shown that if X is a Banach space so that $\mathcal{A}(X)$ is K-amenable then $\mathcal{A}(\ell_p^n(X))$ is K-amenable for every $1\le p\le\infty$ and $n\in\mathbb{N}$. Indeed, let $\mathcal{H}$ be the group of permutation matrices generated by a cyclic permutation of the unit vector basis of ℓ_p^n, and let $\mathcal{G}=\{\mathrm{diag}(t)\sigma : t\in\{\pm1\}^n,\sigma\in\mathcal{H}\}$, so $\frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} g\otimes g^{-1}$ is a diagonal for $\mathcal{A}(\ell_p^n)$ [9, Example 3.3]. Let (d_α) be an approximate diagonal for $\mathcal{A}(X)$ of bound K, and choose for each d_α a representation $\sum_j U_{\alpha,j}\otimes V_{\alpha,j}$ such that $\sum_j\|U_{\alpha,j}\|\|V_{\alpha,j}\|\le K$. Then the elements $\delta_\alpha := \frac{1}{|\mathcal{G}|}\sum_{j,g}(g\otimes U_{\alpha,j})\otimes(g^{-1}\otimes V_{\alpha,j})\in\mathcal{A}(\ell_p^n(X))\widehat{\otimes}\mathcal{A}(\ell_p^n(X))$ form an approximate diagonal for $\mathcal{A}(\ell_p^n(X))$ of bound K. Crucial in establishing this last is the fact that the g's are permutation matrices, since it seems that $\ell_p(X)$ is rarely ever a tight tensor product in the sense of [9, Definition 2.1]. This is better exemplified through our next result, which extends Theorem 2.5 of [9].

Proposition 3.7: *Let E be a Banach space with a 1-unconditional basis $\mathbf{e}=(e_n)$. (Recall $E^n=[e_i]_1^n$.) Suppose there is $K>0$ so that for each $m\in\mathbb{N}$ there exists $n\ge m$ such that $\mathcal{F}(E^n,E^m)\widehat{\otimes}\mathcal{F}(E^m,E^n)$ contains a generalized diagonal Δ_m with the following property: there exists a representation*

$\sum_{i=1}^{k} R_{m,i} \otimes S_{m,i}$ *of* Δ_m *such that* $\sum_i \|R_{m,i}\|\|S_{m,i}\| \leq K$ *and the matrix representation of each* $R_{m,i}$ *(resp.,* $S_{m,i}$*) with respect to the* e_i*'s has at most one non-zero entry in each row and column. If* X *is a Banach space such that* $\mathcal{A}(X)$ *is* M*-amenable then* $\mathcal{A}_0(E(X))$ *is* KM*-amenable.*

Proof. Let (d_α) be an approximate diagonal for $\mathcal{A}(X)$ of bound M. For each d_α choose a representation $\sum_j U_{\alpha,j} \otimes V_{\alpha,j}$ such that $\sum_j \|U_{\alpha,j}\|\|V_{\alpha,j}\| \leq M$. We show that the elements $\delta_{m,\alpha} := \sum_{i,j}(R_{m,i} \otimes U_{\alpha,j}) \otimes (S_{m,i} \otimes V_{\alpha,j})$ form an approximate diagonal of bound KM for $\mathcal{A}_0(E(X))$. First, note that $R_{m,i} \otimes U_{\alpha,j} \in \mathcal{A}(E^n(X), E^m(X)) \subset \mathcal{A}_0(E(X))$ and for any set of vectors $x_1, x_2, \ldots, x_n$ in X we have

$$\begin{aligned}\left\|R_{m,i} \otimes U_{\alpha,j}\Big(\sum_{k=1}^{n} e_k \otimes x_k\Big)\right\| &= \left\|\sum_k R_{m,i}(e_k) \otimes U_{\alpha,j}(x_k)\right\| \\ &= \left\|\sum_k \ \|U_{\alpha,j}(x_k)\| R_{m,i}(e_k)\right\| \\ &\leq \|R_{m,i}\|\|U_{\alpha,j}\|\left\|\sum_k e_k \otimes x_k\right\|,\end{aligned}$$

where the second equality follows from the fact that there is at most one non-zero entry in each row and column of the matrix representation of $R_{m,i}$ with respect to the e_i's. Thus $\|R_{m,i} \otimes U_{\alpha,j}\| \leq \|R_{m,i}\|\|U_{\alpha,j}\|$. Likewise, $S_{m,i} \otimes V_{\alpha,j} \in \mathcal{A}(E^m(X), E^n(X)) \subset \mathcal{A}_0(E(X))$ and $\|S_{m,i} \otimes V_{\alpha,j}\| \leq \|S_{m,i}\|\|V_{\alpha,j}\|$. Combining these estimates we readily obtain that $\|\delta_{m,\alpha}\|_\wedge \leq KM$.

In order to verify that $\pi(\delta_{m,\alpha})W \to W$ and $W \cdot \delta_{m,\alpha} - \delta_{m,\alpha} \cdot W \to 0$ $(W \in \mathcal{A}_0(E(X))$ it is clearly enough to look at operators W of the form $E_{rs} \otimes T$ where $E_{rs} = e_r^* \otimes e_s$ and $T \in \mathcal{A}(X)$. The procedures are standard, so we leave the details to the reader. ∎

An immediate consequence of the above is the following.

Corollary 3.8: *Let* (n_k) *be an increasing sequence of positive integers, let* $1 \leq p, q \leq \infty$*, and let* X *be a Banach space such that* $\mathcal{A}(X)$ *is amenable. Then* $\mathcal{A}_0\big((\bigoplus_k \ell_p^{n_k}(X))_q\big)$ *is amenable. In particular, if* $q > 1$ *then* $\mathcal{A}\big((\bigoplus_k \ell_p^{n_k}(X))_q\big)$ *is amenable.*

Proof. The space $(\bigoplus_k \ell_p^{n_k})_q$ satisfies all hypotheses of Theorem 3.7 (see Example 3.6 above). ∎

Corollary 3.3 essentially reduces the study of amenability of algebras of approximable operators on π-spaces to the problem of finding the minimum among the norms of all generalized diagonals for $\mathcal{F}(X)$ in $\mathcal{F}(Y,X)\widehat{\otimes}\mathcal{F}(X,Y)$ with X and Y finite-dimensional. Here, of course, the main difficulty arises in estimating the projective norm. In some cases, this task can be further simplified. For instance, let the basis $(y_i)_{i=1}^n$ of Y be 1-unconditional. Set $p_{i,j} = \sum_k (y_j^* \otimes x_k) \otimes (x_k^* \otimes y_i)$ $(1 \le i,j \le n)$. Then, while looking for generalized diagonals of minimum norm, we can restrict our attention to convex linear combinations of the $p_{i,i}$'s. Indeed, in this case we have that

$$\Big\| \sum_{i,j} a_{i,j}\, p_{j,i} \Big\|_\wedge \ge \Big\| \sum_i a_{i,i}\, p_{i,i} \Big\|_\wedge = \Big\| \sum_i |a_{i,i}|\, p_{i,i} \Big\|_\wedge . \tag{4}$$

To see this, consider the linear operator

$$\Phi : \mathcal{F}(Y,X)\hat{\otimes}\mathcal{F}(X,Y) \to \mathcal{F}(Y,X)\hat{\otimes}\mathcal{F}(X,Y),\ R\otimes S \mapsto 2^{-n} \sum_{t\in\{-1,1\}^n} RU_t \otimes U_t S,$$

where $U_t \in \mathcal{F}(Y)$ is defined by $U_t(y_j) := t_j y_j$ $(1 \le j \le n)$. It is clear that $\|\Phi\| \le 1$, and it is not difficult to see that $\Phi(\sum_{i,j} a_{i,j}\, p_{j,i}) = \sum_i a_{i,i}\, p_{i,i}$, whence the inequality. As for the equality, let $\Lambda \in \mathcal{F}(Y)$ be defined by $\Lambda(y_i) := \lambda_i y_i$, where $\lambda_i = \overline{a_{i,i}}/|a_{i,i}|$ $(1 \le i \le n)$, and let Φ_Λ be the linear map defined by

$$R \otimes S \mapsto R \otimes \Lambda S \quad (R \in \mathcal{F}(Y,X),\ S \in \mathcal{F}(X,Y)).$$

Then Φ_Λ is an isometry and $\Phi_\Lambda(a_{i,i}\, p_{i,i}) = |a_{i,i}|\, p_{i,i}$ $(1 \le i \le n)$, so the equality follows. The claim that we can restrict our attention to 'convex' linear combinations now follows on combining (4) with the fact that the sum of the diagonal coefficients in the representation (3) must be 1.

Remark 3.9: It is not hard to see that the sequence $(p_{i,i})$ has the same basis, unconditional and symmetric constants as the basis (y_i).

It was asked in [9] whether or not the C_p spaces of W. B. Johnson $(1 < p < \infty)$ carry amenable algebras of compact operators. This question has an interesting interpretation in terms of generalized diagonals. We consider the following more general situation.

