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Two-dimensional Banach spaces with polynomial numerical index zero ☆

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We study two-dimensional Banach spaces with polynomial numerical indices equal to zero.

1. Introduction

The polynomial numerical indices of a Banach space are constants relating the norm and the numerical radius of homogeneous polynomials on the space. Let us present the relevant definitions. For a

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Banach space X, we write B_X for the closed unit ball, S_X for the unit sphere, X^* for the dual space, and $\Pi(X)$ for the subset of $X \times X^*$ given by

$$\Pi(X) = \{(x, x^*) \in S_X \times S_{X^*} : x^*(x) = 1\}.$$

For $k \in \mathbb{N}$ we denote by $\mathscr{P}(^k X; X)$ the space of all k-homogeneous polynomials from X into X endowed with the norm

$$\|P\| = \sup\{\|P(x)\| : x \in B_X\}.$$

We recall that a mapping $P : X \longrightarrow X$ is called a (continuous) k-homogeneous polynomial on X if there is a k-linear continuous mapping $A : X \times \cdots \times X \longrightarrow X$ such that $P(x) = A(x, \ldots, x)$ for every $x \in X$. We refer to the book [6] for background. Given $P \in \mathscr{P}(^k X; X)$, the *numerical range* of P is the subset of the scalar field given by

$$V(P) = \{x^*(P(x)) : (x, x^*) \in \Pi(X)\},$$

and the *numerical radius* of P is

$$v(P) = \sup\{|x^*(P(x))| : (x, x^*) \in \Pi(X)\}.$$

Recently, Choi et al. [2] have introduced the *polynomial numerical index of order k* of a Banach space X as the constant $n^{(k)}(X)$ defined by

$$\begin{aligned} n^{(k)}(X) &= \max\{c \geqslant 0 : c\|P\| \leqslant v(P)\ \forall P \in \mathscr{P}(^k X; X)\} \\ &= \inf\{v(P) : P \in \mathscr{P}(^k X; X),\ \|P\| = 1\} \end{aligned}$$

for every $k \in \mathbb{N}$. This concept is a generalization of the *numerical index* of a Banach space (recovered for $k = 1$) which was first suggested by G. Lumer in 1968 [7].

Let us recall some facts about the polynomial numerical index which are relevant to our discussion. We refer the reader to the already cited [2] and to [4,12,13] for recent results and background. The easiest examples are $n^{(k)}(\mathbb{R}) = 1$ and $n^{(k)}(\mathbb{C}) = 1$ for every $k \in \mathbb{N}$. In the complex case, $n^{(k)}(C(K)) = 1$ for every $k \in \mathbb{N}$ and $n^{(2)}(\ell_1) \leqslant \frac{1}{2}$. The real spaces $\ell_1^m, \ell_\infty^m, c_0, \ell_1$ and ℓ_∞ have polynomial numerical index of order 2 equal to 1/2 [12]. The only finite-dimensional real Banach space X with $n^{(2)}(X) = 1$ is $X = \mathbb{R}$ [13]. The inequality $n^{(k+1)}(X) \leqslant n^{(k)}(X)$ holds for every real or complex Banach space X and every $k \in \mathbb{N}$, giving that $n^{(k)}(H) = 0$ for every $k \in \mathbb{N}$ and every real Hilbert space H of dimension greater than one. This last fact is not true in the complex case in which it follows from an old result by Harris [9] that $n^{(k)}(X) \geqslant k^{\frac{k}{1-k}}$ for every complex Banach space X and every $k \geqslant 2$. Finally, $n^{(k)}(X^{**}) \leqslant n^{(k)}(X)$ for every real or complex Banach space X and every $k \in \mathbb{N}$, and this inequality may be strict.

For a real finite-dimensional space X, the fact $n(X) = 0$ is equivalent to X having infinitely many surjective isometries [15, Theorem 3.8]. In particular, it can be shown that the only two-dimensional space with infinitely many surjective isometries is the Hilbert space. For bigger dimensions the situation is not that easy but it is possible to somehow describe all these spaces (see [14,15]).

We will show in this paper that the situation for numerical indices of higher order is not so tidy, and many different examples of two-dimensional spaces with numerical indices of higher order equal to zero will be given. Namely, we start by showing that $n^{(p-1)}(\ell_p^2) = 0$ if p is an even number and, actually, that $n^{(2k-1)}(X) = 0$ if $(X, \|\cdot\|)$ is a real Banach space of dimension greater than one such that the mapping $x \longmapsto \|x\|^{2k}$ is a $2k$-homogeneous polynomial. Next, we describe all absolute normalized and symmetric norms on $\mathbb{R}^2$ such that the polynomial numerical index of order 3 is zero showing, in particular, that all these norms come from a polynomial. Finally, we present some examples proving that the situation is different for higher orders and for nonsymmetric norms. This is the content of Section 2. We include an appendix (Section 3) where it is shown that the formulae appearing in the examples are actually norms on $\mathbb{R}^2$.

Let us finish the introduction with some notation. We say that a norm $\|\cdot\|$ in $\mathbb{R}^2$ is *absolute* if $\|(x,y)\| = \|(|x|,|y|)\|$ for every $x, y \in \mathbb{R}$, *normalized* if $\|(1,0)\| = \|(0,1)\| = 1$ and *symmetric* whenever $\|(x,y)\| = \|(y,x)\|$ for every $x, y \in \mathbb{R}$. For $1 \leqslant p \leqslant \infty$, we write $\|\cdot\|_p$ to denote the p-norm and ℓ_p^d to denote the d-dimensional ℓ_p-space (i.e. $\mathbb{R}^d$ endowed with $\|\cdot\|_p$).

Let X be a Banach space, $k \in \mathbb{N}$ and let $S \in L(X)$ be a surjective isometry. Given $P \in \mathscr{P}(^kX; X)$, it is clear that $S^{-1} \circ P \circ S \in \mathscr{P}(^kX; X)$ and one has that

$$V(S^{-1} \circ P \circ S) = V(P) \quad \text{and} \quad \|S^{-1} \circ P \circ S\| = \|P\| \tag{1}$$

(indeed, these equalities follow easily from [9, Theorem 2] but they are actually straightforwardly deduced from the definition of numerical range).

