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## The maximal $C^*$ -algebra of quotients as an operator bimodule

PERE ARA, MARTIN MATHIEU, AND EDUARD ORTEGA

*Dedicated to Professor Heinz König on the occasion of his 80<sup>th</sup> birthday*

**Abstract.** We establish a description of the maximal  $C^*$ -algebra of quotients of a unital  $C^*$ -algebra  $A$  as a direct limit of spaces of completely bounded bimodule homomorphisms from certain operator submodules of the Haagerup tensor product  $A \otimes_h A$  labelled by the essential closed right ideals of  $A$  into  $A$ . In addition the invariance of the construction of the maximal  $C^*$ -algebra of quotients under strong Morita equivalence is proved.

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**1. Introduction.** The maximal  $C^*$ -algebra of quotients,  $Q_{\max}(A)$ , of a (unital)  $C^*$ -algebra  $A$  was introduced in [3] as a  $C^*$ -analytic analogue of the maximal symmetric ring of quotients of a non-singular ring studied, e.g., in [9] and [10]. As a  $C^*$ -algebra of quotients it shares some of the properties of the local multiplier algebra  $M_{\text{loc}}(A)$  of  $A$  [2]; for instance, it arises as the completion of the bounded part of its algebraic counterpart and it can be canonically embedded into the injective envelope  $I(A)$  of  $A$ . However, in contrast to the situation of  $M_{\text{loc}}(A)$ , there is no

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direct limit construction for  $Q_{\max}(A)$  in the category of  $C^*$ -algebras. This makes its study somewhat more cumbersome. The purpose of this note is to alleviate this difficulty by providing a direct limit description in a different category, the category of operator modules. From the well-established good properties of the Haagerup tensor product it comes as no surprise that this concept will play an important role.

We will be guided by the construction of the maximal symmetric ring of quotients  $Q_{\max}^s(R)$  of a (non-singular unital) ring  $R$  with involution as a two-sided localisation of the regular bimodule  ${}_R R_R$  with respect to a certain filter of right ideals in [11]. In this situation, for each essential right ideal  $I$  of  $R$ , a certain right ideal  $M_I$  in the ring  $R \otimes R^{\text{op}}$  is used to introduce the filter  $\Omega$  consisting of all right ideals containing some  $M_I$ . It then turns out that  $Q_{\max}^s(R)$  is canonically isomorphic to  $\varinjlim_{\Omega} \text{Hom}(M_I, R)$ , where  $\text{Hom}(M_I, R)$  stands for the space of all  $R$ -bimodule homomorphisms from  $M_I$  into  $R$ . Our principal goal in the present paper is to modify this algebraic construction in such a way that, in the situation of a unital  $C^*$ -algebra  $A$ , the  $C^*$ -algebra  $Q_{\max}(A)$  can be obtained in an analogous manner as a direct limit of spaces of completely bounded bimodule homomorphisms from certain operator submodules of the Haagerup tensor product  $A \otimes_h A$  labelled by the essential closed right ideals of  $A$  into  $A$ .

On our route to establish our main result, Theorem 2.9, we shall obtain several other useful properties of the maximal  $C^*$ -algebra of quotients. These will enable us to show, in Theorem 2.4, that this construction is invariant under strong Morita equivalence of  $C^*$ -algebras (as it is indeed the case for the local multiplier algebra). For a comprehensive discussion of various types of Morita equivalence, we refer to [5].

Throughout, we shall use the terminology and notation of [2], [3], and [6] with the exception that  $A \otimes_h A$  already denotes the completion of the algebraic tensor product  $A \otimes A$  with respect to the Haagerup norm  $\|\cdot\|_h$ . In particular,  $\mathcal{O}_1$  stands for the category of operator spaces with complete contractions as the morphisms, and  $CB_{A-A}(E, F)$  is the operator space of all completely bounded  $A$ -bimodule maps from the operator  $A$ -bimodule  $E$  into the operator  $A$ -bimodule  $F$ .