Let (X_n) be a sequence of finite-dimensional Banach spaces dense in the Banach–Mazur sense in the class of all finite-dimensional Banach spaces, and let $\mathbf{e}$ be an unconditional shrinking Schauder basis. Define $C_{\mathbf{e}} := (\bigoplus_n X_n)_{\mathbf{e}}$.

It is readily seen from Corollary 3.3 that the algebra $\mathcal{A}(C_{\mathbf{e}})$ is amenable if and only if there exists an absolute constant K with the following property:

For every finite-dimensional Banach space X there exists a finite-dimensional Banach space Y so that $\mathcal{F}(Y,X)\widehat{\otimes}\mathcal{F}(X,Y)$ contains a generalized diagonal for $\mathcal{F}(X)$ of norm at most K.

We do not know if one such constant can exist. However, if X is a finite-dimensional Banach space with unconditional constant $< \lambda$ then, by a finite-dimensional version of a well-known result of J. Lindenstrauss [19, Remark 4], there exists a finite-dimensional Banach space Y with symmetric constant $< \lambda$ such that X is a 1-complemented subspace of Y. Thus, $\mathcal{F}(Y)\widehat{\otimes}\mathcal{F}(Y)$ contains a diagonal for $\mathcal{F}(Y)$ of norm $< \lambda$, and in turn $\mathcal{F}(Y,X)\widehat{\otimes}\mathcal{F}(X,Y)$ contains a generalized diagonal for $\mathcal{F}(X)$ of norm $< \lambda$. As a simple consequence of this we quote the following.

Proposition 3.10: *Let (X_n) be a sequence of finite-dimensional Banach spaces with unconditional constant $< \lambda$, dense in the Banach–Mazur sense in the class of all finite-dimensional Banach spaces with unconditional constant $< \lambda$. Let $\mathbf{e}$ be an shrinking 1-unconditional Schauder basis. Then the algebra $\mathcal{A}\big((\bigoplus_n X_n)_{\mathbf{e}}\big)$ has property $\mathbb{A}$.*

We should like to end this section by noting that, there is an analogue of Lindenstrauss's result, due to Johnson, Rosenthal and Zippin, which states that there is a universal constant C ($\leq 16^{12}$) so that every finite-dimensional Banach space is a 1-complemented subspace of a finite-dimensional space with basis constant no greater than C [17, Corollary 4.12(a)].

4. Amenability and equivalence of operator ideal norms

Unfortunately, the characterization found in the previous section is not very helpful when it comes to determine if the algebra of approximable operators on a given Banach space is amenable or not. In this section we take a different approach.

Recall that a Banach space X is called **approximately primary** if, for every projection $P \in \mathcal{B}(X)$, at least one of the product maps

$$\pi : \mathcal{A}(PX,X)\widehat{\otimes}\mathcal{A}(X,PX) \to \mathcal{A}(X)$$

or

$$\pi : \mathcal{A}((I-P)X, X)\widehat{\otimes}\mathcal{A}(X,(I-P)X) \to \mathcal{A}(X)$$

is surjective. This notion was introduced in [9], where it was shown that if $\mathcal{A}(X)$ is amenable then X must be approximately primary. Moreover, also in the same paper (see the proof of [9, Theorem 6.9] and comments after it), it was shown that none of the following spaces is approximately primary: $\ell_p \oplus \ell_q$ $(1 < p, q < \infty$, $p \neq q$ and neither p nor q is equal 2), $\ell_1 \oplus \ell_q$ $(q > 2)$ and $\ell_p \oplus \ell_\infty$ $(p < 2)$.

Essential to the proof of this last result were the following:

Fact 1: *Given Banach spaces X and Y, if the space $X \oplus Y$ is approximately primary, then at least one of the product maps $\pi : \mathcal{A}(Y,X)\widehat{\otimes}\mathcal{A}(X,Y) \to \mathcal{A}(X)$ or $\pi : \mathcal{A}(X,Y)\widehat{\otimes}\mathcal{A}(Y,X) \to \mathcal{A}(Y)$ is surjective.*

Fact 2: *For every Banach space X, the product map $\pi : \mathcal{A}(\ell_p, X)\widehat{\otimes}\mathcal{A}(X,\ell_p) \to \mathcal{A}(X)$ is surjective if and only if the bilinear map $\varphi : \mathcal{A}(\ell_p, X) \times \mathcal{A}(X,\ell_p) \to \mathcal{A}(X)$ is open.*

The results of this section are, to some extent, generalizations of these two facts. We start by recalling some standard terminology.

Let $\mathcal{F}$ be the operator ideal of all finite-rank operators between Banach spaces so, for every pair of Banach spaces (X, Y) we have $\mathcal{F} \cap \mathcal{B}(X,Y) = \mathcal{F}(X,Y)$. Recall that an **operator ideal norm on** $\mathcal{F}$ is a function $\gamma : \mathcal{F} \to [0,\infty[$ that satisfies:

a) $\gamma\big|_{\mathcal{F}(E,F)}$ is a norm for every pair of Banach spaces E and F;
b) $\gamma(I_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}) = 1$;
c) if $A \in \mathcal{B}(Y, Y_0)$, $B \in \mathcal{B}(X_0, X)$ and $T \in \mathcal{F}(X,Y)$ then $\gamma(ATB) \leq \|A\|\gamma(T)\|B\|$.

It is well-known that if γ is as above then $\|T\| \leq \gamma(T)$ for every $T \in \mathcal{F}$. Moreover, if $T = f \otimes x$ then $\|T\| = \|f\|\|x\| = \gamma(T)$ $(x \in X, f \in X^*)$.

In the terminology of [6, §9] the operator ideal $\mathcal{F}$ endowed with an operator ideal norm as in the above definition is a normed operator ideal. Of course, it will not be 'Banach' (i.e., complete). The reason for doing things in this way should become clear later on. Examples of operator ideal norms on $\mathcal{F}$ are the restrictions of the classical operator ideal norms, like nuclear and π-summing norms, to $\mathcal{F}$.

Now the main result of this section reads as follows.

THEOREM 4.1: *Let X and Y be Banach spaces, and let γ and τ be operator ideal norms on $\mathcal{F}$. Suppose that*

i) *$\mathcal{A}(X)$ has a BAI of bound λ;*
ii) *The multiplication $\pi : \mathcal{A}(Y,X)\widehat{\otimes}\mathcal{A}(X,Y) \to \mathcal{A}(X)$ is surjective with inversion constant β;*
iii) *γ and τ are equivalent on one of $\mathcal{F}(Y,X)$ or $\mathcal{F}(X,Y)$, say $c\gamma \leq \tau \leq C\gamma$.*

Then γ and τ are equivalent on $\mathcal{F}(X)$, specifically

$$c\beta^{-2}\lambda^{-2}\gamma \leq \tau \leq C\beta^2\lambda^2\gamma.$$

Proof. Let $F \in \mathcal{F}(X)$ and let (T_α) be a BAI for $\mathcal{A}(X)$ of bound λ. Note that for any operator ideal norm γ we have that $\lim_\alpha \gamma(F - T_\alpha F T_\alpha) = 0$. Indeed, simply write $F = GH$ with $G, H \in \mathcal{F}(X)$. Then

$$\begin{aligned}\gamma(F - T_\alpha F T_\alpha) &= \gamma((G - T_\alpha G)H + T_\alpha G(H - HT_\alpha)) \\ &\leq \|G - T_\alpha G\|\gamma(H) + \lambda\gamma(G)\|H - HT_\alpha\|,\end{aligned}$$

which tends to 0 as $\alpha \to \infty$.

Let $G \in \mathcal{F}(X)$ with $\|G\| \leq \lambda$ and let $L > \beta$. Choose $\sum_i R_i \otimes S_i \in \mathcal{A}(Y,X)\widehat{\otimes}\mathcal{A}(X,Y)$ so that $\sum_i R_i S_i = G$ and $\sum_i \|R_i\|\|S_i\| < \lambda L$. Then for any $F \in \mathcal{F}(X)$ we have that

$$\sum\nolimits_{i,j} \gamma(R_i S_i F R_j S_j) \leq \gamma(F) \sum\nolimits_{i,j} \|R_i S_i\|\|R_j S_j\| \leq \gamma(F) L^2\lambda^2.$$

Moreover,

$$\begin{aligned}&\gamma\Big(\sum_{1\leq i,j\leq n} R_i S_i F R_j S_j - GFG\Big) \\ &\qquad = \gamma\Big(\Big(\sum_1^n R_i S_i - G\Big)F\Big(\sum_1^n R_j S_j\Big) + GF\Big(\sum_1^n R_j S_j - G\Big)\Big) \\ &\qquad \leq \Big\|\sum_1^n R_i S_i - G\Big\|\gamma(F)\lambda L + \|G\|\gamma(F)\Big\|\sum_1^n R_j S_j - G\Big\|,\end{aligned}$$

which tends to 0 as $n \to \infty$. So, the series $\sum_{i,j} R_i S_i F R_j S_j$ is unconditionally γ-convergent in $\mathcal{F}(X)$ to the sum GFG.