Let us also recall that X is a *smooth* space if given $x \in X \setminus \{0\}$ there exists a unique norm-one linear functional $x^* \in X^*$ such that $x^*(x) = \|x\|$. Moreover, this functional is given by the derivative $D_x\| \cdot \|$ of the norm at x. If X is a finite-dimensional space it is known [5, Corollary 1.5 and Remark 1.7] that X is smooth if and only if its norm is Fréchet differentiable on S_X.

2. The results

Our first goal is to discuss the polynomial numerical index of the real spaces ℓ_p^2 for $1 < p < \infty$. Let us recall that $n^{(k)}(\ell_p^2) > 0$ for $p = 1, \infty$ and every $k \in \mathbb{N}$ [12, Corollary 2.5].

Example 2.1. Let $1 < p < \infty$.

(a) If p is an even number and $k \in \mathbb{N}$, then $n^{(k)}(\ell_p^2) = 0$ if $k \geqslant p-1$ and $n^{(k)}(\ell_p^2) > 0$ if $k < p-1$.
(b) If p is not an even number, then $n^{(k)}(\ell_p^2) > 0$ for every $k \in \mathbb{N}$.

Proof

(a) Given $(x, y) \in S_{\ell_p^2}$, the only functional which norms (x, y) is $(x^{p-1}, y^{p-1}) \in \ell_{p/p-1}^2$. If we consider the polynomial $P \in \mathscr{P}(^{p-1}\ell_p^2; \ell_p^2)$ defined by $P(x, y) = (-y^{p-1}, x^{p-1})$ then,

$$(x^{p-1}, y^{p-1})(P(x, y)) = -x^{p-1}y^{p-1} + y^{p-1}x^{p-1} = 0$$

for all $(x, y) \in S_{\ell_p^2}$ implying that $v(P) = 0$ and $n^{(p-1)}(\ell_p^2) = 0$. Therefore, for $k \geqslant p-1$, $n^{(k)}(\ell_p^2) = 0$ by [2, Proposition 2.5]. If $k < p-1$ and $P = (P_1, P_2) \in \mathscr{P}(^k\ell_p^2; \ell_p^2)$ is non-zero, observe that

$$x^{p-1}P_1(x, y) + y^{p-1}P_2(x, y)$$

is a scalar homogeneous polynomial which cannot be constant zero. Indeed, we can assume without loss of generality that P_1 is non-zero and evaluate the above expression at $(x, 1)$ for $x \in \mathbb{R}$ obtaining

$$x^{p-1}P_1(x, 1) + P_2(x, 1).$$

We observe that the first summand is a non-zero polynomial in the variable x of degree at least $p-1$ and the second one has degree at most k. So their sum cannot be equal to zero for every $x \in \mathbb{R}$.

(b) When p is not an even number, the only linear functional which norms $(x, y) \in \ell_p^2$ with $x, y \neq 0$ is $(x|x|^{p-2}, y|y|^{p-2}) \in \ell_{p/p-1}^2$. If $P = (P_1, P_2) \in \mathscr{P}(^k\ell_p^2; \ell_p^2)$ satisfies $v(P) = 0$, then

$$x|x|^{p-2}P_1(x, y) + y|y|^{p-2}P_2(x, y) = 0 \tag{2}$$

for every $x, y \neq 0$. Now, if $p \notin \mathbb{N}$, evaluating at $(x, 1)$ for every $x > 0$, we get

$$x^{p-1}P_1(x, 1) = -P_2(x, 1) \quad (x \in \mathbb{R}^+).$$

If $P_1(x, 1)$ is not zero in $\mathbb{R}^+$, dividing the above equation by $x^{p-1+\deg(P_1(x,1))}$ and taking the limit as $x \to +\infty$, we get a contradiction. Hence, we have that $P_1(x, 1) = 0$ for $x \in \mathbb{R}^+$ which implies $P_2(x, 1) = 0$ for $x \in \mathbb{R}^+$ and, therefore, that $P = 0$. Finally, if $p \in \mathbb{N}$ is odd, we use (2) to obtain

$$x^{p-1}P_1(x, 1) + P_2(x, 1) = 0 \quad (x \in \mathbb{R}^+),$$
$$-x^{p-1}P_1(x, 1) + P_2(x, 1) = 0 \quad (x \in \mathbb{R}^-),$$

which, together with the fact that $x^{p-1}P_1(x, 1) + P_2(x, 1)$ and $-x^{p-1}P_1(x, 1) + P_2(x, 1)$ are polynomials, implies

$$x^{p-1}P_1(x,1) + P_2(x,1) = 0 \quad (x \in \mathbb{R}),$$
$$-x^{p-1}P_1(x,1) + P_2(x,1) = 0 \quad (x \in \mathbb{R}).$$

This obviously gives $P_1(x,1) = 0$ and $P_2(x,1) = 0$ for $x \in \mathbb{R}$, implying that $P = 0$ and finishing the proof. □

Since ℓ_p^2 is an absolute summand of ℓ_p and ℓ_p^d for every $d \geqslant 2$, by [4, Proposition 2.1] we get the following.

Corollary 2.2. *Let p be an even number and $d \geqslant 2$ an integer. Then, $n^{(p-1)}(\ell_p) = n^{(p-1)}(\ell_p^d) = 0$.*

Remark 2.3. It is claimed in [11] that $n^{(k)}(\ell_p^d) > 0$ for every $k \in \mathbb{N}$, every $1 < p < \infty, p \neq 2$, and every integer $d \geqslant 2$. Going into the proof of that result, one realizes that it is needed that p is not an even integer.

It is known that $n(X^*) \leqslant n(X)$ for every Banach space X. Example 2.1 shows that, unlike the linear case, there is no general inequality between the polynomial numerical indices of a Banach space and the ones of its dual.

