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Unconditional constants and polynomial inequalities

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Abstract

If P is a polynomial on $\mathbb{R}^m$ of degree at most n, given by $P(\mathbf{x}) = \sum_{\alpha\in\mathbb{N}^m,|\alpha|\le n} \mathbf{a}_\alpha \mathbf{x}^\alpha$, and $\mathcal{P}_n(\mathbb{R}^m)$ is the space of such polynomials, then we define the polynomial $|P|$ by $|P|(\mathbf{x}) = \sum_{\alpha\in\mathbb{N}^m,|\alpha|\le \mathbf{n}} |\mathbf{a}_\alpha| \mathbf{x}^\alpha$. Now if $\mathsf{B} \subseteq \mathbb{R}^m$ is a convex set, we define the norm $\|P\|_{\mathsf{B}} := \sup\{|P(\mathbf{x})| : \mathbf{x} \in \mathsf{B}\}$ on $\mathcal{P}_n(\mathbb{R}^m)$, and then we investigate the inequality

$$\||P|\|_{\mathsf{B}} \le C_{\mathsf{B}} \|P\|_{\mathsf{B}},$$

providing sharp estimates on C_{B} for some specific spaces of polynomials. These C_{B}'s happen to be the unconditional constants of the canonical bases of the considered spaces.

Keywords: Unconditional constant; Polynomial inequalities; Trinomials; Homogeneous polynomials; Extreme points

1. Preliminaries

If E is a Banach space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), a sequence $(x_n)_{n\in\mathbb{N}}$ of non-zero vectors in E is said to be an unconditional basic sequence if there is a constant $C > 0$ such that for all $n \in \mathbb{N}$, all $a_1, \ldots, a_n$ in $\mathbb{K}$, every choice of signs $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ and all subsets A of $\{1, \ldots, n\}$ we have

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that

$$\left\| \sum_{k \in A} \varepsilon_k a_k x_k \right\| \leq C \left\| \sum_{k=1}^{n} a_k x_k \right\|. \tag{1}$$

The smallest constant C fitting in (1) is known as the *unconditional constant* of $(x_n)_{n\in\mathbb{N}}$ in E. Plenty of literature about this topic has appeared in the past years. In particular we are interested in estimating the unconditional constants of certain bases in spaces of polynomials.

Using the standard notation for multi-indices, let $\mathbf{x}^\alpha$ denote the monomial $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$, where $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $\alpha_k \in \mathbb{N} \cup \{0\}$, $1 \leq k \leq m$. Now let $\mathcal{B}_n = \{\mathbf{x}^\alpha : |\alpha| \leq n\}$ be the canonical basis of $\mathcal{P}_n(\mathbb{R}^m)$, the space of polynomials of degree not greater than n on $\mathbb{R}^m$, and consider $\mathcal{S} = \{\mathbf{x}^{\alpha_k} : |\alpha_k| \leq n, 1 \leq k \leq r\}$ any subset of $\mathcal{B}_n$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ a choice of signs, $P(\mathbf{x}) = a_{\alpha_1}\mathbf{x}^{\alpha_1} + \cdots + a_{\alpha_r}\mathbf{x}^{\alpha_r}$ and $P_\varepsilon(\mathbf{x}) = \varepsilon_1 a_{\alpha_1}\mathbf{x}^{\alpha_1} + \cdots + \varepsilon_r a_{\alpha_r}\mathbf{x}^{\alpha_r}$. Then, if B is a convex set in $\mathbb{R}^m$, we have

$$\|P_\varepsilon\|_{\mathsf{B}} := \sup_{\mathbf{x}\in\mathsf{B}} |P_\varepsilon(\mathbf{x})| \leq \sup_{\mathbf{x}\in\mathsf{B}} |a_{\alpha_1}|\|\mathbf{x}|^{\alpha_1} + \cdots + |a_{\alpha_r}|\|\mathbf{x}|^{\alpha_r} = |||P|||_{\mathsf{B}},$$

where $|P|(\mathbf{x}) = |a_{\alpha_1}|\mathbf{x}^{\alpha_1} + \cdots + |a_{\alpha_r}|\mathbf{x}^{\alpha_r}$. Moreover, if $\varepsilon_k = \operatorname{sign}(a_{\alpha_k})$, then $\|P_\varepsilon\|_{\mathsf{B}} = |||P|||_{\mathsf{B}}$. This shows that the unconditional constant of $\mathcal{S}$ coincides with the best possible constant $C_{\mathsf{B},\mathcal{S}}$ in the inequality

$$|||P|||_{\mathsf{B}} \leq C_{\mathsf{B},\mathcal{S}}\|P\|_{\mathsf{B}},$$

for every P in the space generated by $\mathcal{S}$. If B is the unit ball of a Banach space with norm $\|\cdot\|$, an interesting question related to the problem of finding the constants $C_{\mathsf{B},\mathcal{S}}$ consists of determining the largest radius $r > 0$ so that $|||P|||_{\mathsf{B}_r} := \sup\{|P(x)| : \|x\| \leq r\} \leq 1$ whenever $\|P\|_{\mathsf{B}} \leq 1$ for all P generated by $\mathcal{S}$. It should be noticed that this is a real polynomial version of a multi-dimensional analogue of Bohr's Theorem (see i.e. [1,5–8]).

We provide the exact value of the unconditional constant of the canonical basis in the space of real trinomials endowed with the sup norm over the interval $[-1, 1]$. Related results for complex trigonometric trinomials can be found in [14]. On the other hand, if $\mathcal{P}({}^n\ell_p^m)$ denotes, as usual, the space of n-homogeneous polynomials on ℓ_p^m endowed with the sup norm over the unit ball of ℓ_p^m, then the precise asymptotic order in m of the unconditional basis constants in $\mathcal{P}({}^n\ell_p^m)$ is given in [4]. In particular, for $1 \leq p \leq 2$ the asymptotic order in m is $m^{(n-1)/q}$, where q is the conjugate exponent of p. In Theorem 3.16 we prove that in the case $1 \leq p \leq 2$ the unconditional constant of the canonical basis in the space $\mathcal{P}({}^n\ell_p^m)$ is $m^{(n-1)/q}$, for every $m \in \mathbb{N}$. We also give the exact value of the unconditional constant in the space $\mathcal{P}({}^n\ell_p^m)$ for certain values of $m, n \in \mathbb{N}$ and $1 \leq p \leq \infty$, and in the space of 2-homogeneous polynomials on $\mathbb{R}^2$ endowed with the sup norm over the simplex, always referred to the canonical basis.

Many of the problems studied in this paper involve at most three variables, especially the inequalities that concern trinomials. For this reason one might be tempted to use elementary calculus from scratch. However a direct application of elementary calculus is not recommendable, moreover, most of the times it is not even feasible. Instead, one of the techniques that will be used in order to tackle these problems will rely on the following well known consequence of the Krein–Milman Theorem:

Remark 1.1. If C is a convex body in a Banach space and $f : C \to \mathbb{R}$ is a convex function that attains its maximum, then there is an extreme point $e \in C$ so that $f(e) = \max\{f(x) : x \in C\}$.

For every real polynomial P and every $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ we will often write $\|P\|_{[\alpha,\beta]}$ to denote the sup norm of P over the interval $[\alpha, \beta]$. From now on, $\mathcal{P}_{m,n}(\mathbb{R})$ will denote the three-dimensional space of polynomials of the form $ax^m + bx^n + c$ with $m, n \in \mathbb{N}$, $m > n$ and $a, b, c \in \mathbb{R}$ endowed with the norm $\|\cdot\|_{m,n}$ defined on $\mathbb{R}^3$ by

$$\|ax^m + bx^n + c\|_{m,n} := \max_{x\in[-1,1]} |ax^m + bx^n + c|.$$

Also, $\mathcal{P}(^2\Delta)$ denotes the space of 2-homogeneous polynomials on the simplex (the region enclosed by the triangle in $\mathbb{R}^2$ of vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$) endowed with the sup norm on Δ. If E is a Banach space, B_E and S_E denote, as usual, its closed unit ball and its unit sphere respectively. In the case where $E = \mathcal{P}_{m,n}(\mathbb{R})$ or $\mathcal{P}(^2\Delta)$, the notation B_E will be replaced by $\mathsf{B}_{m,n}$ and B_Δ, respectively. Also, ext (C) denotes the set of extreme points of the convex set C.