Assume that γ and τ are equivalent on $\mathcal{F}(X,Y)$. Then

$$\begin{aligned} c\gamma(GFG) &\leq \sum_{i,j} c\gamma(R_iS_iFR_jS_j) \;\leq\; \sum_{i,j} c\|R_i\|\gamma(S_iF)\|R_jS_j\| \\ &\leq \sum_{i,j} \|R_i\|\|S_i\|\tau(F)\|R_jS_j\| \;\leq\; L^2\lambda^2\tau(F). \end{aligned}$$

Letting $G = T_\alpha$ and $\alpha \to \infty$ we obtain $c\gamma(F) \leq L^2\lambda^2\tau(F)$. Likewise $\tau(F) \leq L^2\lambda^2 C\gamma(F)$. A similar proof working with FR_j rather than S_iF gives the result in case γ and τ are equivalent on $\mathcal{F}(Y,X)$. ∎

We now bring amenability into the picture. We start with the following refinement of [9, Theorem 6.8].

Proposition 4.2: *Let X be a Banach space and let $P : X \to X$ be a bounded projection. Set $Y = \operatorname{rg} P$ and $Z = \operatorname{rg}(I-P)$. If $\mathcal{A}(X)$ is K-amenable then at least one of the maps $\pi_1 : \mathcal{A}(Z,Y)\widehat{\otimes}\mathcal{A}(Y,Z) \to \mathcal{A}(Y)$ or $\pi_2 : \mathcal{A}(Y,Z)\widehat{\otimes}\mathcal{A}(Z,Y) \to \mathcal{A}(Z)$ is surjective with inversion constant no greater than $4K\|P\|\|I-P\|\max\{\|P\|^3, \|I-P\|^3\}$.*

Proof. The proof is almost the same as that of [9, Theorem 6.8], one only needs to keep track of the constants.

Set $P_1 = P$, $P_2 = I - P$, $\mathcal{A} = \mathcal{A}(X)$ and $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$ $(i,j = 1,2)$. Let $A_{ii}^\circ = \pi(\mathcal{A}_{ji}\widehat{\otimes}\mathcal{A}_{ij})$ with the norm $\|\cdot\|^\circ$ inherited from

$$\mathcal{A}_{ji}\widehat{\otimes}\mathcal{A}_{ij}/(\mathcal{A}_{ji}\widehat{\otimes}\mathcal{A}_{ij} \cap \ker \pi)$$

via the natural isomorphism induced by the product map π $(i,j = 1,2,\ i \neq j)$. It is easy to see that $\|a_{ii}a_{ii}^\circ\|^\circ \leq \|a_{ii}\|\|a_{ii}^\circ\|^\circ$ $(a_{ii} \in \mathcal{A}_{ii},\ a_{ii}^\circ \in \mathcal{A}_{ii}^\circ)$, $i = 1,2$. So $\mathcal{A}_{ii}^\circ$ is a Banach $\mathcal{A}_{ii}$-bimodule, and $\mathcal{A}_{ii}$ is a Banach $\mathcal{A}_{ii}^\circ$-bimodule, $i = 1,2$. Let $\mathcal{A}^\circ = \{a \in \mathcal{A} : P_iaP_i \in \mathcal{A}_{ii}^\circ,\ i = 1,2\}$ with the norm $\|a\|^\circ = \max\{\|P_1aP_1\|^\circ, \|P_1aP_2\|, \|P_2aP_1\|, \|P_2aP_2\|^\circ\}$ $(a \in \mathcal{A}^\circ)$. Then $\|aa^\circ\|^\circ \leq M\|a\|\|a^\circ\|^\circ$ and $\|a^\circ a\|^\circ \leq M\|a\|\|a^\circ\|^\circ$ $(a \in \mathcal{A},\ a^\circ \in \mathcal{A}^\circ)$ for some constant $M \leq 2\max\{\|P_1\|^2, \|P_2\|^2\}$, so $(\mathcal{A}^\circ, \|\cdot\|^\circ)$ is a Banach $\mathcal{A}$-bimodule.

The map $D : \mathcal{A} \to \mathcal{A}^\circ$, $a \mapsto P_1aP_2 - P_2aP_1 = P_1a - aP_1$ is a bounded derivation, and so, there is $C \in (\mathcal{A}^\circ)^{**}$ such that $Da = aC - Ca$ $(a \in \mathcal{A})$. Furthermore, we can choose C so that $\|C\|^\circ \leq KM\|D\|$ (see [14, Theorem 1.3]). Let $C_{ii} = P_iCP_i \in (\mathcal{A}_{ii}^\circ)^{**}$ and let $\imath_i : \mathcal{A}_{ii}^\circ \to \mathcal{A}_{ii}$ be the inclusion map $(i = 1,2)$. It can be shown that $a_{ii}(\imath_i^{**}C_{ii}) = \lambda_i a_{ii} = (\imath_i^{**}C_{ii})a_{ii}$ $(a_{ii} \in \mathcal{A}_{ii})$ for some $\lambda_i \in \mathbb{C}$

$(i = 1, 2)$. Moreover, $\lambda_2 - \lambda_1 = 1$ and if $\lambda_i \neq 0$ then $\mathcal{A}_{ii}^{\circ} = \mathcal{A}_{ii}$ and $\imath_i^{**} C_{ii} = C_{ii}$. (See the proof of [9, Theorem 6.8] for details.)

As $\lambda_2 - \lambda_1 = 1$, at least one of λ_1 or λ_2 must have absolute value greater than or equal $1/2$. Without loss of generality, suppose $|\lambda_1| \geq 1/2$, so we have $\mathcal{A}_{11}^{\circ} = \mathcal{A}_{11}$ and $\imath_1^{**} C_{11} = C_{11}$. Let (e_α) be a net in $\mathcal{A}_{11}^{\circ}$ bounded by $\|\lambda_1^{-1} C_{11}\|^{\circ}$ and weak-$*$ convergent to $\lambda_1^{-1} C_{11}$. Since $a_{11}^{\circ}(\lambda_1^{-1} C_{11}) = a_{11}^{\circ} = (\lambda_1^{-1} C_{11}) a_{11}^{\circ}$, it is readily seen that $e_\alpha a_{11}^{\circ} \to a_{11}^{\circ}$ and $a_{11}^{\circ} e_\alpha \to a_{11}^{\circ}$ weakly for every $a_{11}^{\circ} \in \mathcal{A}_{11}^{\circ}$. A standard argument (see [5, Proposition 2.9.14 (iii)]) shows that $\mathcal{A}_{11}^{\circ}$ has BAI of bound $\|\lambda_1^{-1} C_{11}\|^{\circ}$.

Now let $a_{11} \in \mathcal{A}_{11}$ be arbitrary. It is easy to see that $\mathcal{A}_{11}$ is an essential $\mathcal{A}_{11}^{\circ}$-bimodule, so, by [5, Theorem 2.9.24], there exist $e^{\circ} \in \mathcal{A}_{11}^{\circ}$ and $b \in \mathcal{A}_{11}$ such that $a_{11} = e^{\circ} b$, $\|e^{\circ}\|^{\circ} \leq \|\lambda_1^{-1} C_{11}\|^{\circ}$ and $\|b\| \leq \|a_{11}\|$. Thus,

$$\|a_{11}\|^{\circ} \leq \|e^{\circ}\|^{\circ} \|b\| \leq \|\lambda_1^{-1} C_{11}\|^{\circ} \|a_{11}\| \leq 2\|C\|^{\circ} \|a_{11}\| \leq 2KM\|D\| \|a_{11}\|.$$

As $\|P_1 a P_2 - P_2 a P_1\|^{\circ} = \max\{\|P_1 a P_2\|, \|P_2 a P_1\|\} \leq \|P_1\| \|P_2\| \|a\|$ $(a \in \mathcal{A})$, we find that $\|D\| \leq \|P_1\| \|P_2\|$, so

$$\|a_{11}\|^{\circ} \leq 2KM\|P_1\| \|P_2\| \|a_{11}\| \quad (a_{11} \in \mathcal{A}_{11}).$$

To finish the proof of the proposition, one just needs to note that the linear isomorphisms $\mathcal{A}_{21} \widehat{\otimes} \mathcal{A}_{12} \to \mathcal{A}(Z, Y) \widehat{\otimes} \mathcal{A}(Y, Z)$, $R \otimes S \mapsto R|_Z^Y \otimes S|_Y^Z$, and $\mathcal{A}(Y) \to \mathcal{A}_{11}$, $T \mapsto \imath T P_1$, where $\imath : Y \to X$ denotes the inclusion map, have norms no greater than 1 and $\|P_1\|$, respectively. Combining these two last estimates with those previously found, we finally obtain that the inversion constant of π_1 cannot be greater than $2KM\|P_1\| \|P_2\| \max\{\|P_1\|, \|P_2\|\}$, as claimed. ∎

Combining Proposition 4.2 and Theorem 4.1 we obtain the following.