Example 2.4. The reflexive space $X = \ell_4^2$ satisfies $n^{(k)}(X) = 0$ and $n^{(k)}(X^*) > 0$ for all $k \geqslant 3$.

Our next result is a generalization of Corollary 2.2 to every Banach space whose norm raised to an even power is a homogeneous polynomial.

Proposition 2.5. *Let k be a positive integer and let $(X, \|\cdot\|)$ be a real Banach space of dimension greater than one. If the mapping $x \longmapsto \|x\|^{2k}$ is a 2k-homogeneous polynomial, then $n^{(2k-1)}(X) = 0$.*

Proof. Let R and A be, respectively, the $2k$-homogeneous scalar polynomial and the corresponding symmetric $2k$-linear form such that $A(x, \ldots, x) = R(x) = \|x\|^{2k}$ for every $x \in X$. Since R is Gâteaux differentiable on S_X, so is $\|\cdot\|$. Moreover, for fixed $x \in S_X$, we have that

$$2kD_x\|\cdot\|(y) = D_xR(y) = 2kA(x, \ldots, x, y)$$

for every $y \in X$ and, therefore, the functional given by $x^*(y) = A(x, \ldots, x, y)$ is the only norm-one functional satisfying $x^*(x) = 1$. To finish the proof, we fix x_0, y_0 two linearly independent elements of X and we define $P \in \mathscr{P}(^{2k-1}X; X)$ by

$$P(x) = -A(x, \ldots, x, y_0)x_0 + A(x, \ldots, x, x_0)y_0 \quad (x \in X),$$

which clearly satisfies $P \neq 0$. Finally, for $(x, x^*) \in \Pi(X)$ we have that

$$\begin{aligned} x^*(P(x)) &= A(x, \ldots, x, P(x)) \\ &= A(x, \ldots, x, -A(x, \ldots, x, y_0)x_0 + A(x, \ldots, x, x_0)y_0) \\ &= -A(x, \ldots, x, y_0)A(x, \ldots, x, x_0) + A(x, \ldots, x, x_0)A(x, \ldots, x, y_0) = 0. \end{aligned}$$

Therefore, $v(P) = 0$ and, consequently, $n^{(2k-1)}(X) = 0$. □

The rest of the paper is devoted to the two-dimensional case. We start with some facts about two-dimensional spaces with polynomial numerical index 0 which will be useful in this paper.

Theorem 2.6. *Let $(X, \|\cdot\|)$ be a two-dimensional real space such that $n^{(k)}(X) = 0$ for some $k \geqslant 1$, let $k_0 = \min\{k : n^{(k)}(X) = 0\}$, and $P = (P_1, P_2) \in \mathscr{P}(^{k_0}X; X)$ with $v(P) = 0$. The following hold:*

(a) *The $(k_0 + 1)$-homogeneous scalar polynomial defined by*

$$Q(x, y) = yP_1(x, y) - xP_2(x, y) \quad ((x, y) \in X)$$

only vanishes at $(0, 0)$.

(b) *k_0 is odd.*
(c) *$(X,\|\cdot\|)$ is a smooth space. Moreover, for every non-zero $(x,y)\in X$ the unique functional $(x^*,y^*)\in S_{X^*}$ which norms (x,y) is given by*

$$x^*=\frac{-P_2(x,y)\|(x,y)\|}{Q(x,y)}\quad \text{and}\quad y^*=\frac{P_1(x,y)\|(x,y)\|}{Q(x,y)}.$$

(d) *The polynomial P is unique in the following sense: $\widetilde{P}\in\mathscr{P}(^{k_0}X;X)$ satisfies $v(\widetilde{P})=0$ if and only if there exists $\lambda\in\mathbb{R}$ so that $\widetilde{P}=\lambda P$.*

Proof. Given $P=(P_1,P_2)\in\mathscr{P}(^{k_0}X;X)$ with $v(P)=0$, we claim that P_1 and P_2 do not have any factor in common and, in particular, that P only vanishes at $(0,0)$. Indeed, if $k_0\geqslant 2$, suppose that there exist scalar polynomials S,R_1,R_2 with $\deg(R_i)<k_0$ such that $P_i=SR_i$ for $i=1,2$. Since $v(P)=0$, given an element $(x,y)\in S_X$ and a linear functional $(x^*,y^*)\in S_{X^*}$ satisfying $x^*x+y^*y=1$, we have that

$$x^*P_1(x,y)+y^*P_2(x,y)=0$$

and, therefore,

$$S(x,y)(x^*R_1(x,y)+y^*R_2(x,y))=0,$$

which gives us $x^*R_1(x,y)+y^*R_2(x,y)=0$ whenever $S(x,y)\neq 0$. Writing $R=(R_1,R_2)$ and using that $V(R)$ is connected [1, Theorem 1] and that S only has a finite number of zeros in S_X, we deduce $v(R)=0$ and so $n^{(k)}(X)=0$ for some $k<k_0$, contradicting the minimality of k_0. If $k_0=1$, the above argument is immediate.

(a) The fact that $Q(x_0,y_0)=0$ for some $(x_0,y_0)\neq 0$ yields that $P(x_0,y_0)=\lambda(x_0,y_0)$ for some $\lambda\in\mathbb{R}$ which, together with $v(P)=0$, implies that $\lambda=0$ contradicting the fact that P only vanishes at $(0,0)$.
(b) Since Q only vanishes at $(0,0)$, its degree k_0+1 must be even and thus k_0 is odd.
(c) Given $(x,y)\in S_X$, we observe that any functional $(x^*,y^*)\in S_{X^*}$ norming (x,y) satisfies the linear equations $x^*x+y^*y=1$ and $x^*P_1(x,y)+y^*P_2(x,y)=0$ which uniquely determine (x^*,y^*) as

$$x^*=\frac{-P_2(x,y)}{Q(x,y)}\quad\text{and}\quad y^*=\frac{P_1(x,y)}{Q(x,y)},$$