**2. The Results.** For our purposes here, the following description of the maximal  $C^*$ -algebra of quotients as a  $C^*$ -subalgebra of the injective envelope  $I(A)$  of a  $C^*$ -algebra  $A$  is the most expedient. Let

$$Q_{\max}^s(A)_b = \{q \in I(A) \mid qJ + q^*J \subseteq A \text{ for some } J \in \mathcal{J}_{\text{cer}}\},$$

where  $\mathcal{J}_{\text{cer}}$  denotes the set of all closed essential right ideals of  $A$  (so that  $J \in \mathcal{J}_{\text{cer}}$  has non-zero intersection with any non-zero right ideal of  $A$ ). Then  $Q_{\max}(A) = \overline{Q_{\max}^s(A)_b}$ ; see [3, Theorem 4.8]. The proposition below states a property of the maximal  $C^*$ -algebra of quotients that is shared with the maximal symmetric ring of quotients of a semiprime ring, compare [10, Proposition 1.6].

**Lemma 2.1.** *Let  $A$  be a  $C^*$ -algebra, and let  $e \in M(A)$  be a projection. Then*

$$\iota: \mathfrak{I}_{cer}(eAe) \longrightarrow \mathfrak{I}_{cer}(A), \quad J \longmapsto \overline{JA + (1-e)A}$$

*is an injective mapping and*

$$\kappa: \mathfrak{I}_{cer}(A) \longrightarrow \mathfrak{I}_{cer}(eAe), \quad I \longmapsto eIe$$

*is a surjective mapping such that  $\kappa \circ \iota = \text{id}_{\mathfrak{I}_{cer}(eAe)}$ .*

*Proof.* Take  $J \in \mathfrak{I}_{cer}(eAe)$  and note that the annihilator  $(AeA)^\perp$  coincides with  $(\overline{AeA})^\perp$  and is contained in  $(1-e)A(1-e)$ . In order to prove that  $\overline{JA + (1-e)A}$  is an essential right ideal of  $A$ , it thus suffices to show that  $\overline{JA + (1-e)AeA + (AeA)^\perp}$  is an essential right ideal of  $A$ . Since  $\overline{AeA} \oplus (AeA)^\perp$  is essential as a right ideal, it is enough to show that  $\overline{JA + (1-e)AeA}$  is essential in  $\overline{AeA}$ . Let  $z$  be a non-zero element in  $\overline{AeA}$ . If  $ezAe = 0$  then  $ez\overline{AeA} = 0$  and thus  $ez = 0$  entailing that  $z = (1-e)z \in \overline{(1-e)AeA}$ . We can therefore assume that there is  $t \in A$  such that  $ezte$  is a non-zero element of  $eAe$ . Since  $J$  is essential in  $eAe$ , there is  $s \in eAe$  such that  $eztes$  is a non-zero element of  $J$ . It follows that

$$z(tes) = eztes + (1-e)ztes \in J + \overline{(1-e)AeA} \subseteq \overline{JA + (1-e)AeA},$$

as desired.

To prove the second assertion, take  $I \in \mathfrak{I}_{cer}(A)$ . If  $z$  is a non-zero element in  $eAe$ , there is  $a \in A$  such that  $za$  is a non-zero element of  $I$ . Since  $za \in AeA$ , we have  $zaAe \neq 0$ ; thus  $0 \neq z(eaAe) \subseteq eIe$  wherefore  $eIe \in \mathfrak{I}_{cer}(eAe)$ . It is obvious that  $\kappa \circ \iota = \text{id}_{\mathfrak{I}_{cer}(eAe)}$ . In particular,  $\iota$  is injective and  $\kappa$  is surjective.  $\square$

**Proposition 2.2.** *Let  $e$  be a projection in the multiplier algebra of a  $C^*$ -algebra  $A$ . Then  $Q_{\max}(eAe) = eQ_{\max}(A)e$ .*

*Proof.* By [7, Proposition 6.3],  $I(eM(A)e) = eI(M(A))e$ ; combining this with  $I(M(A)) = I(A)$  and  $I(M(eAe)) = I(eAe)$ , we get

$$I(eAe) = I(M(eAe)) = I(eM(A)e) = eI(M(A))e = eI(A)e.$$

Thus we can view  $Q_{\max}(eAe)$  as a  $C^*$ -subalgebra of  $eI(A)e$ . We shall divide the proof of the main statement into three steps.