Corollary 4.3: *Let γ and τ be operator ideal norms on $\mathcal{F}$. Let X be a Banach space such that $\mathcal{A}(X)$ is K-amenable and let $P : X \to X$ be a bounded projection. Set $Y = \operatorname{rg} P$ and $Z = \operatorname{rg}(I - P)$. If γ and τ are equivalent on one of $\mathcal{F}(Y, Z)$ or $\mathcal{F}(Z, Y)$, say $c\gamma \leq \tau \leq C\gamma$, then we must have $c\kappa^{-2}\gamma \leq \tau \leq C\kappa^2\gamma$ on one of $\mathcal{F}(Y)$ or $\mathcal{F}(Z)$, for some $\kappa \leq 4K^2\|P\| \|I - P\| \max\{\|P\|^4, \|I - P\|^4\}$.*

Proof. By Proposition 4.2, at least one of the product maps

$$\pi_1 : \mathcal{A}(Y, Z) \widehat{\otimes} \mathcal{A}(Z, Y) \to \mathcal{A}(Z) \quad \text{or} \quad \pi_2 : \mathcal{A}(Z, Y) \widehat{\otimes} \mathcal{A}(Y, Z) \to \mathcal{A}(Y)$$

is onto with inversion constant no greater than

$$4K\|P\| \|I - P\| \max\{\|P\|^3, \|I - P\|^3\}.$$

To fix ideas, suppose $\pi_1 : \mathcal{A}(Y,Z)\widehat{\otimes}\mathcal{A}(Z,Y) \to \mathcal{A}(Y)$ is onto. As $\mathcal{A}(X)$ is K-amenable it has a BAI of bound K, and so, $\mathcal{A}(Y)$ has a BAI of bound $K\|P\|$. Now one just needs to apply Theorem 4.1. ∎

The estimate for κ given in the last corollary is very unlikely to be sharp. However, to the effects of the present paper, the significant fact about it is that it depends only on the amenability constant and the given projection. The importance of this fact will be fully appreciated in Section 5 when we prove the non-amenability of $\mathcal{A}(T)$ for T the Tsirelson's space.

Following are some important consequences of Corollary 4.3.

In what follows, we denote by Γ_p $(1 \le p \le \infty)$ the operator ideal of all bounded linear maps between Banach spaces that factor through ℓ_p endowed with the operator ideal norm

$$\gamma_p(T : X \to Y) := \inf\left\{\|R\|\|S\| : X \xrightarrow{S} \ell_p \xrightarrow{R} Y \text{ and } RS = T\right\}.$$

Recall also that a Banach space X is said to be of cotype 2 if there exists a constant C such that, for all finite subsets $\{x_1, x_2, \ldots, x_n\}$ of X, we have

$$\Big(\sum_i \|x_i\|^2\Big)^{1/2} \le C2^{-n} \sum_{t\in\{-1,1\}^n} \Big\|\sum_i t_i x_i\Big\|.$$

Corollary 4.4: *Let X and Y be infinite-dimensional Banach spaces with the λ-BAP. If none of them is isomorphic to a Hilbert space and if X^* and Y are both of cotype 2 then $\mathcal{A}(X \oplus Y)$ is not amenable.*

Proof. Since X^* and Y are both of cotype 2, by Pisier's abstract version of Grothendieck's inequality [22, Theorem 4.1], we have that $\mathcal{B}(X,Y) = \Gamma_2(X,Y)$, and hence that $\|\,.\,\| \sim \gamma_2$ on $\mathcal{F}(X,Y)$. Suppose towards a contradiction that $\mathcal{A}(X\oplus Y)$ is amenable. By Corollary 4.3, either $\|\,.\,\| \sim \gamma_2$ on $\mathcal{F}(X)$ or $\|\,.\,\| \sim \gamma_2$ on $\mathcal{F}(Y)$. Suppose $\|\,.\,\| \sim \gamma_2$ on $\mathcal{F}(X)$, so, for some constant M we have $\gamma_2(T) \le M\|T\|$ $(T \in \mathcal{F}(X))$. Let $E \subset X$ be a finite-dimensional subspace. By [6, §16.9, Corollary], there exists $T_E \in \mathcal{F}(X)$ such that $T_E(x) = x$ $(x \in E)$ and $\|T_E\| \le \lambda + 1$. Let $\imath_E : E \to X$ be the inclusion map. Then we have

$$\gamma_2(I\big|_E) = \gamma_2(T_E\imath_E) \le \gamma_2(T_E)\|\imath_E\| \le M\|T_E\| \le M(\lambda+1).$$

This last holds for any E, so, $\sup_E \gamma_2(I\big|_E) \le M(\lambda+1)$. By [20, Proposition 5.2], $\gamma_2(I) \le M(\lambda+1)$, i.e., X is isomorphic to a Hilbert space, contrary to assumption.

Analogously, if $\|\,.\,\| \sim \gamma_2$ on $\mathcal{F}(Y)$, we find that Y must be isomorphic to a Hilbert space, contradicting the hypotheses once again. Thus, $\mathcal{A}(X \oplus Y)$ cannot be amenable. ∎

Let $1 < p < \infty$. The p-th James space, $\mathfrak{J}_p$, is the completion of the linear space of complex sequences with finite support in the norm

$$\|(\alpha_n)\|_{\mathfrak{J}_p} = \sup\Big\{ \Big(\sum_{n=1}^{m-1} |\alpha_{i_n} - \alpha_{i_{n+1}}|^p \Big)^{1/p} : m, i_1, \ldots, i_m \in \mathbb{N}, \\ m \geq 2 \text{ and } i_1 < \ldots < i_m \Big\}.$$

It is unknown if $\mathcal{A}(\mathfrak{J}_p)$ is amenable for any p. However, as a consequence of Corollary 4.4 we have the following.

Corollary 4.5: *The algebra $\mathcal{A}(\mathfrak{J}_p \oplus \mathfrak{J}_p^*)$ is not amenable for any $p \in [2, \infty[$.*

Proof. By [23, Theorem 1], $\mathfrak{J}_p^*$ has cotype 2 and neither $\mathfrak{J}_p$ nor $\mathfrak{J}_p^*$ is isomorphic to a Hilbert space. So we can apply Corollary 4.4. ∎

Recall from [20] that a Banach space X is said to be an $\mathcal{L}_p$-space if it contains a net (X_α) of finite-dimensional subspaces, directed by inclusion, whose union is dense in X, and such that $\sup_\alpha d(X_\alpha, \ell_p^{\dim X_\alpha}) < \infty$.

Our next result generalizes [9, Theorem 6.9].

Corollary 4.6: *Let X be an $\mathcal{L}_p$-space, and let Y be an $\mathcal{L}_q$-space, where $1 \leq p, q \leq \infty$. Then $\mathcal{A}(X \oplus Y)$ is amenable if and only if one of the following holds:*

a) *$p = q$.*
b) *$p = 2$ and $1 < q < \infty$.*
c) *$1 < p < \infty$ and $q = 2$.*

Proof. Since the direct sum of two $\mathcal{L}_p$-spaces (resp., of an $\mathcal{L}_p$-space with $1 < p < \infty$ and an $\mathcal{L}_2$-space) is an $\mathcal{L}_p$-space, and the algebra of approximable operators on an $\mathcal{L}_p$-space is always amenable [9, Theorem 6.4], it is clear that if (a) (resp., (b) or (c)) is satisfied then $\mathcal{A}(X \oplus Y)$ is amenable.

Now suppose that none of (a), (b) or (c) is satisfied. We want to show that $\mathcal{A}(X \oplus Y)$ is not amenable. By [21, Theorem III(a)] and [9, Corollary 5.5], it is enough to consider the following two cases: (i) $p < 2 \leq q$, and (ii) $p < q < 2$. The case (i) follows from Corollary 4.4 above since for $p < 2$ (resp., $2 \leq q$) an $\mathcal{L}_p$-space (resp., the dual of an $\mathcal{L}_q$-space) has cotype 2. In dealing with the second case we use the following result from [18], that we state as in [6, §26.5. Corollary 2]:

THEOREM: *(Kwapień). Let* $1 \le p \le r \le q \le \infty$. *Then* $\mathcal{B}(\ell_q, \ell_p) = \Gamma_r(\ell_q, \ell_p)$.