since $Q(x,y)\neq 0$. For arbitrary $(x,y)\neq(0,0)$ it suffices to use what we have just proved and the homogeneity.
(d) Since $v(\widetilde{P})=0$, for every $((x,y),(x^*,y^*))\in\Pi(X)$ we have $x^*\widetilde{P}_1(x,y)+y^*\widetilde{P}_2(x,y)=0$ which, together with (c), gives

$$\frac{-P_2(x,y)}{Q(x,y)}\widetilde{P}_1(x,y)+\frac{P_1(x,y)}{Q(x,y)}\widetilde{P}_2(x,y)=0$$

and, therefore,

$$P_1(x,y)\widetilde{P}_2(x,y)=P_2(x,y)\widetilde{P}_1(x,y)$$

for every $(x,y)\in S_X$. Now it suffices to recall that P_1 and P_2 do not have any factor in common to get the result. □

We have to restrict ourselves to the two-dimensional case since the above result is not true for higher dimensions.

Remark 2.7. Consider the real Banach space $X=\ell_2^2\oplus_1 Y$, where Y is any non-null Banach space. Then $n^{(k)}(X)\leqslant n^{(k)}(\ell_2^2)=0$ for every $k\in\mathbb{N}$ by [4, Proposition 2.1]. But the norm of X is not smooth at points $(0,y)\in S_X$ with $y\in S_Y$. Also, if we choose Y such that $n^{(k)}(Y)=0$, there are different non-null polynomials whose numerical radii are zero.

A consequence of Theorem 2.6 is the following partial answer to Problem 42 of [10].

Corollary 2.8. *If X is a two-dimensional real Banach space with $n^{(2)}(X) = 0$, then $n(X) = 0$.*

It is a well known result (see [14, Corollary 2.5] and [15, Theorem 3.1]) that the only two-dimensional real space with numerical index 0 is the Euclidean space. The above theorem allows us to give a different and elementary proof of this fact. We include it here since it gives some ideas which we will use later.

Corollary 2.9. *Let X be a two-dimensional real space with $n(X) = 0$. Then, X is the two-dimensional real Euclidean space.*

Proof. Let $e_1, e_2 \in S_X$ and $e_1^*, e_2^* \in S_{X^*}$ be so that $e_i^*(e_j) = \delta_{ij}$ for $i, j \in \{1, 2\}$ (the existence of such elements is guaranteed by [16, Theorem II.2.2]). We fix a linear operator T with $v(T) = 0$ and we write it in the basis $\{e_1, e_2\}$:

$$T(x, y) = (ax + by, cx + dy) \quad ((x, y) \in X).$$

Since $e_i^*(Te_i) = 0$ for $i = 1, 2$ we obtain $a = d = 0$. Given an arbitrary nonzero $(x, y) \in X$, we use Theorem 2.6 to get that the unique linear functional which norms (x, y) is given by

$$\left(\frac{-cx\|(x, y)\|}{by^2 - cx^2}, \frac{by\|(x, y)\|}{by^2 - cx^2}\right),$$

but such a functional must coincide with the differential of the norm, implying that

$$\frac{\partial\|\cdot\|}{\partial x}(x, y) = \frac{-cx\|(x, y)\|}{by^2 - cx^2} \quad \text{and} \quad \frac{\partial\|\cdot\|}{\partial y}(x, y) = \frac{by\|(x, y)\|}{by^2 - cx^2}.$$

We rewrite the first equation as follows:

$$\frac{1}{\|(x, y)\|}\frac{\partial\|\cdot\|}{\partial x}(x, y) = \frac{-cx}{by^2 - cx^2}$$

and we integrate it with respect to x, obtaining

$$\log\|(x, y)\| = \frac{1}{2}\log(by^2 - cx^2) + f(y)$$

for some differentiable function f. Differentiating now with respect to y we get

$$\frac{1}{\|(x, y)\|}\frac{\partial\|\cdot\|}{\partial y}(x, y) = \frac{by}{by^2 - cx^2} + f'(y),$$

so $f'(y) = 0$ and $f(y)$ is constant, say M. Therefore, we can write

$$\|(x, y)\| = e^M(by^2 - cx^2)^{\frac{1}{2}}$$

and deduce that $b > 0$ and $c < 0$. Now, since $\|e_1\| = \|e_2\| = 1$, we get $1 = e^M b^{\frac{1}{2}} = e^M(-c)^{\frac{1}{2}}$ which yields that

$$\|(x, y)\| = e^M(by^2 - cx^2)^{\frac{1}{2}} = e^M b^{\frac{1}{2}}(x^2 + y^2)^{\frac{1}{2}} = (x^2 + y^2)^{\frac{1}{2}}. \qquad \square$$

There are more two-dimensional spaces for which the polynomial numerical index of order 3 is zero since we already know that $n^{(3)}(\ell_4^2) = 0$. However, we are able to completely describe absolute normalized and symmetric norms with polynomial numerical index of order 3 equal to zero showing, in particular, that all of them come from a polynomial. We will see later that the hypothesis of symmetry is necessary.

Theorem 2.10. *Let $X = (\mathbb{R}^2, \|\cdot\|)$ be a two-dimensional Banach space satisfying that $n^{(3)}(X) = 0$ with $\|\cdot\|$ being a normalized absolute symmetric norm. Then, there is $\beta \in [0, 3]$ such that*

$$\|(x,y)\| = (x^4 + 2\beta x^2y^2 + y^4)^{\frac{1}{4}} \quad ((x,y) \in X).$$

In particular, the fourth power of the norm of X is a polynomial.