2. Unconditional constants in spaces of trinomials

This section is devoted to the study of the inequality

$$|||P|||_{m,n} \le C(m, n)\|P\|_{m,n}, \tag{2}$$

where $P \in \mathcal{P}_{m,n}(\mathbb{R})$, $m, n \in \mathbb{N}$, with $m > n$, and $C(m, n)$ is the smallest positive constant in (2). Observe that if $P(x) = ax^m + bx^n + c$ for every $x \in \mathbb{R}$, then

$$|||P|||_{m,n} = \sup\{\big||a|x^m + |b|x^n + |c|\big| : x \in [-1, 1]\} = |a| + |b| + |c|.$$

On the other hand, dividing the inequality $|||P|||_{m,n} \le C(m, n)\|P\|_{m,n}$ by $\|P\|_{m,n}$, we can use without loss of generality polynomials of norm not greater than one in order to calculate $C(m, n)$. Hence

$$C(m, n) = \sup\{|a| + |b| + |c| : ax^m + bx^n + c \in \mathsf{B}_{m,n}\}.$$

Now since the mapping $\mathsf{B}_{m,n} \ni ax^m + bx^n + c \mapsto |a| + |b| + |c|$ is convex, attains its maximum and $\mathsf{B}_{m,n}$ is convex too, we finally obtain from Remark 1.1 that

$$C(m, n) = \sup\{|a| + |b| + |c| : ax^m + bx^n + c \in \text{ext}\,(\mathsf{B}_{m,n})\}. \tag{3}$$

A combination of (3) and the description of ext $(\mathsf{B}_{m,n})$ provided in [13] lets us obtain $C(m, n)$ for all choices of m and n. For this purpose we will also require a technical lemma whose proof, being a direct application of the Implicit Function Theorem, is left to the reader.

Lemma 2.1. *Let $m, n \in \mathbb{N}$ with $m > n$. Then the equation*

$$\frac{(m-n)t}{n}\left(\frac{nx}{mt}\right)^{\frac{m}{m-n}} = 2 - t - x, \tag{4}$$

defines implicitly a unique differentiable curve $x = \Gamma(t)$ defined on $(0, \infty)$ such that $\Gamma(n/m) = 1$, $\Gamma(2) = 0$ and $\Gamma(t) > 0$ whenever $t \in (0, 2)$.

This following lemma will be also necessary in order to complete the proof of the main theorem of this section. Its proof can be also found in [13].

Lemma 2.2. *If $m, n \in \mathbb{N}$ are such that $m > n$ then the equation*

$$|n + mx| = (m - n)|x|^{\frac{m}{m-n}}$$

has only three roots, one at $x = -1$, another one at a point $\lambda_0 \in (-\frac{n}{m}, 0)$ and a third one at a point $\lambda_1 > 0$.

Now we are ready to state and prove the main result of this section.

Theorem 2.3. *If $m, n \in \mathbb{N}$ with $m > n$ then*

$$C(m,n) = \begin{cases} 3 & \text{if } m \text{ and } n \text{ have different parity,} \\ 1 + \dfrac{4}{m-n}\left(\dfrac{m^m}{n^n}\right)^{\frac{1}{m-n}} & \text{if } m, n \text{ are even,} \\ \dfrac{n - \lambda_0 m}{n + \lambda_0 m} & \text{if } m, n \text{ are odd,} \end{cases}$$

where λ_0 is given by Lemma 2.2. Therefore the unconditional constant of the canonical basis of $\mathcal{P}_{m,n}(\mathbb{R})$ is $C(m,n)$.

Proof. The proof will be done by cases. Thus, we will have to prove that:

(a) $C(m,n) = 3$ for every $m, n \in \mathbb{N}$ with different parity.

(b) $C(m,n) = 1 + \frac{4}{m-n}\left(\frac{m^m}{n^n}\right)^{\frac{1}{m-n}}$ for every $m, n \in \mathbb{N}$ even.

(c) $C(m,n) = \frac{n-\lambda_0 m}{n+\lambda_0 m}$ for every $m, n \in \mathbb{N}$ odd.

From now on, and in this proof, we will use Eq. (3) repeatedly without mentioning it explicitly. The proof of the first case, even though the result is the same, depends considerably on whether m is even or odd. If m is odd then [13, Theorem 3.4] states that the extreme polynomials of $\mathsf{B}_{m,n}$ are

$$\left\{\pm(2x^n - 1), \pm(x^m + x^n - 1), \pm(-x^m + x^n - 1), \pm 1\right\}.$$

Since the sum of the absolute values of the coefficients of $\pm(2x^n - 1), \pm(x^m + x^n - 1)$ and $\pm(-x^m + x^n - 1)$ is 3, the result follows. Notice that in this case, equality is attained in (2) for the scalar multiples of the polynomials $2x^n - 1$, $x^m + x^n - 1$ and $-x^m + x^n - 1$. Now, if m is even (recall that n has to be odd), then ([13, Theorem 4.6]) the extreme polynomials of $\mathsf{B}_{m,n}$ are

$$\left\{\pm\left(tx^m \pm \Gamma(t)x^n + 1 - t - \Gamma(t)\right), \pm 1 : \frac{n}{m} \le t \le 2\right\},$$

where $\Gamma(t)$ is defined in Lemma 2.1. Then,

$$C(m,n) = \max\left\{|t| + |\Gamma(t)| + |1 - t - \Gamma(t)|, 1 : \frac{n}{m} \le t \le 2\right\}.$$

To find the above maximum we will find first the critical points of $\Phi(t) = t + \Gamma(t)$ over the interval $[\frac{n}{m}, 2]$. By means of implicit differentiation, some calculations give:

$$\Gamma'(t) = \frac{\Gamma(t)}{t} \cdot \frac{2n - mt - n\Gamma(t)}{2m - mt - n\Gamma(t)}. \tag{5}$$

Hence, if t_0 is a critical point, it is not difficult to check that $\Phi'(t_0) = 0$ is equivalent to

$$mt_0(2 - t_0) + (2n - nt_0 - mt_0)\Gamma(t_0) - n\Gamma(t_0)^2 = 0,$$

from which one can show easily that either $\Gamma(t_0) = 2 - t_0$ or $\Gamma(t_0) = -\frac{mt_0}{n}$. This last solution has to be discarded since $\Gamma(t) \ge 0$ over the considered interval. Therefore we have that $\Gamma(t_0) - 2 + t_0 = 0$, which by means of Eq. (4) implies that

$$\frac{(m-n)t_0}{n}\left(\frac{n\Gamma(t_0)}{mt_0}\right)^{\frac{m}{m-n}} = 0.$$

It follows then that $\Gamma(t_0) = 0$, which gives $t_0 = 2$. Therefore, if $t \in \left[\frac{n}{m}, 2\right]$,

$$t + \Gamma(t) = \Phi(t) \geq \min\left\{\Phi(t) : \frac{n}{m} \leq t \leq 2\right\} = \min\left\{\Phi\left(\frac{n}{m}\right), \Phi(2)\right\} = 1 + \frac{n}{m} > 1.$$

Thus, for all $t \in \left[\frac{n}{m}, 2\right]$ we have $1 - t - \Gamma(t) < 0$, from which

$$C(m,n) = \max\left\{2t + 2\Gamma(t) - 1 : \frac{n}{m} \leq t \leq 2\right\} = \max\left\{2\Phi(t) - 1 : t \in \left[\frac{n}{m}, 2\right]\right\}.$$

Since the mapping $2\Phi - 1$ has the same critical points as Φ we obtain that

$$C(m,n) = \max\left\{2\Phi\left(\frac{n}{m}\right) - 1, 2\Phi(2) - 1\right\} = 3,$$

and, in this case, equality is attained in (2) for the scalar multiples of the polynomial $2x^m - 1$.