Let $p < q < 2$ *and let* $r \in \,]p, q[$. *Using Kwapień's theorem and* [21, Theorem III(c)] *it can be shown that there exists a constant* M *so that*

$$\sup\{\gamma_r(T|_E) : E \subset Y \text{ a finite-dimensional subspace}\} \le M\|T\| \quad (T \in \mathcal{B}(Y, X)).$$

By [22, Corollary 8.9], there is an L_r-space so that $\mathcal{B}(Y, X) = \Gamma_{L_r}(Y, X)$, where $\Gamma_{L_r}(Y, X)$ denotes the space of all operators from X to Y that factor through L_r with the norm

$$\gamma_{L_r}(T) := \inf\{\|R\|\|S\| : X \xrightarrow{S} L_r \xrightarrow{R} Y \text{ and } RS = T\}.$$

Assume towards a contradiction that $\mathcal{A}(X \oplus Y)$ is amenable. Then, by Corollary 4.3, either $\gamma_{L_r} \sim \|\,.\,\|$ on $\mathcal{F}(X)$ or $\gamma_{L_r} \sim \|\,.\,\|$ on $\mathcal{F}(Y)$. We show that none of these can happen. Indeed, suppose, to fix ideas, that $\gamma_{L_r} \sim \|\,.\,\|$ on $\mathcal{F}(X)$. Then, by [21, Theorem 4.3], $\gamma_{L_r}(I_X) < \infty$, and so, X is isomorphic to a complemented subspace of an L_r-space which, by [21, Theorem III(b)], must be an $\mathcal{L}_r$-space. But this is impossible since $p \ne r$. Analogously, if $\gamma_{L_r} \sim \|\,.\,\|$ on $\mathcal{F}(Y)$, we find that Y is an $\mathcal{L}_r$-space as well as an $\mathcal{L}_q$-space reaching again the same absurd. Thus, neither $\gamma_{L_r} \sim \|\,.\,\|$ on $\mathcal{F}(X)$ nor $\gamma_{L_r} \sim \|\,.\,\|$ on $\mathcal{F}(Y)$. It follows that $\mathcal{A}(X \oplus Y)$ cannot be amenable and this concludes the proof. ∎

Remark 4.7: It should be noted that the argument of [9, Theorem 6.9] can be extended without difficulty to cover the more general situation of Corollary 4.6 when $1 < p, q < \infty$.

We now turn our attention to the second fact mentioned at the beginning of this section, namely, the equivalence between surjectivity of

$$\mathcal{A}(\ell_p, X)\widehat{\otimes}\mathcal{A}(X, \ell_p) \to \mathcal{A}(X)$$

and openness of

$$\mathcal{A}(\ell_p, X) \times \mathcal{A}(X, \ell_p) \to \mathcal{A}(X).$$

It is not hard to see that the reason why this last holds is that $\ell_p \cong \ell_p(\ell_p)$ $(1 \le p \le \infty)$, or more precisely, because γ_p, being a norm, must satisfy the triangle inequality. In what remains of this section we look at this in more detail.

Let Z be an infinite-dimensional Banach space. Given any pair of Banach spaces (X,Y) we let

$$\gamma_Z(T) := \inf\left\{\|R\|\|S\| : RS = T,\ S \in \mathcal{F}(X,Z) \text{ and } R \in \mathcal{F}(Z,Y)\right\} \quad (T \in \mathcal{F}(X,Y)).$$

In general, γ_Z need not be a norm on $\mathcal{F}(X,Y)$. For example, let $Z_n = \ell_p^n \oplus \ell_2$ for some $p \in (2,\infty)$ fixed, let $I : \ell_p^{2n} \to \ell_p^{2n}$ be the identity map, and let P_1 (resp., P_2) be the natural projection onto the first (resp., last) n coordinates. Then $\gamma_{Z_n}(P_1 + P_2)$ tends to ∞ with n while $\gamma_{Z_n}(P_1) + \gamma_{Z_n}(P_2) = 2$ for all n. Indeed, suppose towards a contradiction that $\gamma_{Z_n}(P_1 + P_2) < C$ for some constant C independent of n. Then for every n there is $E_n \subset \ell_p^n \oplus \ell_2$ and a linear isomorphism $T_n : \ell_p^{2n} \to E_n$ such that $\|T_n\|\|T_n^{-1}\| < C$. Let Q_n be the natural projection from $\ell_p^n \oplus \ell_2$ onto ℓ_p^n, and let $(x_{n,i})$ be a basis for E_n. Without loss of generality, let $Q_n(x_{n,1}),\ldots,Q_n(x_{n,m})$ be a maximal subset of linearly independent vectors from $\left\{Q_n(x_{n,i}) : 1 \le i \le 2n\right\}$, so $m \le n$. Taking linear combinations if necessary, we can pass to a new basis of E_n, $x_{n,1},\ldots,x_{n,m},y_{n,m+1},\ldots,y_{n,2n}$, in which each $y_{n,i} \in \ell_2$. Thus, E_n contains an isometric copy of ℓ_2^n and in turn ℓ_p^{2n} contains a C-isomorphic copy of ℓ_2^n. But this last should hold for every n, which is impossible by [8, Example 3.1]. Thus, for big enough n, γ_{Z_n} is not a norm.

Let us say that the Banach space Z has the **factorization-norm property** if for every pair of Banach spaces, (X,Y), γ_Z is a norm on $\mathcal{F}(X,Y)$. It is easily verified that if Z has the factorization-norm property then γ_Z is an operator ideal norm on $\mathcal{F}$. Also note from the example of the previous paragraph that the factorization-norm property is an isometric property.

COROLLARY 4.8: *Let X be an infinite dimensional Banach space such that the algebra $\mathcal{A}(X)$ is K-amenable. Let P be a bounded projection on X. Set $Y = \operatorname{rg} P$ and $Z = \operatorname{rg}(I-P)$. If both, Y and Z, have the factorization-norm property, then at least one of the maps*

$$\varphi_Y : \mathcal{F}(Z,Y) \times \mathcal{F}(Y,Z) \to \mathcal{F}(Y) \quad \text{or} \quad \varphi_Z : \mathcal{F}(Y,Z) \times \mathcal{F}(Z,Y) \to \mathcal{F}(Z),$$

is M-open for some constant $M \le \kappa^2 K^2 \|P\|\|I-P\|$ with κ as in Corollary 4.3.

Proof. Since $\mathcal{A}(X)$ is K-amenable it has a BAI of bound K. In turn, $\mathcal{A}(Y)$ has a BAI of bound $C = K\|P\|$ and $\mathcal{A}(Z)$ has a BAI of bound $c^{-1} = K\|I-P\|$. As $\mathcal{A}(Y,Z)$ is an essential Banach left $\mathcal{A}(Z)$-module and an essential Banach right

$\mathcal{A}(Y)$-module we have, by [5, Theorem 2.9.24], that $\gamma_Y \leq C\|\cdot\|$ and $\gamma_Z \leq c^{-1}\|\cdot\|$ on $\mathcal{F}(Y,Z)$, so $c\gamma_Z \leq \gamma_Y \leq C\gamma_Z$ on $\mathcal{F}(Y,Z)$. Thus, by Corollary 4.3, we should have $\kappa^{-2}c\gamma_Z \leq \gamma_Y \leq \kappa^2 C\gamma_Z$ on at least one of $\mathcal{F}(Y)$ or $\mathcal{F}(Z)$ for some constant κ. To fix ideas, suppose $\kappa^{-2}c\gamma_Z \leq \gamma_Y \leq \kappa^2 C\gamma_Z$ holds on $\mathcal{F}(Y)$. Since $\mathcal{A}(Y)$ is an essential $\mathcal{A}(Y)$-module we have, once again by [5, Theorem 2.9.24], that $\gamma_Y \leq C\|\cdot\|$ on $\mathcal{F}(Y)$. This last combined with $\kappa^{-2}c\gamma_Z \leq \gamma_Y$ gives that $\gamma_Z \leq M\|\cdot\|$ on $\mathcal{F}(Y)$, where $M \leq \kappa^2 C/c$, as desired. The case where $\kappa^{-2}c\gamma_Z \leq \gamma_Y \leq \kappa^2 C\gamma_Z$ holds on $\mathcal{F}(Z)$ is treated analogously. ∎

It seems difficult, in general, to determine whether or not a given Banach space has the factorization-norm property. It is well-known, for instance, that any Banach space Z such that $Z \cong \ell_p(Z)$, in particular, any Banach space of the form $\ell_p(E)$, where E is some Banach space and $1 \leq p \leq \infty$, has the factorization-norm property (see [15, Proposition 1]).

The following proposition is analogous to [15, Proposition 1]. It gives a sufficient condition for a Banach space to have the factorization-norm property.