Proof. We can assume that $n^{(2)}(X) \neq 0$ since otherwise X is a Hilbert space and the result holds with $\beta = 1$. We fix $P = (P_1, P_2) \in \mathscr{P}(^3X; X)$ with $v(P) = 0$ and we consider the associated scalar polynomial $Q(x,y) = yP_1(x,y) - xP_2(x,y)$ which only vanishes at $(0,0)$ by Theorem 2.6. Hence we can assume without loss of generality that $Q > 0$ on $\mathbb{R}^2 \setminus \{(0,0)\}$. Next, the norm being absolute, the operator $U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a surjective isometry and so the polynomial $(R_1, R_2) = U^{-1} \circ P \circ U$, which is given by

$$(R_1(x,y), R_2(x,y)) = (P_1(x,-y), -P_2(x,-y)) \quad ((x,y) \in X)$$

satisfies

$$v(R_1, R_2) = 0 \quad \text{and} \quad \|(R_1, R_2)\| = \|P\|$$

by (1). Thus, Theorem 2.6 tells us that there is $\lambda \in \mathbb{R}$ with $|\lambda| = 1$ so that

$$P_1(x,-y) = \lambda P_1(x,y) \quad \text{and} \quad P_2(x,-y) = -\lambda P_2(x,y)$$

for every $(x,y) \in X$. Moreover, we have that $\lambda = -1$. Indeed, it suffices to take a non-zero $x \in \mathbb{R}$ and to observe that

$$Q(x,-x) = -xP_1(x,-x) - xP_2(x,-x) = -\lambda Q(x,x),$$

which implies $\lambda = -1$ since $Q > 0$ on $\mathbb{R}^2 \setminus \{(0,0)\}$. Hence, for every $(x,y) \in X$ we get

$$P_1(x,-y) = -P_1(x,y) \quad \text{and} \quad P_2(x,-y) = P_2(x,y). \tag{3}$$

Analogously, the norm being symmetric, the operator $V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a surjective isometry and so the polynomial $(S_1, S_2) = V^{-1} \circ P \circ V$, which is given by

$$(S_1(x,y), S_2(x,y)) = (P_2(y,x), P_1(y,x)) \quad ((x,y) \in X)$$

satisfies

$$v(S_1, S_2) = 0 \quad \text{and} \quad \|(S_1, S_2)\| = \|P\|.$$

by (1). Therefore, using again Theorem 2.6 and the fact that $Q > 0$ on $\mathbb{R}^2 \setminus \{(0,0)\}$, we deduce that

$$P_2(x,y) = -P_1(y,x) \quad ((x,y) \in X).$$

Therefore, if we write $P_1(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$ for some $a, b, c, d \in \mathbb{R}$, we obtain $P_2(x,y) = -dx^3 - cx^2y - bxy^2 - ay^3$. Further, using (3) we deduce that

$$P_1(x,y) = bx^2y + dy^3 \quad \text{and} \quad P_2(x,y) = -dx^3 - bxy^2$$

for every $(x,y) \in X$. This, together with Theorem 2.6, tells us that the linear functional which norms an arbitrary non-zero $(x,y) \in X$ is given by

$$\left(\frac{(dx^3 + bxy^2)\|(x,y)\|}{dx^4 + 2bx^2y^2 + dy^4}, \frac{(bx^2y + dy^3)\|(x,y)\|}{dx^4 + 2bx^2y^2 + dy^4} \right)$$

thus, we have that

$$\frac{1}{\|(x,y)\|} \frac{\partial \|\cdot\|}{\partial x}(x,y) = \frac{dx^3 + bxy^2}{dx^4 + 2bx^2y^2 + dy^4} \quad \text{and}$$
$$\frac{1}{\|(x,y)\|} \frac{\partial \|\cdot\|}{\partial y}(x,y) = \frac{bx^2y + dy^3}{dx^4 + 2bx^2y^2 + dy^4}.$$

Integrating the first equation with respect to x we obtain

$$\log\|(x,y)\| = \frac{1}{4}\log(dx^4+2bx^2y^2+dy^4)+f(y) \quad (x,y\in\mathbb{R})$$

for some differentiable function f. Differentiating now with respect to y we get

$$\frac{1}{\|(x,y)\|}\frac{\partial\|\cdot\|}{\partial y}(x,y) = \frac{bx^2y+dy^3}{dx^4+2bx^2y^2+dy^4}+f'(y) \quad (x,y\in\mathbb{R}),$$

so $f'(y)=0$ and $f(y)$ is constant, say C. Therefore, we can write

$$\|(x,y)\| = \mathrm{e}^C(dx^4+2bx^2y^2+dy^4)^{\frac{1}{4}} \quad (x,y\in\mathbb{R}).$$

Now, since $\|(1,0)\|=\|(0,1)\|=1, d>0$ and $\mathrm{e}^C d^{\frac{1}{4}}=1$ so, calling $\beta=b\mathrm{e}^{4C}$, we have

$$\|(x,y)\| = (x^4+2\beta x^2y^2+y^4)^{\frac{1}{4}} \quad (x,y\in\mathbb{R}).$$

Finally, this formula defines a norm if and only if $\beta\in[0,3]$ as shown in Proposition 3.1. □

The next example shows that the hypothesis of symmetry of the norm in the above theorem cannot be dropped.

Example 2.11. There are normalized absolute norms $\|\cdot\|$ on $\mathbb{R}^2$ such that the spaces $X=(\mathbb{R}^2,\|\cdot\|)$ satisfy $n^{(3)}(X)=0$ and $\|\cdot\|^\ell$ is not a polynomial for any positive number ℓ. Indeed, for any irrational $0<a<1$, we consider the function $\|\cdot\|_a$ defined by

$$\|(x,y)\|_a = \left(x^2+\left(\frac{a}{1+a}\right)^{1+a}y^2\right)^{\frac{-a}{2}}\left(x^2+\left(\frac{a}{1+a}\right)^{a}y^2\right)^{\frac{1+a}{2}} \quad ((x,y)\in\mathbb{R}^2\setminus\{(0,0)\})$$

and $\|(0,0)\|_a=0$, which is a norm as shown in Proposition 3.6 and obviously satisfies that $\|\cdot\|_a^\ell$ is not a polynomial for any positive number ℓ. We then consider $X=(\mathbb{R}^2,\|\cdot\|_a)$ and the polynomial $P=(P_1,P_2)\in\mathscr{P}(^3X;X)$ given by

$$P(x,y) = \left(\left(\frac{a}{1+a}\right)^{a}\left(\frac{1+2a}{1+a}\right)x^2y+\left(\frac{a}{1+a}\right)^{1+2a}y^3, -x^3\right) \quad ((x,y)\in X).$$