Now, if m and n are both even, then according to [13, Theorem 5.5] the extreme polynomials of $\mathsf{B}_{m,n}$ are

$$\left\{\pm\left(sx^m + \Lambda(s)x^n - 1 - s - \Lambda(s)\right), \pm\left(tx^m - \Upsilon(t)x^n + 1\right), \pm 1\right\},$$

where $s \in [\gamma_0, -2]$, $t \in [-\gamma_1, -\gamma_0]$, $\gamma_0 = -\frac{2}{m-n} \cdot \left(\frac{m^m}{n^n}\right)^{\frac{1}{m-n}}$, $\gamma_1 = \frac{-2n}{m-n}$ and Λ and Υ are defined by

$$\Lambda(s) = -\Gamma(|s|) \quad \text{and} \quad \Upsilon(t) = \left(\frac{2m}{m-n}\right)^{\frac{m-n}{m}} \cdot \left(\frac{mt}{n}\right)^{\frac{n}{m}}.$$

On the one hand

$$\begin{aligned}\max\{|t| + |\Upsilon(t)| + 1 : -\gamma_1 \leq t \leq -\gamma_0\} &= \max\{t + \Upsilon(t) + 1 : -\gamma_1 \leq t \leq -\gamma_0\}\\ &= -2\gamma_0 + 1,\end{aligned}$$

and on the other hand

$$\max\{|s| + |\Lambda(s)| + |1 + s + \Lambda(s)| : \gamma_0 \leq s \leq -2\} = \begin{cases}-1 - 2s & \text{if } 1 + s + \Lambda(s) \leq 0,\\ 1 + 2\Lambda(s) & \text{otherwise.}\end{cases}$$

Since it can be proved (see [13]) that $\Lambda(\gamma_0) = -\gamma_0$ and Λ is decreasing in the considered interval, from the latter formula it follows that

$$\max\{|s| + |\Lambda(s)| + |1 + s + \Lambda(s)| : \gamma_0 \leq s \leq -2\} \leq -2\gamma_0 + 1.$$

Therefore

$$C(m,n) = -2\gamma_0 + 1 = 1 + \frac{4}{m-n}\left(\frac{m^m}{n^n}\right)^{\frac{1}{m-n}},$$

and, in this case, equality is achieved in (2) for the scalar multiples of the polynomial $-\gamma_0 x^m + \gamma_0 x^n + 1$.

To prove (c) notice that from [13, Theorem 2.6], the extreme polynomials of $\mathsf{B}_{m,n}$ are the following:

$$\left\{\pm\left(t\cdot x^m-\frac{mt^{\frac{n}{m}}}{(m-n)^{\frac{m-n}{m}}n^{\frac{n}{m}}}\cdot x^n\right),\pm 1:\frac{n}{m-n}\le t\le\frac{n}{n+m\lambda_0}\right\}. \tag{6}$$

Hence

$$C(m,n)=\max\left\{|t|+\left|\frac{mt^{\frac{n}{m}}}{(m-n)^{\frac{m-n}{m}}n^{\frac{n}{m}}}\right|,1:\frac{n}{m-n}\le t\le\frac{n}{n+m\lambda_0}\right\}$$
$$=\max\left\{t+\frac{mt^{\frac{n}{m}}}{(m-n)^{\frac{m-n}{m}}n^{\frac{n}{m}}}:\frac{n}{m-n}\le t\le\frac{n}{n+m\lambda_0}\right\}.$$

And, since $\Psi(t)=t+\frac{mt^{\frac{n}{m}}}{(m-n)^{\frac{m-n}{m}}n^{\frac{n}{m}}}$ is an increasing function, we will have that

$$C(m,n)=\Psi\left(\frac{n}{n+m\lambda_0}\right),$$

which, after performing some calculations and by using that $|n+m\lambda_0|=(m-n)|\lambda_0|^{\frac{m}{m-n}}$, yields the desired result. Moreover, in this case equality is achieved in (2) for the scalar multiples of the polynomial $\frac{nx^m}{n+m\lambda_0}-\frac{m|\lambda_0|x^n}{n+m\lambda_0}$. ∎

Remark 2.4. In Theorem 2.3 it can be seen that, if $k\in\mathbb{N}$, $k>1$, then for every $n\in\mathbb{N}$ even we have

$$C(kn,n)=1+\frac{4}{k-1}\cdot k^{\frac{k}{k-1}},$$

independent of n.

3. Unconditional constants in spaces of homogeneous polynomials

3.1. Unconditional constants in $\mathcal{P}(^2\Delta)$

The geometry of the unit ball of $\mathcal{P}(^2\Delta)$ is described in detail in [12]. In particular the authors give an explicit description of B_Δ through its extreme points. In the following recall that if $P\in\mathcal{P}(^2\Delta)$, then $\|P\|_\Delta$ denotes the sup norm of P over Δ.

Theorem 3.1. *If $P\in\mathcal{P}(^2\Delta)$ then*

$$|||P|||_\Delta\le 2\|P\|_\Delta,$$

and 2 is optimal in the previous inequality. Therefore the unconditional constant (referred to the canonical basis) of $\mathcal{P}(^2\Delta)$ is 2.

Proof. Let C_Δ be defined as

$$C_\Delta:=\sup\{|||P|||_\Delta:P\in\mathsf{B}_\Delta\}.$$

Then since $\mathsf{B}_\Delta\ni P\mapsto|||P|||_\Delta$ is convex, using Remark 1.1 and the characterization of the extreme points of B_Δ given in [12], namely

$$\pm(x^2-(2+2\sqrt{2(1+t)})xy+ty^2),\qquad\pm(tx^2-(2+2\sqrt{2(1+t)})xy+y^2),$$

where $t \in [-1, 1]$, we obtain

$$\begin{aligned} C_\Delta &= \sup\{|||P|||_\Delta : P \in \operatorname{ext}(\mathsf{B}_\Delta)\} \\ &= \sup\{\max\{\|x^2 + (2 + 2\sqrt{2(1+t)})xy + |t|y^2\|_\Delta, \\ &\qquad \||t|x^2 + (2 + 2\sqrt{2(1+t)})xy + y^2\|_\Delta\} : t \in [-1, 1]\}. \end{aligned}$$

Now using some simple calculations and the fact that (see [12, Theorem 2.1])

$$\|ax^2 + bxy + cy^2\|_\Delta = \begin{cases} \max\left\{|a|, |c|, \left|\dfrac{b^2 - 4ac}{4(a - b + c)}\right|\right\} \\ \quad \text{if } a - b + c \neq 0 \text{ and } 0 < \dfrac{2c - b}{2(a - b + c)} < 1, \\ \max\{|a|, |c|\} \quad \text{otherwise,} \end{cases}$$

we obtain that $C_\Delta = 2$. If $P \in \mathcal{P}(^2\Delta)$ is arbitrary, applying the latter to $P/\|P\|_\Delta$ we infer that $|||P|||_\Delta \leq 2\|P\|_\Delta$. Moreover, the constant 2 is sharp since equality holds for the scalar multiples of the polynomial $P(x, y) = x^2 - 6xy + y^2$. ■

3.2. Unconditional constants in $\mathcal{P}(^n\ell_p^m)$

In this section $C_{p,m,n}$ will denote the unconditional constant of the canonical basis of $\mathcal{P}(^n\ell_p^m)$. If $P \in \mathcal{P}(^n\ell_p^m)$, we will use the notation

$$\|P\|_{\ell_p^m} := \sup\{|P(x)| : \|x\|_p \leq 1\},$$

to denote the sup norm of P over the unit ball of ℓ_p^m. Here and from now on, $\|\cdot\|_p$ denotes the usual ℓ_p norm.

In order to prove the following theorem we will use the characterization of the extreme points of $\mathsf{B}_{\mathcal{P}(^2\ell_p^2)}$ appearing in [10].