Proposition 4.9: *Let Z be an infinite dimensional Banach space such that for every finite-dimensional subspace E of Z and every $\varepsilon > 0$ there exist finite-dimensional subspaces F and G of Z such that $E \subset F$, G is $(1+\varepsilon)$-complemented in Z, and for some set of vectors $\{u_1, u_2\}$ forming a 1-unconditional basis of their $\mathbb{R}$-linear span we have that $d(G, F\oplus_u F) \leq 1+\varepsilon$, where $F\oplus_u F$ denotes the direct sum of two copies of F endowed with the norm $\|(x,y)\| := \big\|\|x\|_Z\, u_1 + \|y\|_Z\, u_2\big\|$ $((x,y) \in F \oplus F)$. Then Z has the factorization-norm property.*

Proof. Of course, only the triangle inequality needs to be verified. For this, let (X,Y) be a pair of Banach spaces, let $T_1, T_2 \in \mathcal{F}(X,Y)$, and let $\varepsilon > 0$ be arbitrary. Let $S_i \in \mathcal{F}(X,Z)$ and $R_i \in \mathcal{F}(Z,Y)$ be such that $R_iS_i = T_i$ and $\|R_i\|\|S_i\| \leq \gamma_Z(T_i) + \varepsilon$ $(i = 1,2)$. Set $E = S_1X + S_2X \subset Z$ and let F and G be finite-dimensional subspaces of Z as in the hypotheses. Let $L : G \to F \oplus_u F$ be a linear isomorphism such that $\|L\|\|L^{-1}\| \leq 1+\varepsilon$, and let $P_G : Z \to G$ be a projection onto G of norm $\leq 1+\varepsilon$. Let $\overline{S} : X \to G$, $x \mapsto L^{-1}(S_1x, S_2x)$, and let $\overline{R} : Z \to Y$, $z \mapsto (R_1P_1 + R_2P_2)LP_Gz$, where P_1 and P_2 denote the canonical coordinate projections onto the first and second components of $F\oplus_u F$, respectively.

It is easily seen that $T_1 + T_2 = \overline{R}\,\overline{S}$, that

$$\|\overline{S}\| \leq \|L^{-1}\|\big\|\|S_1\|u_1 + \|S_2\|u_2\big\|,$$

and that

$$\|\overline{R}\| \leq (1+\varepsilon)\|L\|\big\|\|R_1\|u_1^* + \|R_2\|u_2^*\big\|,$$

where u_1^*, u_2^* is the basis dual to u_1, u_2. By [16, Main Lemma], we can assume that

$$\big\|\|S_1\|u_1 + \|S_2\|u_2\big\|\big\|\|R_1\|u_1^* + \|R_2\|u_2^*\big\| = \|R_1\|\|S_1\| + \|R_2\|\|S_2\|.$$

Then

$$\begin{aligned}\gamma_Z(T_1+T_2) \leq \|\overline{R}\|\|\overline{S}\| \ &\leq \ (1+\varepsilon)^2(\|R_1\|\|S_1\| + \|R_2\|\|S_2\|)\\ &\leq (1+\varepsilon)^2(\gamma_Z(T_1) + \gamma_Z(T_2) + 2\varepsilon).\end{aligned}$$

Since ε is arbitrary the desired conclusion follows. ∎

All the following Banach spaces are easily seen to satisfy the condition of Proposition 4.9 and hence have the factorization-norm property.

Example 4.10: Any Banach space Z with a 1-unconditional Schauder basis $\mathbf{z} = (z_n)$ such that (i) $\liminf_n d\big([z_i]_1^n, [z_i]_{n+1}^{2n}\big) = 1$; and (ii) $\|\sum_{i=1}^{2n} a_i z_i\| = \|\sum_{i=1}^{2n} b_i z_i\|$ for all scalar sequences $a_1, a_2, \ldots, a_{2n}$ and $b_1, b_2, \ldots, b_{2n}$ such that $\|\sum_{i=1}^{n} a_i z_i\| = \|\sum_{i=1}^{n} b_i z_i\|$ and $\|\sum_{i=n+1}^{2n} a_i z_i\| = \|\sum_{i=n+1}^{2n} b_i z_i\|$ $(n \in \mathbb{N})$.

Example 4.11: Any Banach space of the form $(\bigoplus_k E_k)_{\mathbf{z}}$, where $\mathbf{z}$ is as in the previous example and $E_k = E$ $(k \in \mathbb{N})$ for some Banach space E.

Example 4.12: Any *Johnson space* in the sense of [1, Definition 3.1].

It is unclear, however, whether or not the factorization-norm property is an essential hypothesis in Corollary 4.8. In fact, the following argument suggests that the same conclusion or at least a similar one might hold without this assumption.

Let $1 < p \leq \infty$ and let X be an infinite dimensional Banach space such that $\mathcal{A}(X)$ is K-amenable. Then $\mathcal{A}(\ell_p(X))(= \mathcal{A}_0(\ell_p(X)))$ is K-amenable as well. Let $P : X \to X$ be a bounded projection. Set $Y = \operatorname{rg} P$ and $Z = \operatorname{rg}(I-P)$. As $\ell_p(Y)$ and $\ell_p(Z)$ have the factorization-norm property and $\|\bigoplus_{i=1}^{\infty} P : \ell_p(X) \to \ell_p(Y)\| = \|P\|$ there is, by Corollary 4.8, a constant M, depending only on K and P, so that, at least one of the maps,

$$\varphi_{\ell_p(Y)} : \mathcal{F}(\ell_p(Z), \ell_p(Y)) \times \mathcal{F}(\ell_p(Y), \ell_p(Z)) \to \mathcal{F}(\ell_p(Y)),$$

or

$$\varphi_{\ell_p(Z)} : \mathcal{F}(\ell_p(Y), \ell_p(Z)) \times \mathcal{F}(\ell_p(Z), \ell_p(Y)) \to \mathcal{F}(\ell_p(Z)),$$

is M-open. To fix ideas, suppose $\varphi_{\ell_p(Y)}$ is M-open. Then

$$\varphi_Y : \mathcal{F}(\ell_p(Z), Y) \times \mathcal{F}(Y, \ell_p(Z)) \to \mathcal{F}(Y),$$

is M-open too.

Remark 4.13: The fact that at least one of the maps $\varphi_{\ell_p(Y)}$ or $\varphi_{\ell_p(Z)}$ above is M-open if $\mathcal{A}(X)$ is amenable, still holds for $p = 1$, but this case needs to be treated separately as $\mathcal{A}(\ell_1(X)) \neq \mathcal{A}_0(\ell_1(X))$ (see Lemma 5.4 below).

Now let (T_α) be a BAI for $\mathcal{F}(Y)$. Then, by the above, we have that for every $1 \leq p \leq \infty$ and every α there are operators $S_{\alpha,p} : Y \to \ell_p(Z)$ and $R_{\alpha,p} : \ell_p(Z) \to Y$ such that $R_{\alpha,p} S_{\alpha,p} = T_\alpha$ and $\|R_{\alpha,p}\|\|S_{\alpha,p}\| \leq M\|T_\alpha\|$. The fact that M is independent of α and p suggests the following might be true:

There exist $k \in \mathbb{N}$ and a positive constant $\widetilde{M}$ such that for every index α, there are operators $R_\alpha : \bigoplus_{i=1}^k Z \to Y$ and $S_\alpha : Y \to \bigoplus_{i=1}^k Z$ so that $R_\alpha S_\alpha = T_\alpha$ and $\|R_\alpha\|\|S_\alpha\| \leq \widetilde{M}\|T_\alpha\|$, that is, the product map

$$\varphi : \mathcal{F}\Big(\bigoplus_{i=1}^k Z, Y\Big) \times \mathcal{F}\Big(Y, \bigoplus_{i=1}^k Z\Big) \to \mathcal{F}(Y)$$

is $\widetilde{M}$-open.

5. Tsirelson-like spaces

As announced earlier, in the final section of this paper we establish the non-amenability of the algebra of approximable operators on the Tsirelson space. In fact, we shall obtain this as a consequence of a more general result (see Theorem 5.6 below).

We start with a definition. It is closely related to the old notion of crude finite representability introduced in [12].

Definition 5.1: A Banach space Y is said to be M**-crudely** π**-finitely representable** in a Banach space Z if for every finite-dimensional subspace E of Y, there exist a finite-rank projection $P : Y \to Y$ whose range contains E, and operators $S : Y \to Z$ and $R : Z \to Y$ such that $RS = P$ and $\|R\|\|S\| \leq M$.

This last definition is justified by the following.

LEMMA 5.2: *Let Y be a π_1-space and let Z be a Banach space. Then all the following are equivalent:*

a) *Y is $(M+\varepsilon)$-crudely π-finitely representable in Z for every $\varepsilon > 0$.*
b) *$\mathcal{F}(Z,Y) \times \mathcal{F}(Y,Z) \to \mathcal{F}(Y)$ is $(M+\varepsilon)$-open for every $\varepsilon > 0$.*
c) *For every $\varepsilon > 0$ there exists in $\mathcal{F}(Y)$ a bounded net of projections, (P_α), converging strongly to the identity operator on Y, and such that $\sup_\alpha \gamma_Z(P_\alpha) \le M+\varepsilon$, that is, such that Z contains $P_\alpha Y$'s uniformly $(M+\varepsilon)$-complemented.*

Proof. It is easy to see that (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c). That (c) $\Rightarrow$ (a) follows from [17, Lemma 2.4]. ∎

Remark 5.3: If we simply assume in the last lemma that Y is a π_λ-space, then still (a) $\iff$ (c) and (a) $\Rightarrow$ (b), but (b) implies that Y is $(\lambda M+\varepsilon)$-crudely π-finitely representable in Z for every $\varepsilon > 0$.

Before passing to the main result of the section we need another result that we collect as a lemma.