Since $\|\cdot\|_a$ is differentiable on S_X, for $(x,y)\in S_X$, the only functional $(x^*,y^*)\in S_{X^*}$ norming (x,y) is given by $\left(\frac{\partial\|\cdot\|_a}{\partial x}(x,y), \frac{\partial\|\cdot\|_a}{\partial y}(x,y)\right)$. It is easy to check that

$$\frac{\partial\|\cdot\|_a}{\partial x}(x,y) = x^3A(x,y,a),$$
$$\frac{\partial\|\cdot\|_a}{\partial y}(x,y) = \left(\left(\frac{a}{1+a}\right)^{a}\left(\frac{1+2a}{1+a}\right)x^2y+\left(\frac{a}{1+a}\right)^{1+2a}y^3\right)A(x,y,a),$$

where

$$A(x,y,a) = \left(x^2+\left(\frac{a}{1+a}\right)^{1+a}y^2\right)^{\frac{-a}{2}-1}\left(x^2+\left(\frac{a}{1+a}\right)^{a}y^2\right)^{\frac{1+a}{2}-1}.$$

Therefore, $x^*P_1(x,y)+y^*P_2(x,y)=0$ which implies $v(P)=0$.

For higher order, there are examples of absolute normalized and symmetric norms with polynomial numerical indices equal to zero which do not come from polynomials.

Example 2.12. For every positive integer $m\geqslant 3$, there are absolute normalized and symmetric norms $\|\cdot\|_{m,\theta}$ such that the spaces $X_{m,\theta}=(\mathbb{R}^2,\|\cdot\|_{m,\theta})$ satisfy $n^{(2m-1)}(X_{m,\theta})=0$ and $\|\cdot\|_{m,\theta}^{2\ell}$ is not a polynomial for any positive number ℓ. Indeed, let $\|\cdot\|_{m,\theta}$ be defined by

$$\|(x,y)\|_{m,\theta} = (x^2+y^2)^{\frac{\theta}{2}}(x^{2m-2}+y^{2m-2})^{\frac{1-\theta}{2m-2}} \quad ((x,y)\in\mathbb{R}^2),$$

where $\theta \in [0,1]$. This formula defines a norm as shown in Proposition 3.5. To prove that $n^{(2m-1)}(X_{m,\theta}) = 0$, we define the polynomial $P=(P_1,P_2)\in\mathscr{P}(^{2m-1}X_{m,\theta};X_{m,\theta})$ by

$$P_1(x,y) = \theta y(x^{2m-2}+y^{2m-2}) + (1-\theta)y^{2m-3}(x^2+y^2),$$
$$P_2(x,y) = -\theta x(x^{2m-2}+y^{2m-2}) - (1-\theta)x^{2m-3}(x^2+y^2)$$

and we show that $v(P)=0$. Since $\|\cdot\|_{m,\theta}$ is differentiable on $S_{X_{m,\theta}}$, for $(x,y)\in S_{X_{m,\theta}}$ the only functional $(x^*,y^*)\in S_{X^*_{m,\theta}}$ norming (x,y) is given by $\left(\frac{\partial\|\cdot\|_{m,\theta}}{\partial x}(x,y), \frac{\partial\|\cdot\|_{m,\theta}}{\partial y}(x,y)\right)$ and, therefore,

$$x^* = [\theta x(x^{2m-2}+y^{2m-2}) + (1-\theta)x^{2m-3}(x^2+y^2)]B(x,y,m,\theta),$$
$$y^* = [\theta y(x^{2m-2}+y^{2m-2}) + (1-\theta)y^{2m-3}(x^2+y^2)]B(x,y,m,\theta),$$

where

$$B(x,y,m,\theta) = (x^2+y^2)^{\frac{\theta}{2}-1}(x^{2m-2}+y^{2m-2})^{\frac{1-\theta}{2m-2}-1}.$$

Now, it is routine to check that $x^*P_1(x,y)+y^*P_2(x,y)=0$. Finally, if $\theta\in[0,1]$ is chosen irrational, then $\|\cdot\|^{2\ell}_{m,\theta}$ is not a polynomial for any positive integer ℓ.

3. Appendix: some norms in the plane

The aim of this last section is to justify that some formulae appearing in the past section are really norms. We start with the norms given in Theorem 2.10 for which the justification is direct.

Proposition 3.1. *For $\beta\in\mathbb{R}$, the formula*

$$\|(x,y)\| = (x^4+2\beta x^2y^2+y^4)^{\frac{1}{4}} \quad ((x,y)\in\mathbb{R}^2)$$

defines a norm in $\mathbb{R}^2$ if and only if $\beta\in[0,3]$.

Proof. We start by observing that for $0\leqslant\beta\leqslant 1$ we can write

$$\|(x,y)\| = (\beta(x^2+y^2)^2+(1-\beta)(x^4+y^4))^{\frac{1}{4}} = \left\|\left(\beta^{\frac{1}{4}}\|(x,y)\|_2, (1-\beta)^{\frac{1}{4}}\|(x,y)\|_4\right)\right\|_4$$

and so it defines a norm on $\mathbb{R}^2$. In case $\beta<0$, it is easy to check that the set

$$A = \{(x,y)\in\mathbb{R}^2 : x^4+2\beta x^2y^2+y^4\leqslant 1\}$$

is not convex and thus $\|\cdot\|$ is not a norm. Indeed, fix $0<\delta<(-2\beta)^{\frac{1}{2}}$ and observe that the points

$$\left(\frac{1}{(1+2\beta\delta^2+\delta^4)^{\frac{1}{4}}}, \frac{\delta}{(1+2\beta\delta^2+\delta^4)^{\frac{1}{4}}}\right) \quad\text{and}\quad \left(\frac{1}{(1+2\beta\delta^2+\delta^4)^{\frac{1}{4}}}, \frac{-\delta}{(1+2\beta\delta^2+\delta^4)^{\frac{1}{4}}}\right)$$

belong to A while their midpoint $\left(\frac{1}{(1+2\beta\delta^2+\delta^4)^{\frac{1}{4}}}, 0\right)$ does not.