Theorem 3.2. *Let $1 < p < \infty$, $p \neq 2$ and*

$$f(\alpha) := \frac{2^{\frac{p-2}{p}}\left[\alpha(1-\alpha^p)\left(\alpha - (1-\alpha^p)^{\frac{1}{p}}\right) + \alpha^p(1-\alpha^p)^{\frac{1}{p}}\left(\alpha + (1-\alpha^p)^{\frac{1}{p}}\right)\right]}{\alpha(1-\alpha^p)^{\frac{1}{p}}\left(\alpha^2 + (1-\alpha^p)^{\frac{2}{p}}\right)}.$$

If we set $M_f := \sup\{f(\alpha) : \alpha \in [2^{-\frac{1}{p}}, 1]\}$ then, for every $P \in \mathcal{P}(^2\ell_p^2)$, we have

$$|||P|||_{\ell_p^2} \leq M_f\|P\|_{\ell_p^2}.$$

Moreover, $C_{p,2,2} = M_f$.

Proof. Since $\mathsf{B}_{\mathcal{P}(^2\ell_p^2)} \ni P \mapsto |||P|||_{\ell_p^2}$ is convex, by Remark 1.1 we have that

$$C_{p,2,2} = \sup\{|||P|||_{\ell_p^2} : P \in \operatorname{ext}(\mathsf{B}_{\mathcal{P}(^2\ell_p^2)})\}.$$

We will divide this proof into two cases. Let us first start with the case $p > 2$. From Propositions 2.1 and 2.3 in [10] we have that the extreme polynomials of $\mathsf{B}_{\mathcal{P}(^2\ell_p^2)}$ are of the form

(a) $P(x, y) = ax^2 + cy^2$ where $ac \geq 0$ and $|a|^{\frac{p}{p-2}} + |c|^{\frac{p}{p-2}} = 1$, or
(b) $P(x, y) = \pm(x^2 - y^2)$, or
(c) $P(x, y) = \pm\left(\frac{\alpha^p - \beta^p}{\alpha^2+\beta^2}(x^2 - y^2) + 2\alpha\beta\frac{\alpha^{p-2}+\beta^{p-2}}{\alpha^2+\beta^2}xy\right)$, with $\alpha, \beta \geq 0$ and $\alpha^p + \beta^p = 1$.

If P is as in (a) or (b), then $\|\|P\|\|_{\ell_p^2} = \|P\|_{\ell_p^2} = 1$. Thus we will focus our attention on the polynomials of type (c). It is easy to see that if P is as in (c) the polynomial $|P|$ attains its ℓ_p^2 norm at $(2^{-\frac{1}{p}}, 2^{-\frac{1}{p}})$, from which we obtain

$$\begin{aligned} C_{p,2,2} &= \sup\left\{\left\|\frac{\alpha^p - \beta^p}{\alpha^2+\beta^2}(x^2+y^2) + 2\alpha\beta\frac{\alpha^{p-2}+\beta^{p-2}}{\alpha^2+\beta^2}xy\right\|_{\ell_p^2} : \alpha \geq \beta \geq 0, \alpha^p + \beta^p = 1\right\} \\ &= 2^{1-\frac{2}{p}} \sup\left\{\frac{\alpha\beta^p(\alpha-\beta) + \alpha^p\beta(\alpha+\beta)}{\alpha\beta(\alpha^2+\beta^2)} : \alpha \geq \beta \geq 0, \alpha^p+\beta^p = 1\right\} \\ &= \sup\{f(\alpha) : 2^{-\frac{1}{p}} \leq \alpha \leq 1\} = M_f. \end{aligned}$$

This concludes the proof for the case $p > 2$. In a further remark we will give a few particular values for M_f.

Secondly, if $p < 2$, then the extreme polynomials of $\mathsf{B}_{\mathcal{P}(^2\ell_p^2)}$ are as in (a), (b), (c) or

(d) $P(x, y) = \pm\left(\frac{\alpha^p-\beta^p}{\alpha^2-\beta^2}(x^2+y^2) \pm 2\alpha\beta\frac{\alpha^{p-2}-\beta^{p-2}}{\alpha^2-\beta^2}xy\right)$, with $\alpha, \beta \geq 0$ and $\alpha^p + \beta^p = 1$.

Notice that when $\alpha \to \beta$ we also obtain extreme polynomials from (d), namely $P(x, y) = 2^{\frac{2}{p-2}}(x^2+y^2) + 2^{\frac{2}{p}-1}(2-p)xy$. Anyway, if P is as in (d), then $|P|$ is again a polynomial as in (d), and therefore $\|\|P\|\|_{\ell_p^2}/\|P\|_{\ell_p^2} = 1$. This shows that in this case $C_{p,2,2} = M_f$ holds too. ∎

Remark 3.3. It will be seen in Theorem 3.19 that $C_{2,2,2} = \sqrt{2}$, which completes Theorem 3.2.

Remark 3.4. In the previous theorem, the value of $C_{p,2,2}$ cannot be obtained explicitly in general, however, using symbolic calculus software a few exact values for $C_{p,2,2}$ can be obtained for some specific values of p, as shown in the table below.

p	$C_{p,2,2}$
$\frac{3}{2}$	$\frac{1}{2}\left(3+5\sqrt{2}+2\sqrt{2+10\sqrt{2}}\right)^{1/3} \simeq 1.31$
3	$\left(\frac{1}{2}+\frac{5}{\sqrt{2}}\right)^{1/3} \simeq 1.59$
4	$\sqrt{3} \simeq 1.73$
5	$\left(\frac{1}{2}+\frac{29}{\sqrt{2}}\right)^{1/5} \simeq 1.83$
6	$2^{\frac{1}{6}}5^{\frac{1}{3}} \simeq 1.91$
8	$17^{\frac{1}{4}} \simeq 2.03$

The constants $C_{1,2,2}$ and $C_{\infty,2,2}$ can also be obtained by using the same technique and the characterization of the extreme points of the unit ball of $\mathcal{P}(^2\ell_1^2)$ and $\mathcal{P}(^2\ell_\infty^2)$ that appears in [3, Theorem 2.6] and [2, Page 477] respectively.

Theorem 3.5. *If* $P \in \mathcal{P}(^2\ell_1^2)$ *then*

$$\|\|P\|\|_{\ell_1^2} \leq \frac{1+\sqrt{2}}{2}\|P\|_{\ell_1^2}$$

and equality is attained for the scalar multiples of the polynomials $\pm\frac{\sqrt{2}}{2}(x^2-y^2)\pm(2+\sqrt{2})xy$. *Moreover,* $C_{1,2,2}=\frac{1+\sqrt{2}}{2}$.

Proof. As we did before we have that

$$C_{1,2,2}=\sup\{|||P|||_{\ell_1^2} : P\in \operatorname{ext}(\mathsf{B}_{\mathcal{P}(^2\ell_1^2)})\}.$$

On the other hand, from Theorem 2.6 in [3] the extreme polynomials of $\mathsf{B}_{\mathcal{P}(^2\ell_1^2)}$ are of the form:

(a) $P(x,y)=\pm x^2\pm 2xy\pm y^2$, or
(b) $P(x,y)=\pm\frac{\sqrt{4|t|-t^2}}{2}(x^2-y^2)+txy$, where $|t|\in(2,4]$.

If P is as in (a), then $|||P|||_{\ell_1^2}=\|P\|_{\ell_1^2}=1$. Thus we will focus our attention on the polynomials of type (b). It is easy to see that if P is as in (b), then the polynomial $|P|$ attains its ℓ_1^2 norm at $\left(\frac{1}{2},\frac{1}{2}\right)$. From this we obtain

$$\begin{aligned} C_{1,2,2} &= \sup\left\{\left\|\frac{\sqrt{4|t|-t^2}}{2}(x^2+y^2)+|t|xy\right\|_{\ell_1^2} : |t|\in(2,4]\right\} \\ &= \sup\left\{\left\|\frac{\sqrt{4s-s^2}}{2}(x^2+y^2)+sxy\right\|_{\ell_1^2} : s\in(2,4]\right\} \\ &= \sup\left\{\frac{\sqrt{4s-s^2}+s}{4} : s\in(2,4]\right\}. \end{aligned}$$

Notice that we have performed above the change of variables $s=|t|$. A simple calculus exercise shows that $(2,4]\ni s\mapsto\frac{\sqrt{4s-s^2}+s}{4}$ attains its maximum at $s=2+\sqrt{2}$, ($t=\pm(2+\sqrt{2})$) and that the maximum is $\frac{1+\sqrt{2}}{2}$. This concludes the proof. ■

Theorem 3.6. *If* $P\in\mathcal{P}(^2\ell_\infty^2)$ *then*

$$|||P|||_{\ell_\infty^2}\le(1+\sqrt{2})\|P\|_{\ell_\infty^2}$$

and equality is attained for the scalar multiples of the polynomials $\frac{2+\sqrt{2}}{4}(x^2-y^2)\pm\frac{\sqrt{2}}{2}xy$. *Moreover,* $C_{\infty,2,2}=1+\sqrt{2}$.

