HAMILTONIAN APPROACH TO THE DYNAMICS OF PARTICLES IN NON SCALING FFAG ACCELERATORS

B.D. Muratori, S.L. Smith and S.I. Tzenov*, STFC/DL/ASTeC, Daresbury, Warrington, U.K.

Abstract

Starting from first principle the Hamiltonian formalism for the description of the dynamics of particles in non scaling FFAG machines has been developed. The stationary reference (closed) orbit has been found within the Hamiltonian framework. The dependence of the path length on the energy deviation has been described in terms of higher order dispersion functions. The latter have been used subsequently to specify the longitudinal part of the Hamiltonian. It has been shown that higher order phase slip coefficients should be taken into account to adequately describe the acceleration in non scaling FFAG accelerators.

INTRODUCTION

Fixed field alternating gradient (FFAG) accelerators were proposed half century ago [1, 2], when acceleration of electrons was first demonstrated. In addition to that, acceleration of protons has been recently achieved at the KEK Proof-of-Principle (PoP) proton FFAG [3]. Machines of this type use conventional magnets with the bending and focusing field being kept constant during acceleration. The latter alternate in sign, thus providing a more compact radial extension and consequently smaller aperture as compared to the AVF cyclotrons. The ring essentially consists of a sequence of short periods with very large periodicity.

Non scaling FFAG machines have until recently been considered as an alternative. The bending and the focusing is provided simultaneously by focusing and defocusing quadrupole magnets repeating in alternating sequence. There is a number of advantages of the non scaling FFAG lattice as compared to the scaling one, among which are the relatively small magnet aperture and the lower field strength. Unfortunately this lattice leads to a large betatron tune variation across the required energy range for acceleration as opposed to the scaling lattice, where the betatron tune stays constant. As a consequence several resonances are crossed during the acceleration cycle, some of them nonlinear created by the magnetic field imperfections, as well as half-integer and integer ones.

Because non scaling FFAG accelerators have otherwise very desirable features, it is important to investigate analytically and numerically some of the peculiarities of the beam dynamics, the new type of fast acceleration regime (so-called serpentine acceleration) and the effects of crossing of linear as well as nonlinear resonances. Some of these problems will be discussed in the present paper.

* E-mail: s.i.tzenov@dl.ac.uk

HAMILTONIAN FORMALISM

General Description

The Hamiltonian describing the motion of a particle in a natural coordinate system associated with a planar reference curve with curvature K is [4]

$$\widetilde{H} = -(1 + K\widetilde{x})$$
$$\times \left[\sqrt{\gamma^2 - 1 - \left(\widetilde{P}_x - \widetilde{q}A_x \right)^2 - \left(\widetilde{P}_z - \widetilde{q}A_z \right)^2} + \widetilde{q}A_s \right]. \quad (1)$$

Here $\mathbf{A} = (A_x, A_z, A_s)$ is the electromagnetic vector potential, while $\widetilde{x}$ and $\widetilde{z}$ are the horizontal and vertical deviations from the reference orbit with canonical conjugate momenta $\widetilde{P}_{x,z} = P_{x,z}/(m_0 c) = P_{x,z}/p_0$. Furthermore, the distance s along the circumference of the machine is chosen as an independent variable, m_0 is the particle rest mass and q is the charge, where $\widetilde{q} = q/(m_0 c) = q/p_0$. The longitudinal canonical coordinate Θ and its canonical conjugate variable γ comprising the third degree of freedom are given by

$$\Theta = -ct, \qquad \gamma = \frac{E}{E_0} = \frac{E}{m_0 c^2}. \quad (2)$$

Since the longitudinal quantities are dominant, one can expand the square root in power series in the transverse canonical coordinates. Tedious but straightforward algebra yields [4]

$$\widetilde{H} = \widetilde{H}_0 + \widetilde{H}_1 + \widetilde{H}_2 + \widetilde{H}_3 + \widetilde{H}_4 + \dots, \quad (3)$$

$$\widetilde{H}_0 = -\beta\gamma + \frac{1}{E_0} \left(\frac{\mathrm{d}\Delta E}{\mathrm{d}s} \right) \int \mathrm{d}\Theta \sin\phi(\Theta), \quad (4)$$

$$\widetilde{H}_1 = -(\beta\gamma - \beta_e\gamma_e)K\widetilde{x}, \quad (5)$$

$$\widetilde{H}_2 = \frac{\widetilde{P}_x^2 + \widetilde{P}_z^2}{2\beta\gamma} + \frac{1}{2}\left[(g + \beta_e\gamma_e K^2)\widetilde{x}^2 - g\widetilde{z}^2 \right], \quad (6)$$