Finally, for $\beta\geqslant 1$, we consider the change of variables given by

$$x = \frac{u+v}{(2+2\beta)^{\frac{1}{4}}} \quad\text{and}\quad y = \frac{u-v}{(2+2\beta)^{\frac{1}{4}}};$$

we observe that

$$(x^4+2\beta x^2y^2+y^4)^{\frac{1}{4}} = \left(u^4+2\,\frac{3-\beta}{1+\beta}\,u^2v^2+v^4\right)^{\frac{1}{4}}$$

and that the mapping $g : [1, +\infty[\longrightarrow] - 1, 1]$ given by $g(\beta) = \frac{3-\beta}{1+\beta}$ satisfies

$$g([1,3]) = [0,1] \quad \text{and} \quad g(]3,+\infty[) =]-1,0[.$$

So the remaining cases $1 \leqslant \beta \leqslant 3$ and $3 < \beta$ are covered respectively by the previous ones $0 \leqslant \beta \leqslant 1$ and $\beta < 0$. □

The study of the functions appearing in Examples 2.11 and 2.12 is more difficult and requires some tricky arguments. We would like to thank Vladimir Kadets for providing us with some crucial ideas.

We start with some folklore lemmata on convex functions. Recall that a function $f : A \longrightarrow \mathbb{R}$ on a convex set A is said to be *convex* if

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y) \quad (x, y \in A,\ \lambda \in [0,1]).$$

A subset C of a vector space is said to be a *cone* if $\alpha x + \beta y \in C$ for every $x, y \in C$ and every $\alpha, \beta \in \mathbb{R}^+$. If $f : C \longrightarrow \mathbb{R}$ is positive homogeneous, then f is convex if and only if f is *sublinear*, i.e.

$$f(x+y) \leqslant f(x) + f(y) \quad (x, y \in A).$$

Lemma 3.2. *Let $(X, \|\cdot\|)$ be a normed space, $C \subseteq X$ a cone and let $f : C \longrightarrow \mathbb{R}$ be a positive homogeneous function. If*

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y) \quad (x, y \in C \cap S_X,\ \lambda \in [0,1]),$$

then f is convex on C.

Proof. Since f is positive homogeneous, it is enough to show that it is sublinear. If $x, y \in C$ are non-null elements, then $x/\|x\|$ and $y/\|y\|$ belong to $C \cap S_X$ and so

$$\frac{1}{\|x\| + \|y\|} f(x+y) = f\left(\frac{\|x\|}{\|x\| + \|y\|} \frac{x}{\|x\|} + \frac{\|y\|}{\|x\| + \|y\|} \frac{y}{\|y\|} \right) \leqslant \frac{1}{\|x\| + \|y\|} (f(x) + f(y)).$$

If $x = 0$ or $y = 0$, the result is trivial. □

It is well known (see [17, Proposition 2.2], for instance) that a twice differentiable function $f : A \longrightarrow \mathbb{R}$ defined on an open convex subset A of $\mathbb{R}^d$ is convex if and only if the Hessian matrix of f is semi-definite positive. With this in mind, the following result is completely evident.

Lemma 3.3. *Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a continuous function which is twice differentiable with the partial derivatives of second order continuous on $\mathbb{R}^d \setminus \{0\}$. If there are open convex subsets $A_1, \ldots, A_m$ such that $\bigcup_{i=1}^m A_i$ is dense in $\mathbb{R}^d$ and $f|_{A_i}$ is convex for $i = 1, \ldots, m$, then f is convex on $\mathbb{R}^d$.*

Proof. Since $f|_{A_i}$ is convex, the Hessian matrix of f is semi-definite positive on A_i. Since $\bigcup_{i=1}^m A_i$ is dense in $\mathbb{R}^n$ and the partial derivatives of second order of f are continuous, we get that the Hessian matrix of f is semi-definite positive on $\mathbb{R}^n \setminus \{0\}$. Now, for fixed $x, y \in \mathbb{R}^d$ such that the segment $[x, y]$ does not contain 0, there is an open halfplane S such that $0 \notin S$ and $[x, y] \subset S$. Since the Hessian matrix of f is semi-definite positive on S, we get that f is convex on S and so on $[x, y]$. The remainder case in which $0 \in [x, y]$ reduces to the above one by the continuity of f. □

We finish the list of preliminary results with an obvious lemma on convex real functions.

Lemma 3.4. *Let $I \subset \mathbb{R}$ be an interval, let $\gamma, \gamma_0, \gamma_1 : I \longrightarrow \mathbb{R}$ be twice differentiable positive functions, and let $\varphi = \log(\gamma), \varphi_i = \log(\gamma_i)$ for $i = 0, 1$.*

(a) *γ is convex if and only if $\varphi'' + [\varphi']^2 \geqslant 0$. In particular, if $\varphi'' \geqslant 0$, then γ is convex.*
(b) *If φ_0'' and φ_1'' are nonnegative, then for each $\theta \in [0,1]$ the function*

$$\gamma_\theta(t) = [\gamma_1(t)]^\theta[\gamma_0(t)]^{1-\theta} \quad (t \in I)$$

is convex.

Proof

(a) We have clearly that

$$\varphi' = \frac{\gamma'}{\gamma} \quad \text{and} \quad \varphi'' = \frac{\gamma''\gamma - [\gamma']^2}{\gamma^2}, \quad \text{so } \varphi'' + [\varphi']^2 = \frac{\gamma''\gamma}{\gamma^2}.$$

Now, γ is convex if and only if $\gamma'' \geqslant 0$ and, since γ is positive, this is equivalent to $\varphi'' + [\varphi']^2 \geqslant 0$.