**Step 1.**  $eQ_{\max}^s(A)_be \subseteq Q_{\max}^s(eAe)_b$ .

In order to show this take  $q \in Q_{\max}^s(A)_b$ . There is  $I \in \mathfrak{I}_{cer}(A)$  such that  $eqeI$  and  $eq^*eI$  are both contained in  $A$ . By Lemma 2.1,  $eIe \in \mathfrak{I}_{cer}(eAe)$  and, since  $eqeIe \subseteq eAe$  and  $eq^*eIe \subseteq eAe$ , and  $eqe \in I(eAe)$ , we conclude that  $eqe \in Q_{\max}^s(eAe)_b$ .

**Step 2.**  $Q_{\max}^s(eAe)_b \subseteq eQ_{\max}^s(A)_be$ .

To see this, let  $q \in Q_{\max}^s(eAe)_b \subseteq I(eAe) = eI(A)e \subseteq I(A)$  and take  $J \in \mathfrak{I}_{\text{cer}}(eAe)$  such that  $qJ, q^*J \subseteq eAe$ . Putting  $J' = \overline{JA} + (1-e)\overline{A}$  we obtain an essential right ideal of  $A$  (Lemma 2.1). Note that

$$qJ' + q^*J' \subseteq \overline{qJA} + \overline{q^*JA} \subseteq A$$

and thus  $q \in eQ_{\max}^s(A)_be$ .

**Step 3.** From the first two steps we conclude, taking closures in  $I(A)$ , that

$$eQ_{\max}(A)e = e\overline{Q_{\max}^s(A)_be} = \overline{eQ_{\max}^s(A)_be} = \overline{Q_{\max}^s(eAe)_b} = Q_{\max}(eAe),$$

since  $eQ_{\max}(A)e$  is closed.  $\square$

**Corollary 2.3.** *Let  $A$  be a unital  $C^*$ -algebra. Then  $M_n(Q_{\max}(A)) = Q_{\max}(M_n(A))$  for each  $n \in \mathbb{N}$ .*

*Proof.* This follows from a standard argument, see, e.g., [8, Remark 17.6], so we merely sketch the essential part for completeness. Let  $\{e_{ij} \mid 1 \leq i, j \leq n\}$  be the canonical matrix units in  $M_n(A) \subseteq Q_{\max}(M_n(A))$  and denote by  $e_i = e_{ii}$  the  $n$  mutually orthogonal projections with  $\sum_{i=1}^n e_i = 1$ . For  $x \in Q_{\max}(M_n(A))$  set  $a_{ij} = \sum_{m=1}^n e_{mi} x e_{jm}$ . Then  $x = \sum_{i,j=1}^n a_{ij} e_{ij}$ , since

$$(2.1) \quad a_{ij} e_{k\ell} = e_{ki} x e_{j\ell} = e_{k\ell} a_{ij} \quad (1 \leq i, j, k, \ell \leq n)$$

and thus  $a_{ij} e_{ij} = e_{ii} x e_{jj} = e_i x e_j$ . Letting  $q_{ij}^{(m)} = e_{mi} x e_{jm}$  we infer from (2.1) that  $q_{ij}^{(m)} = q_{ij}$  for all  $1 \leq m \leq n$ . By Proposition 2.2,

$$q_{ij} = e_m q_{ij} e_m \in e_m Q_{\max}(M_n(A)) e_m = Q_{\max}(e_m M_n(A) e_m) = Q_{\max}(A)$$

as each  $e_m$  is a full projection in  $M_n(A)$ . Consequently, with  $q_x = \sum_{i,j} q_{ij} e_{ij} \in M_n(Q_{\max}(A))$ , we establish a  $*$ -isomorphism  $x \mapsto q_x$  from  $Q_{\max}(M_n(A))$  onto  $M_n(Q_{\max}(A))$  (which also satisfies  $Q_{\max}^s(M_n(A))_b = M_n(Q_{\max}^s(A)_b)$ ).  $\square$

We obtain the following very useful consequence.