$$\widetilde{H}_3 = \frac{K\widetilde{x}}{2\beta\gamma}\left(\widetilde{P}_x^2 + \widetilde{P}_z^2 \right) + \frac{Kg}{6}\left(2\widetilde{x}^3 - 3\widetilde{x}\widetilde{z}^2 \right), \quad (7)$$

$$\widetilde{H}_4 = \frac{\left(\widetilde{P}_x^2 + \widetilde{P}_z^2 \right)^2}{8\beta^3\gamma^3} - \frac{K^2 g}{24}\widetilde{z}^4, \;\; g = \widetilde{q}\left(\frac{\partial B_z}{\partial x} \right)_{x=z=0}, \quad (8)$$

where $\mathrm{d}\Delta E/\mathrm{d}s$ is the energy gain per unit longitudinal distance s, which in thin lens approximation scales as $\Delta E/\Delta s$, where Δs is the length of the cavity. In addition, γ_e is the energy corresponding to the reference orbit.

Reference Orbit

The FFAG lattice with polygonal structure in the horizontal plane will be considered here. To define and subsequently determine the stationary reference orbit, it is convenient to use a global Cartesian coordinate system whose origin is located in the centre of the polygon. To describe step by step the fraction of the reference orbit related to a particular side of the polygon, we rotate each time the axes of the coordinate system by an angle $\Theta_p = 2\pi/N_p$, where N_p is the number of sides of the polygon.

Let X_e and P_e denote the horizontal position along the reference orbit and the reference momentum, respectively. The vertical component of the magnetic field in the median plane of a perfectly linear machine can be written as

$$B_z(X_e, s) = a_1(s)[X_e - X_c - d(s)], \qquad a_1 = \frac{g}{\widetilde{q}}, \quad (9)$$

where s is the distance along the polygon side. The latter is at a distance

$$X_c = \frac{L_p}{2\tan(\Theta_p/2)}, \quad (10)$$

from the centre of the polygon, where the length of the polygon side L_p is actually representing the periodicity of the lattice. The quantity $d(s)$ in Eq. (9) is the relative offset of the magnetic centre from the polygon centre line.

A design (reference) orbit corresponding to a local curvature $K(X_e, s)$ can be defined according to the relation

$$K(X_e, s) = \frac{q}{p_0\beta_e\gamma_e} B_z(X_e, s). \quad (11)$$

In terms of the reference orbit position $X_e(s)$ the equation for the curvature can be written as [5]

$$X_e'' = \frac{q}{p_0\beta_e\gamma_e}\left(1 + X_e'^2\right)^{3/2} B_z(X_e, s), \quad (12)$$

where the prime implies differentiation with respect to s. Note that Eq. (12) parameterizing the local curvature can be derived from a Hamiltonian

$$H_e(X_e, P_e; s) = -\sqrt{\beta_e^2\gamma_e^2 - P_e^2} - \widetilde{q}\int \mathrm{d}X_e B_z(X_e, s), \quad (13)$$

which is nothing but the stationary part of the Hamiltonian (1) evaluated on the reference trajectory ($\widetilde{x} = 0$ and the accelerating cavities being switched off, respectively).

DISPERSION AND BETATRON MOTION

It is convenient to pass to new scaled variables as follows

$$\widetilde{p}_{x,z} = \frac{\widetilde{P}_{x,z}}{\beta_e\gamma_e}, \quad h = \frac{\gamma}{\beta_e^2\gamma_e}, \quad \tau = \beta_e\Theta, \quad \Gamma_e = \frac{\beta\gamma}{\beta_e\gamma_e}. \quad (14)$$

Thus, expressions (4) – (8) become

$$\widetilde{H}_0 = -\Gamma_e + \frac{1}{\beta_e^2 E_e}\left(\frac{\mathrm{d}\Delta E}{\mathrm{d}s}\right)\int \mathrm{d}\tau \sin\phi(\tau), \quad (15)$$

$$\widetilde{H}_1 = -(\Gamma_e - 1)K\widetilde{x}, \quad (16)$$

$$\widetilde{H}_2 = \frac{\widetilde{p}_x^2 + \widetilde{p}_z^2}{2\Gamma_e} + \frac{1}{2}\left[\left(g_e + K^2\right)\widetilde{x}^2 - g_e\widetilde{z}^2\right], \quad (17)$$

$$\widetilde{H}_3 = \frac{K\widetilde{x}}{2\Gamma_e}\left(\widetilde{p}_x^2 + \widetilde{p}_z^2\right) + \frac{Kg_e}{6}\left(2\widetilde{x}^3 - 3\widetilde{x}\widetilde{z}^2\right), \quad (18)$$

$$\widetilde{H}_4 = \frac{\left(\widetilde{p}_x^2 + \widetilde{p}_z^2\right)^2}{8\Gamma_e^3} - \frac{K^2 g_e}{24}\widetilde{z}^4, \qquad g_e = \frac{g}{\beta_e\gamma_e}. \quad (19)$$