Proof. As usual

$$C_{\infty,2,2}=\sup\{|||P|||_{\ell_\infty^2} : P\in \operatorname{ext}(\mathsf{B}_{\mathcal{P}(^2\ell_\infty^2)})\}.$$

On the other hand, the extreme polynomials of $\mathsf{B}_{\mathcal{P}(^2\ell_\infty^2)}$ are described in [2] and are of the form:

(a) $P(x,y)=\pm x^2$, or
(b) $P(x,y)=\pm y^2$, or
(c) $P(x,y)=\pm\left(tx^2-ty^2\pm 2\sqrt{t(1-t)}xy\right)$, where $t\in\left[\frac{1}{2},1\right]$.

If P is as in (a) or (b), then $|||P|||_{\ell_\infty^2}=\|P\|_{\ell_\infty^2}=1$. Thus we will focus our attention on the polynomials of type (c). It is easy to see that if P is as in (c), then the polynomial $|P|$ attains its ℓ_∞^2 norm at $(1,1)$. From this we obtain

$$\begin{aligned} C_{\infty,2,2} &= \sup\left\{\left\|tx^2+ty^2+2\sqrt{t(1-t)}xy\right\|_{\ell_\infty^2} : t\in\left[\frac{1}{2},1\right]\right\} \\ &= \sup\left\{2t+2\sqrt{t(1-t)} : t\in\left[\frac{1}{2},1\right]\right\} \\ &= 1+\sqrt{2}. \end{aligned}$$

Having in mind that the above supremum is attained at $t=\frac{2+\sqrt{2}}{4}$, the proof is concluded. ■

Remark 3.7. For $m,n\in\mathbb{N}$, notice that if $\|\cdot\|_a$ and $\|\cdot\|_b$ are two norms on $\mathbb{R}^m$ with unit balls B_a and B_b respectively and, as usual, $C_{i,m,n}$ is the smallest constant in $|||P|||_{\mathsf{B}_i}\le C_{i,m,n}\|P\|_{\mathsf{B}_i}$ for all n-homogeneous polynomials P on $\mathbb{R}^m$, with $i=a,b$, then the fact that the spaces $(\mathbb{R}^m,\|\cdot\|_a)$ and $(\mathbb{R}^m,\|\cdot\|_b)$ are isometric do not imply that $C_{a,m,n}=C_{b,m,n}$. In order to prove this we just need to consider the isometric spaces $(\mathbb{R}^2,\|\cdot\|_1)$ and $(\mathbb{R}^2,\|\cdot\|_\infty)$ together with Theorems 3.5 and 3.6.

Little can be said about $C_{p,m,n}$ for $m,n\ge 3$. In the following results we give estimates on the ratio $|||P|||_{\ell_i^2}/\|P\|_{\ell_i^2}$ for specific families of polynomials P in $\mathcal{P}({}^3\ell_i^2)$ for $i=1,\infty$. Those estimates provide a lower bound for $C_{1,2,3}$ and $C_{\infty,2,3}$.

Lemma 3.8. *If for every $a,b\in\mathbb{R}$ we define $P_{a,b}(x,y)=ax^3+bx^2y+bxy^2+ay^3$ then*

$$\|P_{a,b}\|_{\ell_1^2}=\begin{cases}\dfrac{1}{18}\sqrt{3b^2-6ab-9a^2}\left|\dfrac{3a-b}{a+b}\right| & \text{if } a\neq 0 \text{ and } b_0\le\dfrac{b}{a}\le -3,\\ |a| & \text{if } a\neq 0 \text{ and } -3\le\dfrac{b}{a}\le 3,\\ \dfrac{|a+b|}{4} & \text{otherwise,}\end{cases}$$

where

$$b_0=-\frac{36\sqrt[3]{4}+24\sqrt[3]{2}+39}{23}\approx -5.49.$$

Proof. The reader can easily check that

$$\|P_{0,b}\|_{\ell_1^2}=|b|\|x^2y+xy^2\|_{\ell_1^2}=\frac{|b|}{4}.$$

Now if $a\neq 0$ then $P_{a,b}=aP_{1,\frac{b}{a}}$. For simplicity let $\alpha=\frac{b}{a}$ and $P_\alpha=P_{1,\frac{b}{a}}$. By symmetry of $\mathsf{B}_{\ell_1^2}$ we have

$$\|P_\alpha\|_{\ell_1^2}=\max\{\|Q_\alpha\|_{[0,1]},\|R_\alpha\|_{[-1,0]}\},$$

where

$$\begin{aligned} Q_\alpha(x)&=P_\alpha(x,1-x)=-(\alpha-3)x^2+(\alpha-3)x+1\\ R_\alpha(x)&=P_\alpha(x,1+x)=2(\alpha+1)x^3+3(\alpha+1)x^2+(\alpha+3)x+1. \end{aligned}$$

Taking into consideration the fact that Q_α has its vertex at $x=\frac{1}{2}$, we have that

$$\begin{aligned} \|Q_\alpha\|_{[0,1]} &= \max\left\{|Q_\alpha(0)|,|Q_\alpha(1)|,\left|Q_\alpha\left(\frac{1}{2}\right)\right|\right\}\\ &= \max\left\{1,\frac{|1+\alpha|}{4}\right\}=\begin{cases}\dfrac{|1+\alpha|}{4} & \text{if } \alpha\le -5 \text{ or } \alpha\ge 3,\\ 1 & \text{otherwise.}\end{cases} \end{aligned}$$

On the other hand, the reader can check that R_α has two critical points in $[-1, 0]$ at

$$\alpha_1 = -\frac{1}{6}\frac{3\alpha + 3 - \sqrt{3\alpha^2 - 6\alpha - 9}}{\alpha + 1} \quad \text{and} \quad \alpha_2 = -\frac{1}{6}\frac{3\alpha + 3 + \sqrt{3\alpha^2 - 6\alpha - 9}}{\alpha + 1}$$

if and only if $|\alpha| \geq 3$. In addition to this

$$|R_\alpha(\alpha_k)| = \frac{1}{18}\sqrt{3\alpha^2 - 6\alpha - 9}\left|\frac{\alpha - 3}{\alpha + 1}\right|,$$

for $k = 1, 2$. Hence, taking into consideration the fact that the equation

$$\frac{1}{18}\sqrt{3\alpha^2 - 6\alpha - 9}\left|\frac{\alpha - 3}{\alpha + 1}\right| = 1,$$

has only two roots at -3 and 15, it is not difficult to see that

$$\begin{aligned}
\|R_\alpha\|_{[-1,0]} &= \begin{cases} \max\{|R_\alpha(-1)|, |R_\alpha(0)|, |R_\alpha(\alpha_1)|, |R_\alpha(\alpha_2)|\} & \text{if } |\alpha| \geq 3, \\ \max\{|R_\alpha(-1)|, |R_\alpha(0)|\} & \text{otherwise} \end{cases} \\
&= \begin{cases} \max\left\{1, \dfrac{1}{18}\sqrt{3\alpha^2 - 6\alpha - 9}\left|\dfrac{\alpha - 3}{\alpha + 1}\right|\right\} & \text{if } |\alpha| \geq 3, \\ 1 & \text{otherwise} \end{cases} \\
&= \begin{cases} \dfrac{1}{18}\sqrt{3\alpha^2 - 6\alpha - 9}\left|\dfrac{\alpha - 3}{\alpha + 1}\right| & \text{if } \alpha \leq -3 \text{ or } \alpha \geq 15, \\ 1 & \text{otherwise.} \end{cases}
\end{aligned}$$