**Theorem 2.4.** *Let  $A$  and  $B$  be two unital strongly Morita equivalent  $C^*$ -algebras. Then their maximal  $C^*$ -algebras of quotients  $Q_{\max}(A)$  and  $Q_{\max}(B)$  are strongly Morita equivalent.*

*Proof.* Suppose that  $A$  and  $B$  are strongly Morita equivalent. Then there is a full projection  $e$  in some matrix algebra  $M_n(A)$  such that  $B \cong eM_n(A)e$ . Therefore

$$(2.2) \quad Q_{\max}(B) \cong Q_{\max}(eM_n(A)e) = eQ_{\max}(M_n(A))e = eM_n(Q_{\max}(A))e$$

by Proposition 2.2 and Corollary 2.3, since  $e$  is a full projection in  $Q_{\max}(M_n(A))$ . As a result,  $Q_{\max}(B)$  and  $Q_{\max}(A)$  are strongly Morita equivalent.  $\square$

**Corollary 2.5.** *Let  $A$  and  $B$  be two unital strongly Morita equivalent  $C^*$ -algebras. If  $A = Q_{\max}(A)$  then  $B = Q_{\max}(B)$ .*

**Remark 2.6.** It is evident that the analogues of Theorem 2.4 and its corollary hold for the local multiplier algebra once the analogue of the statement in Proposition 2.2 is verified. In fact,  $M_{\text{loc}}(eAe) = eM_{\text{loc}}(A)e$  for any projection  $e$  in a unital  $C^*$ -algebra  $A$ . If  $e$  is full, then  $A$  and the hereditary  $C^*$ -subalgebra  $eAe$  are strongly Morita equivalent and therefore the lattices of their closed essential ideals are isomorphic [2, Proposition 1.2.38]. Since  $eM(I)e = M(eIe)$  for every closed essential ideal  $I$  [2, Corollary 1.2.37], the equality  $M_{\text{loc}}(eAe) = eM_{\text{loc}}(A)e$  follows from the direct limit formula for the local multiplier algebra. If  $e$  is arbitrary, then, using [2, Proposition 2.3.6], we have

$$\begin{aligned} eM_{\text{loc}}(A)e &= eM_{\text{loc}}(J \oplus J^\perp)e \\ &= eM_{\text{loc}}(J)e \oplus eM_{\text{loc}}(J^\perp)e = eM_{\text{loc}}(J)e = M_{\text{loc}}(eAe), \end{aligned}$$

where  $J = \overline{AeA}$  is the closed ideal generated by  $e$ ,  $J^\perp$  is its annihilator (so that  $J + J^\perp$  is essential) and we apply the above argument to the full hereditary  $C^*$ -subalgebra  $eAe$  of  $J$ .

The Morita invariance of the local multiplier algebra has several pleasant consequences. For instance, any unital  $C^*$ -algebra  $B$  which is strongly Morita equivalent to a boundedly centrally closed unital  $C^*$ -algebra  $A$  (i.e.,  $Z(A) = Z(M_{\text{loc}}(A))$ ) [2, Definition 3.2.1]) is boundedly centrally closed. For, the centres of  $A$  and  $B$  and the centres of  $M_{\text{loc}}(A)$  and  $M_{\text{loc}}(B)$  are isomorphic. In addition, the analogue of Corollary 2.5 can be used to study iterated local multiplier algebras. For example, suppose that  $A$  is strongly Morita equivalent to a commutative  $C^*$ -algebra  $B$ . Since  $M_{\text{loc}}(M_{\text{loc}}(B)) = M_{\text{loc}}(B)$  as  $M_{\text{loc}}(B)$  is an  $AW^*$ -algebra, it follows that  $M_{\text{loc}}(M_{\text{loc}}(A)) = M_{\text{loc}}(A)$ .