The longitudinal part (τ_e, γ_e) of the reference orbit can be extracted by means of the canonical transformation

$$F_2\left(\widetilde{x}, \widetilde{\widetilde{p}}_x, \widetilde{z}, \widetilde{\widetilde{p}}_z, \tau, \eta; s\right) = \widetilde{x}\widetilde{\widetilde{p}}_x + \widetilde{z}\widetilde{\widetilde{p}}_z + (\tau + s)\left(\eta + \frac{1}{\beta_e^2}\right), \quad (20)$$

$$\sigma = \tau + s, \qquad \eta = h - \frac{1}{\beta_e^2}. \quad (21)$$

The linear and higher order dispersion can be introduced via a canonical transformation aimed to cancel the first order Hamiltonian $\widetilde{H}_1$ in all orders of η. The explicit form of the generating function is

$$G_2(\widetilde{x}, \widehat{p}_x, \widetilde{z}, \widehat{p}_z, \sigma, \widehat{\eta}; s) = \sigma\widehat{\eta} + \widetilde{z}\widehat{p}_z + \widetilde{x}\widehat{p}_x + \sum_{k=1}^{\infty}\widehat{\eta}^k\left[\widetilde{x}\mathcal{X}_k(s) - \widehat{p}_x\mathcal{P}_k(s) + \mathcal{S}_k(s)\right], \quad (22)$$

$$\widetilde{x} = \widehat{x} + \sum_{k=1}^{\infty}\widehat{\eta}^k\mathcal{P}_k, \qquad \widetilde{p}_x = \widehat{p}_x + \sum_{k=1}^{\infty}\widehat{\eta}^k\mathcal{X}_k, \quad (23)$$

$$\sigma = \widehat{\sigma} + \sum_{k=1}^{\infty} k\widehat{\eta}^{k-1}\left(\mathcal{P}_k\widehat{p}_x - \mathcal{X}_k\widehat{x}\right) - \sum_{k=1}^{\infty} k\widehat{\eta}^{k-1}\left(\mathcal{S}_k + \mathcal{X}_k\sum_{m=1}^{\infty}\widehat{\eta}^m\mathcal{P}_m\right). \quad (24)$$

Equating terms of the form $\widehat{x}\widehat{\eta}^n$ and $\widehat{p}_x\widehat{\eta}^n$ in the new transformed Hamiltonian, we determine order by order the conventional (first order) and higher order dispersions. The first order in $\widehat{\eta}$ (terms proportional to $\widehat{x}\widehat{\eta}$ and $\widehat{p}_x\widehat{\eta}$) yields the well-known result

$$\mathcal{P}_1' = \mathcal{X}_1, \qquad \mathcal{X}_1' + \left(g_e + K^2\right)\mathcal{P}_1 = K. \quad (25)$$

From the requirement that the second sum in equation (24) is identically zero, we readily obtain $\mathcal{S}_1 = 0$, and

$$\mathcal{S}_2 = -\frac{\mathcal{X}_1\mathcal{P}_1}{2}. \quad (26)$$

In second order the dispersion equations take the form

$$\mathcal{P}_2' = \mathcal{X}_2 - \mathcal{X}_1 + K\mathcal{X}_1\mathcal{P}_1, \quad (27)$$

$$\mathcal{X}_2' + \left(g_e + K^2\right)\mathcal{P}_2 = -Kg_e\mathcal{P}_1^2 - \frac{K\mathcal{X}_1^2}{2} - \frac{K}{2\gamma_e^2}. \quad (28)$$

In addition, we have

$$\mathcal{S}_3 = -\frac{1}{3}(\mathcal{X}_1\mathcal{P}_2 + 2\mathcal{X}_2\mathcal{P}_1). \tag{29}$$

Up to third order in $\widehat{\eta}$ the new Hamiltonian describing the longitudinal motion and the linear transverse motion acquires the form

$$\widehat{H}_0 = -\frac{\widetilde{\mathcal{K}}_1\widehat{\eta}^2}{2} + \frac{\widetilde{\mathcal{K}}_2\widehat{\eta}^3}{3} + \frac{1}{\beta_e^2 E_e}\left(\frac{\mathrm{d}\Delta E}{\mathrm{d}s}\right)\int \mathrm{d}\tau \sin\phi(\tau), \tag{30}$$

$$\widehat{H}_2 = \frac{\widehat{p}_x^2 + \widehat{p}_z^2}{2} + \frac{1}{2}\big[(g_e + K^2)\widehat{x}^2 - g_e\widehat{z}^2\big], \tag{31}$$

where

$$\widetilde{\mathcal{K}}_1 = K\mathcal{P}_1 - \frac{1}{\gamma_e^2}, \quad \widetilde{\mathcal{K}}_2 = \frac{K\mathcal{P}_1}{\gamma_e^2} - K\mathcal{P}_2 - \frac{\mathcal{X}_1^2}{2} - \frac{3}{2\gamma_e^2}. \tag{32}$$