Let E, F be Banach spaces and let $1 \le p \le \infty$. Recall that we have identified ℓ_p^m with the linear span of the first m vectors of the unit vector basis of ℓ_p, so we have a natural linear isometry $\mathcal{A}(\ell_p^m(E), \ell_p^k(F)) \hookrightarrow \mathcal{A}(\ell_p^n(E), \ell_p^l(F))$, whenever $n \ge m$ and $l \ge k$. We denote by $\mathcal{A}_0(\ell_p(E), \ell_p^k(F))$ (resp., $\mathcal{A}_0(\ell_p^m(E), \ell_p(F))$) the inductive limit of the direct system formed by the spaces $\mathcal{A}(\ell_p^m(E), \ell_p^k(F))$ ($m \in \mathbb{N}$) (resp., $\mathcal{A}(\ell_p^m(E), \ell_p^k(F))$ ($k \in \mathbb{N}$)) together with the corresponding isometric embeddings. There are also natural linear isometries $\mathcal{A}_0(\ell_p(E), \ell_p^k(F)) \hookrightarrow \mathcal{A}_0(\ell_p(E), \ell_p^l(F))$ and $\mathcal{A}_0(\ell_p^m(E), \ell_p(F)) \hookrightarrow \mathcal{A}_0(\ell_p^n(E), \ell_p(F))$ ($l \ge k$, $n \ge m$). We denote by $\mathcal{A}_0(\ell_p(E), \ell_p(F))$ the common inductive limit of the direct systems formed by $\{\mathcal{A}_0(\ell_p(E), \ell_p^k(F)) : k \in \mathbb{N}\}$ and $\{\mathcal{A}_0(\ell_p^m(E), \ell_p(F)) : m \in \mathbb{N}\}$ and their respective families of isometric embeddings. It is not hard to see that $\mathcal{A}_0(\ell_p(E), \ell_p(F))$ is also the inductive limit of the direct system formed by all spaces $\mathcal{A}_0(\ell_p^m(E), \ell_p^k(F))$ and the isometric embeddings $\mathcal{A}(\ell_p^m(E), \ell_p^k(F)) \hookrightarrow \mathcal{A}(\ell_p^n(E), \ell_p^l(F))$ ($n \ge m$, $l \ge k$).

LEMMA 5.4: *Let $1 \le p \le \infty$ and let X be a Banach space such that $\mathcal{A}(X)$ is K-amenable. Let $P : X \to X$ be a bounded projection. Set $Y = \operatorname{rg} P$ and $Z = \operatorname{rg}(I-P)$. Then at least one of the maps*

$$\varphi_1 : \mathcal{A}_0(\ell_p(Z), \ell_p(Y)) \times \mathcal{A}_0(\ell_p(Y), \ell_p(Z)) \to \mathcal{A}_0(\ell_p(Y)),$$

or

$$\varphi_2 : \mathcal{A}_0(\ell_p(Y), \ell_p(Z)) \times \mathcal{A}_0(\ell_p(Z), \ell_p(Y)) \to \mathcal{A}_0(\ell_p(Z)),$$

is $(M+\delta)$-open for every $\delta > 0$ and some constant M that depends only on K and $\|P\|$.

Proof. Let X be a Banach space such that $\mathcal{A}(X)$ is K-amenable. Then $\mathcal{A}(\ell_p^n(X))$ is K-amenable for every $n \in \mathbb{N}$ and every $1 \le p \le \infty$. Let P, Y and Z be as in the hypotheses. As $\| \bigoplus_1^n P : \ell_p^n(X) \to \ell_p^n(Y)\| = \|P\|$ $(n \in \mathbb{N})$, there exists, by Proposition 4.2, a constant $M = M(K, \|P\|)$ so that for each $n \in \mathbb{N}$ at least one of the product maps

$$\pi_{1,n} : \mathcal{A}(\ell_p^n(Z), \ell_p^n(Y)) \widehat{\otimes} \mathcal{A}(\ell_p^n(Y), \ell_p^n(Z)) \to \mathcal{A}(\ell_p^n(Y)), \tag{5}$$

or

$$\pi_{2,n} : \mathcal{A}(\ell_p^n(Y), \ell_p^n(Z)) \widehat{\otimes} \mathcal{A}(\ell_p^n(Z), \ell_p^n(Y)) \to \mathcal{A}(\ell_p^n(Z)), \tag{6}$$

is onto with inversion constant no greater than M.

Without loss of generality, assume there is an increasing sequence of positive integers, (n_k), so that π_{1,n_k} is onto with inversion constant no greater than M for all k. Fix $k \in \mathbb{N}$, let $\varepsilon > 0$ and let $T \in \mathcal{A}(\ell_p^{n_k}(Y))$. There is $\sum_i R_i \otimes S_i \in \mathcal{A}(\ell_p^{n_k}(Z), \ell_p^{n_k}(Y)) \widehat{\otimes} \mathcal{A}(\ell_p^{n_k}(Y), \ell_p^{n_k}(Z))$ such that $\sum_i R_i S_i = T$ and $\sum_i \|R_i\|\|S_i\| \le (M+\varepsilon)\|T\|$. Moreover, we can assume $\lim_i \|R_i\| = 0 = \lim_i \|S_i\|$. For each $i \in \mathbb{N}$, let P_i denote the coordinate projection of $\ell_p(\ell_p^{n_k}(Y))$ onto its i-th summand, and let $\imath_i$ denote the embedding of the i-th summand into $\ell_p(\ell_p^{n_k}(Y)))$. Let $R = \sum_i R_i P_i$ and $S = \sum_i \imath_i S_i$. Then $RS = T$, $\|R\|^q \le \sum_i \|R_i\|^q$ (resp., $\le \max_i \|R_i\|$ if $q = \infty$) and $\|S\|^p \le \sum_i \|S_i\|^p$ (resp., $\le \max_i \|S_i\|$ if $p = \infty$). Furthermore, by [16, §2], we can choose the R_i's and S_i's in such a way that $\|R\|\|S\| \le (1+\varepsilon)\sum_i \|R_i\|\|S_i\|$, and hence $\|R\|\|S\| \le (1+\varepsilon)(M+\varepsilon)\|T\|$. As this last holds for arbitrary ε, the bilinear map

$$\varphi_{n_k} : \mathcal{A}_0(\ell_p(Z), \ell_p^{n_k}(Y)) \times \mathcal{A}_0(\ell_p^{n_k}(Y), \ell_p(Z)) \to \mathcal{A}(\ell_p^{n_k}(Y)) \quad (k \in \mathbb{N}),$$

is $(M+\delta)$-open for any $\delta > 0$.

Now let $\mathcal{T} \in \mathcal{A}_0(\ell_p(Y))$ and $\varepsilon > 0$. There exists a sequence (T_k) in $\mathcal{A}_0(\ell_p(Y))$ such that $T_k \in \mathcal{A}(\ell_p^{n_k}(Y))$ for every k, $\sum_k T_k = \mathcal{T}$ and $\sum_k \|T_k\| \le \|\mathcal{T}\| + \varepsilon$. By the previous part, there exist $R_k \in \mathcal{A}_0(\ell_p(Z), \ell_p^{n_k}(Y))$ and $S_k \in \mathcal{A}_0(\ell_p^{n_k}(Y), \ell_p(Z))$ such that $R_k S_k = T_k$ and $\|R_k\|\|S_k\| \le (M+\varepsilon)\|T_k\|$ $(k \in \mathbb{N})$. Let π_k denote the projection of $\ell_p(\ell_p(Z))$ onto its k-th summand, and let $\jmath_k$

denote the natural embedding of the k-th summand back into $\ell_p(\ell_p(Z))$. Define $\mathcal{R} = \sum_k R_k \pi_k$ and $\mathcal{S} = \sum_k \jmath_k S_k$. Then $\mathcal{RS} = \mathcal{T}$ and an argument similar to the one of the previous paragraph shows that $\|\mathcal{R}\|\|\mathcal{S}\| \le (1+\varepsilon)(M+\varepsilon)(\|\mathcal{T}\|+\varepsilon)$. As $\ell_p(\ell_p(Z)) \cong \ell_p(Z)$, it follows that the product

$$\varphi_1 : \mathcal{A}_0(\ell_p(Z), \ell_p(Y)) \times \mathcal{A}_0(\ell_p(Y), \ell_p(Z)) \to \mathcal{A}_0(\ell_p(Y)),$$

is $(M+\delta)$-open for any $\delta > 0$. ∎

Remark 5.5: When $p > 1$ things are much simpler. Indeed, in this case the claim of the lemma can be easily obtained from Corollary 4.8.

Theorem 5.6: *Let X be a Banach space with a 1-unconditional basis (x_i). Suppose there exist $\delta \ge 1$ and $1 \le p \le \infty$ such that*

(i) *For each $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ so that if $F \subseteq [x_i]_{i=m}^{\infty}$ is a subspace spanned by n disjointly supported vectors then $\inf\{d(F,E) : E \text{ is a subspace of } \ell_p\} \le \delta$.*

(ii) *$\inf\{d([x_i]_{i=1}^n, E) : E \text{ is a subspace of } \ell_p\} \underset{n}{\to} \infty$.*

Then $\mathcal{A}(X)$ is not amenable.