We like to point out that the above arguments are restricted to the situation of unital  $C^*$ -algebras, as the example  $A = C[0, 1] \otimes K(\ell^2)$  studied in [3] and [4] shows; there,  $M_{\text{loc}}(M_{\text{loc}}(A)) \neq M_{\text{loc}}(A)$ .

We now turn our attention to the description of the maximal  $C^*$ -algebra of quotients as a direct limit of spaces of completely bounded bimodule homomorphisms. To this end we shall consider the Haagerup tensor product  $A \otimes_h A$  of a unital  $C^*$ -algebra  $A$  as an operator  $A$ -bimodule with the operations  $a(x \otimes y)b = ax \otimes yb$ ,  $a, b \in A$  and  $x, y \in A$ . For an essential closed right ideal  $I$  in  $A$ , we define the closed sub-bimodule

$$M_I = \overline{A \otimes I + I^* \otimes A}.$$

If  $I$  is two-sided, the closure is not needed [1, Theorem 3.8] but we do not have an analogous result available for one-sided ideals.

**Proposition 2.7.** *Let  $A$  be a unital  $C^*$ -algebra and let  $I \in \mathfrak{I}_{\text{cer}}$ . For each completely bounded  $A$ -bimodule homomorphism  $\psi: M_I \rightarrow A$  there exists a unique  $q \in Q_{\text{max}}^s(A)_b$  such that  $\psi|_{1 \otimes I} = L_{q|I}$  and  $\|\psi\|_{cb} = \|q\|$ .*

We prepare the proof by the following simple lemma.

**Lemma 2.8.** *Let  $A$  be a unital  $C^*$ -subalgebra of the unital  $C^*$ -algebra  $B$ . For each  $q \in B$ , by  $\psi_q(a \otimes b) = aqb$ ,  $a, b \in A$  we can define a completely bounded  $A$ -bimodule homomorphism  $\psi_q: A \otimes_h A \rightarrow B$  such that  $\|\psi_q\|_{cb} = \|q\|$ .*

*Proof.* Evidently, the assignment  $a \otimes b \mapsto aqb$  yields an  $A$ -bimodule homomorphism  $\psi_q$  from  $A \otimes A$  into  $B$ . Since  $\psi_q(1 \otimes 1) = q$ , we have  $\|\psi_q\| \geq \|q\|$ . For the reverse inequality, take  $u \in M_n(A \otimes A)$  with  $\|u\|_h = 1$ . Let  $\varepsilon > 0$ . Then there exist  $v \in M_{n,r}(A)$ ,  $w \in M_{r,n}(A)$ , for some  $r \in \mathbb{N}$ , such that  $u = v \odot w$  and  $\|v\| \|w\| < 1 + \varepsilon$ . If  $v = (v_{k\ell})$ ,  $w = (w_{k\ell})$  then

$$\psi_q^{(n)}(u) = \psi_q^{(n)}(v \odot w) = \sum_{\ell=1}^r (\psi_q(v_{i\ell} \otimes w_{\ell j}))_{ij} = \sum_{\ell=1}^r (v_{i\ell} q w_{\ell j})_{ij} = (v_{k\ell}) \cdot q \cdot (w_{k\ell}),$$

where, by abuse of notation, we also denote by  $q$  the  $r \times r$  diagonal matrix with  $q$  along the diagonal. Therefore,  $\|\psi_q^{(n)}(u)\| \leq \|v\| \|q\| \|w\| < (1 + \varepsilon) \|q\|$ . It follows that  $\|\psi_q^{(n)}\| \leq \|q\|$  for all  $n \in \mathbb{N}$  wherefore  $\|\psi_q\|_{cb} = \|q\|$ . Extending  $\psi_q$  to  $A \otimes_h A$  thus yields the result.  $\square$