Putting all this together we arrive at

$$\|P_\alpha\|_{\ell_1^2} = \begin{cases} \max\left\{\dfrac{|1+\alpha|}{4}, \dfrac{1}{18}\sqrt{3\alpha^2 - 6\alpha - 9}\left|\dfrac{\alpha - 3}{\alpha + 1}\right|\right\} & \text{if } \alpha \leq -5, \\ \dfrac{1}{18}\sqrt{3\alpha^2 - 6\alpha - 9}\left|\dfrac{\alpha - 3}{\alpha + 1}\right| & \text{if } -5 \leq \alpha \leq -3, \\ 1 & \text{if } -3 \leq \alpha \leq 3, \\ \dfrac{|1+\alpha|}{4} & \text{if } \alpha \geq 3. \end{cases}$$

Finally the equation

$$\frac{|1+\alpha|}{4} = \frac{1}{18}\sqrt{3\alpha^2 - 6\alpha - 9}\left|\frac{\alpha - 3}{\alpha + 1}\right|,$$

has a unique root in $(-\infty, -5]$ at b_0 and it is not difficult to see that

$$\frac{|1+\alpha|}{4} < \frac{1}{18}\sqrt{3\alpha^2 - 6\alpha - 9}\left|\frac{\alpha - 3}{\alpha + 1}\right|,$$

if $\alpha \in (b_0, -5]$, which leads us to

$$\|P_\alpha\|_{\ell_1^2} = \begin{cases} \dfrac{1}{18}\sqrt{3\alpha^2 - 6\alpha - 9}\left|\dfrac{\alpha - 3}{\alpha + 1}\right| & \text{if } b_0 \leq \alpha \leq -3, \\ 1 & \text{if } -3 \leq \alpha \leq 3, \\ \dfrac{|1+\alpha|}{4} & \text{otherwise.} \end{cases}$$

This concludes the proof. ■

Proposition 3.9. *If $P_{a,b}(x,y) = ax^3 + bx^2y + bxy^2 + ay^3$ then for every $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$ we have*

$$\frac{\|\!|P_{a,b}|\!\|_{\ell_1^2}}{\|P_{a,b}\|_{\ell_1^2}} = \begin{cases} \dfrac{9|a^2-b^2|}{2|3a-b|\sqrt{3b^2-6ab-9a^2}} & \text{if } a \neq 0 \text{ and } b_0 \leq \dfrac{b}{a} \leq -3, \\ 1 & \text{if } a \neq 0 \text{ and } -3 \leq \dfrac{b}{a}, \\ \dfrac{|a|+|b|}{|a+b|} & \text{otherwise}, \end{cases}$$

where

$$b_0 = -\frac{36\sqrt[3]{4} + 24\sqrt[3]{2} + 39}{23} \approx -5.49.$$

Proof. From Lemma 3.8 it follows that

$$\|\!|P_{a,b}|\!\|_{\ell_1^2} = \|P_{|a|,|b|}\|_{\ell_1^2} = \begin{cases} |a| & \text{if } a \neq 0 \text{ and } \left|\dfrac{b}{a}\right| \leq 3, \\ \dfrac{|a|+|b|}{4} & \text{otherwise.} \end{cases}$$

The desired result is obtained by combining the latter with the facts that if $a \neq 0$ and $b_0 \leq \frac{b}{a} < -3$ then $(|a|+|b|)|a+b| = |a^2-b^2|$ and that if $a \neq 0$ and $\frac{b}{a} > 3$ then $\frac{|a|+|b|}{|a+b|} = 1$. ∎

Corollary 3.10. *If $P_{a,b}(x,y) = ax^3 + bx^2y + bxy^2 + ay^3$ then for every $(a,b) \in \mathbb{R}^2$ we have*

$$\|\!|P_{a,b}|\!\|_{\ell_1^2} \leq \frac{b_0 - 1}{b_0 + 1}\|P_{a,b}\|_{\ell_1^2},$$

where

$$b_0 = -\frac{36\sqrt[3]{4} + 24\sqrt[3]{2} + 39}{23} \approx -5.49$$

and

$$\frac{b_0 - 1}{b_0 + 1} = \frac{18\sqrt[3]{4} + 12\sqrt[3]{2} + 31}{18\sqrt[3]{4} + 12\sqrt[3]{2} + 8} \approx 1.44.$$

Moreover, equality is attained for the scalar multiples of the polynomial P_{1,b_0}.

Proof. If $a = 0$ clearly we have $\frac{\|\!|P_{0,b}|\!\|_{\ell_1^2}}{\|P_{0,b}\|_{\ell_1^2}} = 1$. Otherwise, by Proposition 3.9 we have

$$\frac{\|\!|P_{a,b}|\!\|_{\ell_1^2}}{\|P_{a,b}\|_{\ell_1^2}} = \begin{cases} \dfrac{9|1-\alpha^2|}{2|3-\alpha|\sqrt{3\alpha^2-6\alpha-9}} & \text{if } b_0 \leq \alpha \leq -3, \\ 1 & \text{if } -3 \leq \alpha, \\ \dfrac{1+|\alpha|}{|1+\alpha|} & \text{otherwise}, \end{cases}$$

where $\alpha = \frac{b}{a}$. But this is a mapping in one real variable that can be proved to attain its absolute maximum at b_0, which proves the result. ∎

Remark 3.11. As an immediate consequence of the previous result we have that $C_{1,2,3} \geq \frac{b_0-1}{b_0+1} \approx 1.44$. The authors have numerical evidence showing that, in fact, $C_{1,2,3} = \frac{b_0-1}{b_0+1}$.

A similar result can be proved for polynomials in $\mathcal{P}(^3\ell_\infty^2)$.

Lemma 3.12. *If for every $a, b \in \mathbb{R}$ we define $P_{a,b}(x, y) = ax^3 + bx^2y + bxy^2 + ay^3$ then*

$$\|P_{a,b}\|_{\ell_\infty^2} = \begin{cases} \left| a - \dfrac{b^2}{3a} + \dfrac{2b^3}{27a^2} + \dfrac{2a}{27}\left(-\dfrac{3b}{a} + \dfrac{b^2}{a^2}\right)^{\frac{3}{2}} \right| \\ \quad \text{if } a \neq 0 \text{ and } b_1 \leq \dfrac{b}{a} \leq 3 - 2\sqrt{3}, \\ |2a + 2b| \quad \text{otherwise}, \end{cases}$$

where

$$b_1 = -\frac{3}{7}\left(3 - \frac{2\sqrt[3]{9}}{\sqrt{-12 + 7\sqrt{3}}} + 2\sqrt[3]{-36 + 21\sqrt{3}}\right) \approx -1.66.$$

Proof. The reader can easily check that

$$\|P_{0,b}\|_{\ell_\infty^2} = |b|\|x^2y + xy^2\|_{\ell_\infty^2} = 2|b|.$$

Now if $a \neq 0$ then $P_{a,b} = aP_{1,\frac{b}{a}}$. For simplicity let $\alpha = \frac{b}{a}$ and $P_\alpha = P_{1,\frac{b}{a}}$. By symmetry of $\mathsf{B}_{\ell_\infty^2}$, the norm $\|P_\alpha\|_{\ell_\infty^2}$ would be the maximum of the absolute value of P_α over the segments $\{(x, 1) : |x| \leq 1\}$ and $\{(1, y) : |y| \leq 1\}$. However the symmetry of the coefficients of P_α lets us write

$$\|P_\alpha\|_{\ell_\infty^2} = \max\{|P_\alpha(x, 1)| : |x| \leq 1\}.$$

In other words, if we put

$$Q_\alpha(x) = P_\alpha(x, 1) = x^3 + \alpha x^2 + \alpha x + 1,$$

then $\|P_\alpha\|_{\ell_\infty^2} = \|Q_\alpha\|_{[-1,1]}$. On the other hand, Q_α has two critical points at

$$\alpha_1 = \frac{-\alpha - \sqrt{\alpha^2 - 3\alpha}}{3} \quad \text{and} \quad \alpha_2 = \frac{-\alpha + \sqrt{\alpha^2 - 3\alpha}}{3}.$$