(b) Writing $\varphi_\theta = \log(\gamma_\theta)$, we have that

$$\varphi_\theta'' = \theta\varphi_1'' + (1-\theta)\varphi_0''$$

and the result follows from (a). □

We are now ready to state the convexity of the norms of Examples 2.11 and 2.12.

Proposition 3.5. *For every $p_0, p_1 \in [2, +\infty[$ and every $\theta \in [0,1]$, the function*

$$f_\theta(x,y) = \|(x,y)\|_{p_1}^\theta \, \|(x,y)\|_{p_0}^{1-\theta} \quad (x, y \in \mathbb{R})$$

is a norm on $\mathbb{R}^2$.

Proof. Let us define $\varphi(t) = \log(f_\theta(t,1))$ and $\varphi_i(t) = \log\|(t,1)\|_{p_i}$ for $i = 0,1$ and every $t \in [0,1]$, and observe that

$$\varphi_i'(t) = \frac{t^{p_i-1}}{1+t^{p_i}} \quad \text{and} \quad \varphi_i''(t) = \frac{t^{p_i-2}(p_i - 1 - t^{p_i})}{(1+t^{p_i})^2} \quad (t \in [0,1],\ i = 0,1).$$

If $p_i \geqslant 2$, then $\varphi_i'' \geqslant 0$ for $i = 0,1$ and Lemma 3.4 gives us that the function $t \longmapsto f_\theta(t,1)$ for $t \in [0,1]$ is convex. Using Lemma 3.2 for $(\mathbb{R}^2, \|\cdot\|_\infty)$ we have that f is convex on the cone

$$\{(x,y) \in \mathbb{R}^2 : x \geqslant 0, y \geqslant 0, x \leqslant y\}.$$

Since the function f_θ is absolute and symmetric, the same argument is valid in any of the other seven cones wherein we can divide $\mathbb{R}^2$. Now, since f_θ is twice differentiable with partial derivatives of second order continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$, Lemma 3.3 gives us that it is convex on $\mathbb{R}^2$. Finally, since f_θ is positive homogeneous and it is zero only at zero, it is a norm on $\mathbb{R}^2$. □

Proposition 3.6. *For any $0 < a < 1$, the function $\|\cdot\|_a$ defined by*

$$\|(x,y)\|_a = \left(x^2 + \left(\frac{a}{1+a}\right)^{1+a} y^2\right)^{\frac{-a}{2}} \left(x^2 + \left(\frac{a}{1+a}\right)^{a} y^2\right)^{\frac{1+a}{2}} \quad ((x,y) \in \mathbb{R}^2 \setminus \{(0,0)\})$$

and $\|(0,0)\|_a = 0$, is a norm on $\mathbb{R}^2$.

Proof. First of all, $\|\cdot\|_a$ is positive homogeneous, it is obviously continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$ and it is also continuous at $(0,0)$ by homogeneity. We consider the function

$$\varphi(t) = \log(\|(t,1)\|_a) \quad (t \in \mathbb{R})$$

and observe that

$$\varphi'(t) = \frac{t^3}{\left(\left(\frac{a}{1+a}\right)^a + t^2\right)\left(\left(\frac{a}{1+a}\right)^{1+a} + t^2\right)} \quad (t \in \mathbb{R}),$$

$$\varphi''(t) = \frac{3t^2\left(\frac{a}{1+a}\right)^{1+2a} + t^4\left(\frac{a}{1+a}\right)^{a} + t^4\left(\frac{a}{1+a}\right)^{1+a} - t^6}{\left(\left(\frac{a}{1+a}\right)^{a} + t^2\right)^2\left(\left(\frac{a}{1+a}\right)^{1+a} + t^2\right)^2} \quad (t \in \mathbb{R}),$$

so we obviously obtain that

$$\varphi''(t) + (\varphi'(t))^2 = \frac{3t^2\left(\frac{a}{1+a}\right)^{1+2a} + t^4\left(\frac{a}{1+a}\right)^{a} + t^4\left(\frac{a}{1+a}\right)^{1+a}}{\left(\left(\frac{a}{1+a}\right)^{a} + t^2\right)^2\left(\left(\frac{a}{1+a}\right)^{1+a} + t^2\right)^2} \quad (t \in \mathbb{R}).$$

Therefore, Lemma 3.4 gives us that the function $t \longmapsto \|(t,1)\|_a$ for $t \in \mathbb{R}$ is convex and using now Lemma 3.2 for $(\mathbb{R}^2, |\cdot|_\varepsilon)$ where $|(x,y)|_\varepsilon = \max\{\varepsilon|x|, |y|\}$, and taking $\varepsilon \to 0$, this implies that $\|\cdot\|_a$ is convex on the upper halfplane. Repeating the argument by interchanging 1 by -1, we get that $\|\cdot\|_a$ is also convex on the lower halfplane. Now, Lemma 3.3 gives us that it is convex on $\mathbb{R}^2$ and the homogeneity shows that $\|\cdot\|_a$ is a norm on $\mathbb{R}^2$. □

One may wonder whether Proposition 3.5 is true for every pair of norms on $\mathbb{R}^2$. The following example shows that this is not the case even when working with C^∞ norms.

Example 3.7. For every $\theta \in]0,1[$, there is $\varepsilon > 0$ such that the positive homogeneous function

$$n(x,y) = (x^2 + \varepsilon y^2)^{\frac{\theta}{2}} (\varepsilon x^2 + y^2)^{\frac{1-\theta}{2}}$$

is not a norm. Indeed, just observe that

$$n(1,0) = \varepsilon^{\frac{1-\theta}{2}}, \quad n(0,1) = \varepsilon^{\frac{\theta}{2}} \quad \text{and} \quad n(1,1) = (1+\varepsilon)^{\frac{1}{2}}.$$

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