*Proof of Proposition 2.7.* Let  $x \in I$  and define  $g(x) = \psi(1 \otimes x)$  and  $f(x^*) = \psi(x^* \otimes 1)$ . Then  $g: I \rightarrow A$  and  $f: I^* \rightarrow A$  are completely bounded right and left module homomorphisms, respectively, such that  $y^*g(x) = f(y^*)x$  for all  $x, y \in I$ . By [3, Lemma 4.7] and the subsequent remarks, there is a unique element  $q \in I(A)$  such that  $g = \psi|_{1 \otimes I} = L_{q|I}$  and  $f = \psi|_{I^* \otimes 1} = R_{q|I^*}$ . As  $qI + q^*I \subseteq A$ , it follows that  $q \in Q_{\max}^s(A)_b$  and  $\|q\| = \|L_{q|I}\| \leq \|\psi\|_{cb}$ .

For  $u = a \otimes x + y^* \otimes b \in M_I$ ,  $x, y \in I$ ,  $a, b \in A$  we thus obtain

$$\psi(u) = \psi(a \otimes x + y^* \otimes b) = aqx + y^*qb.$$

Apply Lemma 2.8 with  $B = Q_{\max}(A)$  to obtain  $\psi_q$  on  $A \otimes_h A$  with the property that  $\psi_q|_{M_I} = \psi$ . It follows that  $\|q\| \leq \|\psi\|_{cb} \leq \|\psi_q\|_{cb} = \|q\|$ , as desired.  $\square$

For  $I, J \in \mathfrak{I}_{cer}$  with  $J \subseteq I$  we have an embedding  $M_J \subseteq M_I$  and thus we can define

$$\rho_{JI}: CB_{A-A}(M_I, A) \longrightarrow CB_{A-A}(M_J, A), \quad \psi \longmapsto \psi|_{M_J}.$$

The fundamental property that  $\|L_{q|J}\| = \|L_{q|I}\| = \|q\|$  together with Proposition 2.7 entails that  $\|\psi\|_{cb} = \|\psi|_{1 \otimes I}\| = \|\psi|_{1 \otimes J}\| = \|q\|$ ; i.e., each  $\rho_{JI}$  is isometric. To show that  $\rho_{JI}$  is indeed completely isometric, that is,

$$\rho_{JI}^{(n)}: M_n(CB_{A-A}(M_I, A)) \longrightarrow M_n(CB_{A-A}(M_J, A)), \quad (\psi_{k\ell}) \longmapsto (\psi_{k\ell}|_{M_J}),$$

$I, J \in \mathfrak{I}_{cer}$ ,  $J \subseteq I$  is isometric for each  $n \in \mathbb{N}$ , we use that  $M_n(CB_{A-A}(M_I, A)) = CB_{A-A}(M_I, M_n(A))$  and that every completely bounded  $A$ -bimodule map  $\psi: M_I \rightarrow M_n(A)$  gives rise to a unique element  $q \in M_n(Q_{\max}^s(A)_b)$  via  $q = (q_{k\ell})$ , where  $q_{k\ell}$  is the element in  $Q_{\max}^s(A)_b$  with  $\psi_{k\ell}|_{1 \otimes I} = L_{q_{k\ell}|I}$  given by Proposition 2.7. Letting  $\psi_{q_{k\ell}}$  denote the extension of  $\psi_{k\ell}$  to  $A \otimes_h A$  with values in  $Q_{\max}(A)$  as above we

take  $u \in A \otimes A$  with  $\|u\|_h = 1$ . Given  $\varepsilon > 0$ , write it as  $u = v \odot w$  for some  $v \in M_{1,r}(A)$ ,  $w \in M_{r,1}(A)$  such that  $\|v\| \|w\| < 1 + \varepsilon$ . Denoting by  $q_{k\ell}$  once again the  $r \times r$  diagonal matrix with  $q_{k\ell}$  along the diagonal we find

$$(\psi_{q_{k\ell}}(u)) = (v \cdot q_{k\ell} \cdot w) = \begin{pmatrix} v & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & v \end{pmatrix} \begin{pmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & & \vdots \\ q_{n1} & \cdots & q_{nn} \end{pmatrix} \begin{pmatrix} w & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & w \end{pmatrix}$$

entailing

$$\|(\psi_{q_{k\ell}}(u))\| \leq \|v\| \|w\| \left\| \begin{pmatrix} (q_{k\ell}) & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & (q_{k\ell}) \end{pmatrix} \right\|,$$

by the canonical shuffle.