We also have that

$$\alpha_1 \in [-1, 1] \Leftrightarrow \alpha \leq 0 \quad \text{or } \alpha = 3$$
$$\alpha_2 \in [-1, 1] \Leftrightarrow \alpha \geq 3 \quad \text{or } -1 \leq \alpha \leq 0.$$

Notice that if $\alpha = -1$, then $\alpha_2 = 1$ and if $\alpha = 3$, then $\alpha_1 = \alpha_2 = -1$. Hence

$$\|P_\alpha\|_{\ell_\infty^2} = \begin{cases} \max\{|Q_\alpha(-1)|, |Q_\alpha(1)|, |Q_\alpha(\alpha_1)|\} & \text{if } \alpha \leq -1, \\ \max\{|Q_\alpha(-1)|, |Q_\alpha(1)|, |Q_\alpha(\alpha_1)|, |Q_\alpha(\alpha_2)|\} & \text{if } -1 < \alpha \leq 0, \\ \max\{|Q_\alpha(-1)|, |Q_\alpha(1)|\} & \text{if } 0 < \alpha < 3, \\ \max\{|Q_\alpha(-1)|, |Q_\alpha(1)|, |Q_\alpha(\alpha_2)|\} & \text{if } \alpha \geq 3. \end{cases} \tag{7}$$

Since $Q_\alpha(-1) = 0$, $|Q_\alpha(-1)|$ can be removed from (7). We also have that

$$Q_\alpha(1) = 2 + 2\alpha,$$

$$Q_\alpha(\alpha_1) = 1 - \frac{\alpha^2}{3} + \frac{2\alpha^3}{27} + \frac{2}{27}(-3\alpha + \alpha^2)^{\frac{3}{2}},$$
$$Q_\alpha(\alpha_2) = 1 - \frac{\alpha^2}{3} + \frac{2\alpha^3}{27} - \frac{2}{27}(-3\alpha + \alpha^2)^{\frac{3}{2}}.$$

It can be proved that the equation $|Q_\alpha(1)| = |Q_\alpha(\alpha_1)|$ has only two roots in the interval $(-\infty, 0]$ at b_1 and $3 - 2\sqrt{3}$. Using this it is not difficult to see that for $\alpha \in (-\infty, 0]$, $|Q_\alpha(1)| \leq |Q_\alpha(\alpha_1)|$ if and only if $\alpha \in [b_1, 3 - 2\sqrt{3}]$. On the other hand, the equation $|Q_\alpha(\alpha_1)| = |Q_\alpha(\alpha_2)|$ has only one root in the interval $[-1, 0]$ at 0. This lets us show easily that $|Q_\alpha(\alpha_1)| \geq |Q_\alpha(\alpha_2)|$ whenever $\alpha \in [-1, 0]$. To conclude, the equation $|Q_\alpha(1)| = |Q_\alpha(\alpha_2)|$ has no roots in $[3, \infty)$, from which it can be easily be inferred that $|Q_\alpha(1)| \geq |Q_\alpha(\alpha_2)|$ for every $\alpha \geq 3$. This concludes the proof. ■

Applying the previous result together with the fact that $|||P_{a,b}|||_{\ell_\infty^2} = 2|a| + 2|b|$, we easily arrive at the following:

Proposition 3.13. *If $P_{a,b}(x, y) = ax^3 + bx^2y + bxy^2 + ay^3$ then for every $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ we have*

$$\frac{|||P_{a,b}|||_{\ell_\infty^2}}{\|P_{a,b}\|_{\ell_\infty^2}} = \begin{cases} \dfrac{54a^2(|a| + |b|)}{\left|27a^3 - 9ab^2 + 2b^3 + 2\operatorname{sign}(a)\left(-3ab + b^2\right)^{\frac{3}{2}}\right|} \\ \quad \text{if } a \neq 0 \text{ and } b_1 \leq \dfrac{b}{a} \leq 3 - 2\sqrt{3}, \\ \dfrac{|a| + |b|}{|a + b|} \quad \text{otherwise}, \end{cases}$$

where

$$b_1 = -\frac{3}{7}\left(3 - \frac{2\sqrt[3]{9}}{\sqrt{-12 + 7\sqrt{3}}} + 2\sqrt[3]{-36 + 21\sqrt{3}}\right) \approx -1.66.$$

Corollary 3.14. *If $P_{a,b}(x, y) = ax^3 + bx^2y + bxy^2 + ay^3$ then for every $(a, b) \in \mathbb{R}^2$ we have*

$$|||P_{a,b}|||_{\ell_\infty^2} \leq \frac{b_1 - 1}{b_1 + 1}\|P_{a,b}\|_{\ell_\infty^2},$$

where

$$b_1 = -\frac{3}{7}\left(3 - \frac{2\sqrt[3]{9}}{\sqrt{-12 + 7\sqrt{3}}} + 2\sqrt[3]{-36 + 21\sqrt{3}}\right) \approx -1.66,$$

and $\frac{b_1 - 1}{b_1 + 1} \approx 3.98$. Moreover, equality is attained for the scalar multiples of the polynomial P_{1,b_1}.

Proof. If $a = 0$ clearly we have $\frac{|||P_{0,b}|||_{\ell_\infty^2}}{\|P_{0,b}\|_{\ell_\infty^2}} = 1$. Otherwise, by Proposition 3.13 we have

$$\frac{|||P_{a,b}|||_{\ell_\infty^2}}{\|P_{a,b}\|_{\ell_\infty^2}} = \begin{cases} \dfrac{54(1 + |\alpha|)}{\left|27 - 9\alpha^2 + 2\alpha^3 + 2\left(-3\alpha + \alpha^2\right)^{\frac{3}{2}}\right|} & \text{if } b_1 \leq \alpha \leq 3 - 2\sqrt{3}, \\ \dfrac{1 + |\alpha|}{|1 + \alpha|} & \text{otherwise}, \end{cases}$$

where $\alpha = \frac{b}{a}$. But this is a mapping in one real variable that can be proved to attain its absolute maximum at b_1, which proves the result. ■

Remark 3.15. As an immediate consequence of the previous result we have that $C_{\infty,2,3} \geq \frac{b_1-1}{b_1+1} \approx 3.98$. The authors have numerical evidence showing that, in fact, $C_{\infty,2,3} = \frac{b_1-1}{b_1+1}$.

The following result provides a general estimate on $C_{p,m,n}$ for any $p \geq 1$ and any $m, n \in \mathbb{N}$. This estimate can be proved to be optimal when $p = m = 2$. If $\mathcal{L}^s({}^n\ell_p^m)$ stands for the space of n-linear symmetric forms on $\ell_p^m \times \overset{(n)}{\cdots} \times \ell_p^m$, endowed with the norm given by

$$\|A\|_{\ell_p^m} := \sup\{|A(x_1, \ldots, x_n)| : \|x_k\|_p = 1,\ 1 \leq k \leq n\},$$

for all $A \in \mathcal{L}^s({}^n\ell_p^m)$, then we also obtain an estimate on $|||A|||$. If $\{e_1, \ldots, e_m\}$ is the canonical basis of $\mathbb{R}^m$, notice that here $|A| \in \mathcal{L}^s({}^n\ell_p^m)$ is defined by $|A|(e_{k_1}^{\alpha_1}, \ldots, e_{k_n}^{\alpha_n}) := |A(e_{k_1}^{\alpha_1}, \ldots, e_{k_n}^{\alpha_n})|$ for all $1 \leq k_1, \ldots, k_n \leq m$ and all $\alpha_1, \ldots, \alpha_n \in \mathbb{N} \cup \{0\}$ with $\alpha_1 + \cdots + \alpha_n = n$, where $e_{k_i}^{\alpha_i}$ $(1 \leq i \leq n)$ means that e_{k_i} appears α_i times in the n-tuple $(e_{k_1}^{\alpha_1}, \ldots, e_{k_n}^{\alpha_n})$. The concept of polarization constant will be required: If $P \in \mathcal{P}({}^n\ell_p^m)$ then, according to a standard result, there is a unique $A \in \mathcal{L}^s({}^n\ell_p^m)$ such that $P(x) = \hat{A}(x) := A(x, \ldots, x)$ for all $x \in \ell_p^m$. The set

$$\{\|A\|_{\ell_p^m}/\|\hat{A}\|_{\ell_p^m} : A \in \mathcal{L}^s({}^n\ell_p^m) \setminus \{0\}\} = \{1/\|\hat{A}\|_{\ell_p^m} : A \in \mathsf{S}_{\mathcal{L}^s({}^n\ell_p^m)}\}$$

is bounded by compactness of $\mathsf{S}_{\mathcal{L}^s({}^n\ell_p^m)}$ and its supremum is called the polarization constant of ℓ_p^m. These constants do not depend on m and will be denoted by $c_n(p)$. The exact value of the polarization constants $c_n(p)$ has been obtained by Sarantopoulos in [15] for some choices of p. In particular, according to an old well known result, $c_n(2) = 1$ for every $n \in \mathbb{N}$.