The averaged over one lattice period

$$\mathcal{K}_1 = \frac{1}{L_p}\int\limits_s^{s+L_p} \mathrm{d}\lambda\widetilde{\mathcal{K}}_1(\lambda), \qquad \mathcal{K}_2 = \frac{1}{L_p}\int\limits_s^{s+L_p} \mathrm{d}\lambda\widetilde{\mathcal{K}}_2(\lambda), \tag{33}$$

represent the first and higher order phase slip coefficients, respectively.

The Twiss parameters corresponding to a fixed reference energy γ_e can be introduced in a conventional manner. For the sake of generality instead of (31) let us consider a Hamiltonian of the type

$$\widehat{H}_b = \sum_{u=(x,z)}\left[\frac{\mathcal{F}_u\widehat{p}_u^2}{2} + \mathcal{R}_u\widehat{u}\widehat{p}_u + \frac{\mathcal{G}_u\widehat{u}^2}{2}\right]. \tag{34}$$

A generic Hamiltonian of the type (34) can be transformed to the normal form

$$\mathcal{H}_b = \sum_{u=(x,z)}\frac{\chi_u'}{2}\left(\overline{P}_u^2 + \overline{U}^2\right), \tag{35}$$

by means of a canonical transformation specified by the generating function

$$\mathcal{F}_2(\widehat{x}, \overline{P}_x, \widehat{z}, \overline{P}_z; s) = \sum_{u=(x,z)}\left(\frac{\widehat{u}\overline{P}_u}{\sqrt{\beta_u}} - \frac{\alpha_u\widehat{u}^2}{2\beta_u}\right). \tag{36}$$

The old and the new canonical variables are related through the expressions

$$\widehat{u} = \overline{U}\sqrt{\beta_u}, \qquad \widehat{p}_u = \frac{1}{\sqrt{\beta_u}}(\overline{P}_u - \alpha_u\overline{U}). \tag{37}$$

The phase advance $\chi_u(s)$ and the generalized Twiss parameters $\alpha_u(s)$, $\beta_u(s)$ and $\gamma_u(s)$ are defined as

$$\chi_u' = \frac{\mathrm{d}\chi_u}{\mathrm{d}s} = \frac{\mathcal{F}_u}{\beta_u}, \tag{38}$$

$$\alpha_u' = \frac{\mathrm{d}\alpha_u}{\mathrm{d}s} = \mathcal{G}_u\beta_u - \mathcal{F}_u\gamma_u, \tag{39}$$

$$\beta_u' = \frac{\mathrm{d}\beta_u}{\mathrm{d}s} = -2\mathcal{F}_u\alpha_u + 2\mathcal{R}_u\beta_u. \tag{40}$$

The third Twiss parameter $\gamma_u(s)$ is introduced according to the well-known expression

$$\beta_u\gamma_u - \alpha_u^2 = 1. \tag{41}$$

The corresponding betatron tunes are determined as

$$\nu_u = \frac{N_p}{2\pi}\int\limits_s^{s+L_p}\frac{\mathrm{d}\lambda\mathcal{F}_u(\lambda)}{\beta_u(\lambda)}. \tag{42}$$

It is worthwhile to note that if one takes into account higher order nonlinear dispersion function contributions into the longitudinal part of the Hamiltonian (30), the time of flight variable τ becomes a polynomial function of the energy γ. Moreover, the approach described here provides a systematic perturbative tool to uniquely determine the polynomial coefficients up to arbitrary order. However, in practice a parabolic approximation of the time of flight as a function of energy is sufficient [6].

CONCLUDING REMARKS

Based on the Hamiltonian formalism, the synchro-betatron approach for the description of the dynamics of particles in non scaling FFAG machines has been developed. Its starting point is the specification of the static reference (closed) orbit for a fixed energy as a solution of the equations of motion in the machine reference system. The problem of acceleration and dynamical stability can be sequentially studied in the natural coordinate system associated with the reference orbit thus determined.

It has been further shown that the dependence of the path length on the energy deviation can be described in terms of higher order (nonlinear) dispersion functions. The method provides a systematic tool to determine the dispersion functions to every desirable order.

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