As before, this implies that the norm of the mapping  $\psi = (\psi_{k\ell})$  is dominated by  $\|(q_{k\ell})\|$ , wherefore, in  $M_n(CB_{A-A}(M_I, A))$ ,  $\|(\psi_{k\ell})\| \leq \|(q_{k\ell})\|$ . (Note that, by Corollary 2.3, the mapping  $\psi_q$  given by Lemma 2.8 applied to  $B = Q_{\max}(M_n(A))$  agrees with  $(\psi_{q_{k\ell}})$ .)

On the other hand, from the fact that  $\psi_{k\ell}|_{1 \otimes I} = L_{q_{k\ell}|_I}$ , we conclude that

$$\|(\psi_{k\ell})\| \geq \|(\psi_{k\ell}|_{1 \otimes I})\| = \|(L_{q_{k\ell}|_I})\| = \|(q_{k\ell})\|$$

by Lemma 3.9 in [3]. As a result, the norm of  $\psi = (\psi_{k\ell}) \in M_n(CB_{A-A}(M_I, A))$  coincides with the norm of  $q = (q_{k\ell}) \in M_n(Q_{\max}^s(A)_b)$  which, in particular, implies that the restriction homomorphisms  $\rho_{JI}^{(n)}$  are isometric for each  $n$ .

We are now in a position to prove our main result.

**Theorem 2.9.** *Let  $A$  be a unital  $C^*$ -algebra. The operator  $A$ -bimodules  $Q_{\max}^s(A)_b$  and  $\varinjlim_{I \in \mathfrak{J}_{cer}} CB_{A-A}(M_I, A)$  are completely isometrically isomorphic. As a result,  $Q_{\max}^s(A) = \varinjlim_{I \in \mathfrak{J}_{cer}} CB_{A-A}(M_I, A)$  in the category  $\mathcal{O}_1$ .*

*Proof.* By the above arguments, the mapping

$$\tau_I: CB_{A-A}(M_I, A) \longrightarrow Q_{\max}^s(A)_b, \quad \psi \longmapsto q_\psi,$$

where  $q_\psi \in Q_{\max}^s(A)_b$  is the unique element determined by Proposition 2.7, is completely isometric for each  $I \in \mathfrak{J}_{cer}$ . From the construction, it is clear that, for



$J \subseteq I$ , the following diagram is commutative

$$\begin{array}{ccc} CB_{A-A}(M_I, A) & \xrightarrow{\rho_{JI}} & CB_{A-A}(M_J, A) \\ & \searrow \tau_I \quad \swarrow \tau_J & \\ & Q_{\max}^s(A)_b & \end{array}$$

Since each  $\rho_{JI}$  is completely isometric, there exists a complete isometry

$$\tau: \varinjlim_{I \in \mathfrak{I}_{cer}} CB_{A-A}(M_I, A) \rightarrow Q_{\max}^s(A)_b$$

such that  $\tau|_{CB_{A-A}(M_I, A)} = \tau_I$ . Let  $q \in Q_{\max}^s(A)_b$  and choose  $I \in \mathfrak{I}_{cer}$  with the property that  $qI, q^*I \subseteq A$ . Then

$$\sum_{i=1}^r a_i \otimes x_i + \sum_{j=1}^s y_j^* \otimes b_j \longmapsto \sum_{i=1}^r a_i q x_i + \sum_{j=1}^s y_j^* q b_j$$

defines a completely bounded  $A$ -bimodule homomorphism  $\psi_q$  which clearly satisfies  $\tau_I(\psi_q) = q$ . This yields a right inverse of  $\tau$ , which is therefore surjective. This complete isometry extends to a complete isometry from  $\varinjlim_{I \in \mathfrak{I}_{cer}} CB_{A-A}(M_I, A)$  onto  $Q_{\max}(A)$ .