Theorem 3.16. *If $m, n \in \mathbb{N}$, $p \geq 1$ and $1 < q \leq \infty$ is the conjugate exponent of p, then for every $A \in \mathcal{L}^s({}^n\ell_p^m)$ we have*

$$|||A|||_{\ell_p^m} \leq m^{\frac{n-1}{q}} \|A\|_{\ell_p^m}. \tag{8}$$

In particular, if $P \in \mathcal{P}({}^n\ell_p^m)$ then

$$|||P|||_{\ell_p^m} \leq m^{\frac{n-1}{q}} c_n(p) \|P\|_{\ell_p^m}, \tag{9}$$

from which $C_{p,m,n} \leq m^{\frac{n-1}{q}} c_n(p)$. In particular $C_{2,m,n} \leq m^{\frac{n-1}{2}}$. Here we understand that $m^{\frac{n-1}{q}} = 1$ when $p = 1$ $(q = \infty)$.

Proof. Let us assume first that $p > 1$. We prove (8) by induction on n. It is obvious that (8) follows for every linear form on ℓ_p^m. Now assume the result holds for a given $n \in \mathbb{N}$ and take $A \in \mathcal{L}^s({}^{n+1}\ell_p^m)$. If $x \in \mathbb{R}^m$ is fixed, let us denote by $A(x, \cdot)$ the element of $\mathcal{L}^s({}^n\ell_p^m)$ defined by $(x_2, \ldots, x_{n+1}) \mapsto A(x, x_2, \ldots, x_{n+1})$ for every $x_2, \ldots, x_{n+1} \in \mathbb{R}^m$. If $x_1, \ldots, x_n, x_{n+1}$ are unit vectors in ℓ_p^m and $x_1 = (\lambda_1, \ldots, \lambda_m)$ in the canonical basis $\{e_1, \ldots, e_m\}$ of $\mathbb{R}^m$, then applying the induction hypothesis to every $A(e_k, \cdot)$ together with the fact that $\|A(e_k, \cdot)\|_{\ell_p^m} \leq \|A\|_{\ell_p^m}$ with $1 \leq k \leq m$ and Hölder's Inequality we obtain

$$||A|(x_1, \ldots, x_{n+1})| = \left| |A| \left(\sum_{k=1}^{m} \lambda_k e_k, x_2, \ldots, x_{n+1} \right) \right|$$

$$\begin{aligned}
&\leq \sum_{k=1}^{m} |\lambda_k| \cdot ||A|(e_k, x_2, \ldots, x_{n+1})| \\
&\leq m^{\frac{n-1}{q}} \left(\sum_{k=1}^{m} |\lambda_k| \cdot \|A(e_k, \cdot)\|_{\ell_p^m} \right) \\
&\leq m^{\frac{n-1}{q}} \left(\sum_{k=1}^{m} |\lambda_k| \right) \|A\|_{\ell_p^m} \\
&\leq m^{\frac{n-1}{q}} m^{\frac{1}{q}} \left(\sum_{k=1}^{m} |\lambda_k|^p \right)^{\frac{1}{p}} \|A\|_{\ell_p^m} = m^{\frac{n}{q}} \|A\|_{\ell_p^m},
\end{aligned}$$

from which the induction step follows. A slight modification of the previous reasoning also proves that $|||A|||_{\ell_1^m} = \|A\|_{\ell_1^m}$.

Now if $P \in \mathcal{P}(^n\ell_p^m)$ and $A \in \mathcal{L}^s(^n\ell_p^m)$ is such that $P = \hat{A}$, then (9) follows from (8) and the fact that $|||P|||_{\ell_p^m} \leq |||A|||_{\ell_p^m}$ and $\|A\|_{\ell_p^m} \leq c_n(p)\|P\|_{\ell_p^m}$. ■

Remark 3.17. If $m, n \in \mathbb{N}$, $1 \leq p \leq 2$ and $2 \leq q \leq \infty$ is the conjugate exponent of p, then it has been proved in [4, Theorem 3] that the precise asymptotic order in m of the unconditional basis constants of all n-homogeneous polynomials on ℓ_p^m is $m^{\frac{n-1}{q}}$. In Theorem 3.16 we have shown that in the case $1 \leq p \leq 2$ the unconditional constant of the canonical basis in the space $\mathcal{P}(^n\ell_p^m)$ is at most $m^{\frac{n-1}{q}} c_n(p)$, for every $m \in \mathbb{N}$. In particular, it follows from Theorem 3.16 that $C_{1,2,2} \leq c_2(1) = 2$. However, as we have shown in Theorem 3.5, $C_{1,2,2} = (1 + \sqrt{2})/2 < 2$.

Remark 3.18. Inequality (8) can be proved too using a result by Feng and Tonge when $n = 2$ and $1 \leq p \leq 2$. Indeed, it is proved in [9] that if $1 \leq p \leq 2$ and $2 \leq q \leq \infty$ is the conjugate exponent of p, then for every $A = (a_{ij}) \in \mathcal{L}(\ell_p^m; \ell_q^m)$ we have

$$|A|_q \leq m^{\frac{1}{q}} \|A\|_{pq},$$

where $|A|_q = \left(\sum_{j=1}^m \sum_{i=1}^m |a_{ij}|^q \right)^{\frac{1}{q}}$ and $\|A\|_{pq} = \max\{\|Ax\|_q : \|x\|_p \leq 1\}$. Notice that the previous inequality was also proved by Li [11] and Taşci [16] when $p = 2$. Now since $A = (a_{ij})$, seen as an element of $\mathcal{L}(^2\ell_p^m)$, satisfies $\|A\|_{\ell_p^m} = \|A\|_{pq}$ and $|||A|||_{\ell_p^m} \leq |A|_q$, it follows that

$$|||A|||_{\ell_p^m} \leq m^{\frac{1}{q}} \|A\|_{\ell_p^m}.$$

The estimate $C_{p,m,n} \leq m^{\frac{n-1}{q}} c_n(p)$ obtained in Theorem 3.16 is sharp at least for $p = m = 2$, as we can see in the following:

Theorem 3.19. *For every $n \in \mathbb{N}$ we have $C_{2,2,n} = 2^{\frac{n-1}{2}}$.*

Proof. We just have to consider the polynomial $P(x, y) = \mathrm{Re}\,((x + \mathrm{i}y)^n) + \mathrm{Im}\,((x + \mathrm{i}y)^n)$. Indeed, we have that $\|P\|_{\ell_2^2} = \sqrt{2}$ since

$$|P(x, y)| \leq |\mathrm{Re}\,((x + \mathrm{i}y)^n)| + |\mathrm{Im}\,((x + \mathrm{i}y)^n)| \leq \sqrt{2}|x + \mathrm{i}y|^n = \sqrt{2}\|(x, y)\|_2^n,$$

and if z_0 is an nth root of $\frac{1}{\sqrt{2}} + \frac{\mathrm{i}}{\sqrt{2}}$ then $P(\mathrm{Re}\,(z_0), \mathrm{Im}\,(z_0)) = \sqrt{2}$. On the other hand, we have that $|P|(x, y) = (x + y)^n$ from which it follows that $\|\!|P|\!\|_{\ell_2^2} = 2^{\frac{n}{2}}$. This concludes the proof. ■

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