Proof. Suppose towards a contradiction that $\mathcal{A}(X)$ is K-amenable for some $K \ge 1$. Let $\delta \ge 1$ and $1 \le p \le \infty$ be as in the hypotheses. By Lemma 5.4, for every $m \in \mathbb{N}$ at least one of the maps

$$\varphi_{1,m} : \mathcal{A}_0\big(\ell_p([x_i]_{m+1}^{\infty}), \ell_p([x_i]_1^m)\big) \times \mathcal{A}_0\big(\ell_p([x_i]_1^m), \ell_p([x_i]_{m+1}^{\infty})\big) \to \mathcal{A}_0\big(\ell_p([x_i]_1^m)\big),$$

or

$$\varphi_{2,m} : \mathcal{A}_0\big(\ell_p([x_i]_1^m), \ell_p([x_i]_{m+1}^{\infty})\big) \times \mathcal{A}_0\big(\ell_p([x_i]_{m+1}^{\infty}), \ell_p([x_i]_1^m)\big) \to \mathcal{A}_0\big(\ell_p([x_i]_{m+1}^{\infty})\big),$$

is M-open, where M depends only on K and the norm of the natural projection $P_m : X \to [x_i]_{i=1}^m$. But $\varphi_{2,m}$ cannot be open since otherwise, by Lemma 5.2, $\ell_p([x_i]_{m+1}^{\infty})$ would be crudely π-finitely representable in $\ell_p([x_i]_1^m) \simeq \ell_p$, which is impossible by (ii). Thus, $\varphi_{1,m}$ is M-open and by Lemma 5.2, $\ell_p([x_i]_1^m)$ is $(M+1)$-crudely π-finitely representable in $\ell_p([x_i]_{m+1}^{\infty})$. Note that, as $\|P_m\| = 1$ for every m, M is also independent of m. Thus, for every $m \in \mathbb{N}$ and every $1 \le n \le m$ there exist operators

$$S_{m,n} : [x_i]_{i=1}^n \to \ell_p([x_i]_{m+1}^{\infty}) \quad \text{and} \quad R_{m,n} : \ell_p([x_i]_{m+1}^{\infty}) \to [x_i]_{i=1}^n,$$

so that $R_{m,n}\,S_{m,n} = I_{[x_i]_1^n}$ and $\|R_{m,n}\|\|S_{m,n}\| \leq M+1$.

What remains follows closely the proof of [3, Prop. VI.b.3]. Let $F_{m,n} := S_{m,n}([x_i]_1^n)$. By [3, Prop. V.6], there are disjointly supported vectors $y_1, y_2, \ldots, y_N$ in $\ell_p([x_i]_{m+1}^\infty)$ such that $F_{m,n}$ is 2-isomorphic to a subspace of $F := [y_j]_1^N$. Clearly, we can assume that all the y_j's have finite support. Let P_k be the projection of $\ell_p([x_i]_{m+1}^\infty)$ onto its k-th summand. By (i), there exists m so that $\inf\big\{d([P_k y_j]_{j=1}^N, E) : E \text{ a subspace of } \ell_p\big\} \leq \delta$ for every $k \in \mathbb{N}$. As $F \subseteq \big(\bigoplus_k [P_k y_j]_{j=1}^N\big)_p$, it follows from this last and the fact that $\ell_p \cong \ell_p(\ell_p)$, that $\inf\big\{d(F,E) : E \text{ a subspace of } \ell_p\big\} \leq \delta$, and in turn that $\inf\big\{d([x_i]_1^n, E) : E \text{ a subspace of } \ell_p\big\} \leq 2\delta C$, contradicting (ii). Thus $\mathcal{A}(X)$ cannot be amenable. ∎

We apply Theorem 5.6 to a class of 'Tsirelson-like' spaces introduced in [7], which contains the dual of the original Tsirelson's space as a particular case.

Let us recall briefly the definition of the dual of Tsirelson's space, T, as given in [7, § 2]. Let (t_n) denote the unit vector basis of c_{00} (the space of scalar sequences with finite support). If E, F are finite, non-empty subsets of $\mathbb{N}$, we write $E < F$ to mean that $\max E < \min F$. For any $E \subset \mathbb{N}$ and any $x = \sum_n \alpha_n t_n \in c_{00}$, define $Ex := \sum_{n\in E} \alpha_n t_n$. Set $\|\,.\,\|_0 := \|\,.\,\|_{c_0}$ and, for $m \geq 0$, define

$$\|x\|_{m+1} := \max\left\{\|x\|_m, 2^{-1}\max\Big[\sum_{j=1}^k \|E_j x\|_m\Big]\right\} \quad (x \in c_{00}),$$

where the inner maximum is taken over all possible choices of finite subsets $E_1, E_2, \ldots, E_k$ of $\mathbb{N}$, such that: $\{k\} \leq E_1 < E_2 < \ldots < E_k$. It is easily verified that $\|\,.\,\|_m$ is a norm on c_{00} for every m, and that, for each $x \in c_{00}$ the sequence $(\|\,.\,\|_m)$ is non-decreasing and majorized by $\|x\|_{\ell_1}$. Thus we can define

$$\|x\| := \lim_{m\to\infty} \|x\|_m \quad (x \in c_{00}).$$

The latter is a norm on c_{00}. The dual T^* of Tsirelson's space is defined as the completion of c_{00} in the last norm. It is well-known that the sequence (t_n) is a normalized 1-unconditional basis for T^*.

For $1 \leq p < \infty$, $T^{(p)}$ is defined as the set of all $x = \sum_n \alpha_n t_n$ such that $\sum_n |\alpha_n|^p t_n \in T^*$, endowed with the norm

$$\|x\|_{(p)} = \Big\|\sum_n |\alpha_n|^p t_n\Big\|^{\frac{1}{p}} \quad (x \in T^{(p)}).$$

When $1 < p < \infty$, $T^{(p)}$ is the so called p-convexified Tsirelson's space. Clearly, $T^{(1)}$ is nothing but T^* itself.

Many important facts about $T^*(= T^{(1)})$ are shared by the p-convexified Tsirelson's spaces. Among them we have the following:

a) Each $T^{(p)}$ is reflexive (actually, they are all uniformly convex for $p > 1$);
b) Each $T^{(p)}$ contains ℓ_p^n's uniformly $(1 < p < \infty)$;
c) No $T^{(p)}$ contains an isomorphic copy of ℓ_r $(1 \leq r \leq \infty)$.

Moreover, for every $p \geq 1$ the norm on $T^{(p)}$ satisfies

$$\|x\|_{(p)} = \max\left\{\|x\|_0, 2^{-\frac{1}{p}} \sup\left[\sum_{j=1}^{k} \|E_j x\|_{(p)}^p\right]^{\frac{1}{p}}\right\} \quad (x \in T^{(p)}), \tag{7}$$

where the inner supremum is taken over all choices of finite subsets of $\mathbb{N}$, $E_1, E_2, \ldots, E_k$, such that: $\{k\} \leq E_1 < E_2 < \cdots < E_k$. Property (b) above follows easily from (7).

We need one more fact about these spaces that we collect in the next lemma.

Lemma 5.7: *Let $1 \leq p < \infty$. Then $T^{(p)}$ is not crudely finitely representable in ℓ_p.*

As explained in [3, VI.B], this follows on combining results of Lindenstrauss and Pelczynski [20], and Lindenstrauss and Rosenthal [21]. Precisely, it follows from [20, Remark after Prop. 5.2] (see [22, Corollary 8.9] for a proof of this) that if X is a Banach space complemented in its bidual such that for some $1 \leq p < \infty$, $\sup\{\gamma_p(I_E) : E \text{ a finite-dimensional subspace of } X\} < \infty$, then X is isomorphic to a complemented subspace of an $L_p(\mu)$ space. On the other hand, if X is a complemented subspace of an $L_p(\mu)$ space, which is not isomorphic to a Hilbert space, then it must be an $\mathcal{L}_p$ space [21, Theorem III(b)], and hence it must contain a complemented subspace isomorphic to ℓ_p [20, Proposition 7.3]. Thus, if $T^{(p)}$ $(1 \leq p < \infty)$ were crudely finitely representable in ℓ_p then it would embed complementably in some $L_p(\mu)$. But $T^{(p)}$ contains ℓ_p^n's uniformly, and so it would contain a complemented copy of ℓ_p, which is an absurd.

Corollary 5.8: *The algebra $\mathcal{A}(T^{(p)})$ $(1 \leq p < \infty)$ is not amenable.*

Proof. We simply note that $T^{(p)}$ satisfies conditions (i) and (ii) of Theorem 5.6 $(1 \leq p < \infty)$. Indeed, (i) is an immediate consequence of [2, Proposition 7.3] and (7) above, while (ii) follows easily from Lemma 5.7. ∎

Remark 5.9: By [9, Corollary 5.5], $\mathcal{A}(T)$ cannot be amenable either.

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