Although each individual space  $CB_{A-A}(M_I, A)$  is not necessarily an  $A$ -bimodule, the direct limit  $\varinjlim_{I \in \mathfrak{I}_{cer}} CB_{A-A}(M_I, A)$  is. To see this note that, whenever  $\psi \in CB_{A-A}(M_I, A)$  is given as  $\psi = \psi_q$  with  $qI, q^*I \subseteq A$  and  $c \in A$ , we can define  $\psi c \in CB_{A-A}(M_J, A)$  via

$$(\psi c)(a \otimes x + y^* \otimes b) = a q c x + y^* q c b \quad (a, b \in A, x, y \in J),$$

where  $J \in \mathfrak{I}_{cer}$  is given by  $J = \{u \in I \mid cu \in I\}$ . A similar expression yields  $c\psi$ , and these module operations are evidently compatible with the connecting maps  $\rho_{JI}$ . Therefore, we obtain an  $A$ -bimodule structure on  $\varinjlim_{I \in \mathfrak{I}_{cer}} CB_{A-A}(M_I, A)$  turning it into an operator  $A$ -bimodule, which is completely isometrically isomorphic to  $Q_{\max}^s(A)_b$ .  $\square$

**Remark 2.10.** What is the involution on  $Q_{\max}(A)$  in the above picture? Endow  $A \otimes_h A$  with an involution  $*$  defined by  $(x \otimes y)^* = y^* \otimes x^*$ . Then the involution on  $CB_{A-A}(A \otimes_h A, A) \cong A$  is given by  $(\psi^*)(u) = (\psi(u^*))^*$ ,  $u \in A \otimes_h A$ , and similarly for  $CB_{A-A}(M_I, A)$  noting that  $M_I$  is  $*$ -invariant.

## References

- [1] S. D. ALLEN, A. M. SINCLAIR, AND R. R. SMITH, The ideal structure of the Haagerup tensor product of  $C^*$ -algebras, *J. Reine Angew. Math.* **442**, 111–148 (1993).
- [2] P. ARA AND M. MATHIEU, *Local multipliers of  $C^*$ -algebras*, Springer-Verlag, London, 2003.
- [3] P. ARA AND M. MATHIEU, Maximal  $C^*$ -algebras of quotients and injective envelopes of  $C^*$ -algebras, *Houston J. Math.* **34**, 827–872 (2008).

- [4] M. ARGERAMI, D. R. FARENICK, AND P. MASSEY, The gap between local multiplier algebras of  $C^*$ -algebras, *Quart. J. Math.*, to appear.
- [5] D. P. BLECHER, P. S. MUHLY, AND V. I. PAULSEN, Categories of operator modules (Morita equivalence and projective modules), *Mem. Amer. Math. Soc.* **681**, Providence, RI, 2000.
- [6] E. G. EFFROS AND Z.-J. RUAN, *Operator spaces*, London Math. Soc. Monographs **23**, Clarendon Press, Oxford, 2000.
- [7] M. HAMANA, Tensor products for monotone complete  $C^*$ -algebras, I, *Japan. J. Math.* **8**, 259–283 (1982).
- [8] T. Y. LAM, *Lectures on modules and rings*, Graduate texts in Maths. **189**, Springer-Verlag, New York, 1998.
- [9] S. LANNING, The maximal symmetric ring of quotients, *J. Algebra* **179**, 47–91 (1996).
- [10] E. ORTEGA, Rings of quotients of incidence algebras and path algebras, *J. Algebra* **303**, 225–243 (2006).
- [11] E. ORTEGA, Two-sided localization of bimodules, *Comm. Algebra* **36**, 1911–1926 (2008).

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