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***K*-Theory of non-linear projective toric varieties**

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(Communicated by Andrew Ranicki)

Abstract. We define a category of quasi-coherent sheaves of topological spaces on projective toric varieties and prove a splitting result for its algebraic K -theory, generalising earlier results for projective spaces. The splitting is expressed in terms of the number of interior lattice points of dilations of a polytope associated to the variety. The proof uses combinatorial and geometrical results on polytopal complexes. The same methods also give an elementary explicit calculation of the cohomology groups of a projective toric variety over any commutative ring.

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Let X denote a scheme with a preferred covering by open affine subschemes. A vector bundle on X can be described by a collection of finitely generated projective modules, one for each open affine of the chosen covering, and “restriction” maps between them, satisfying a certain gluing condition. For toric varieties it is possible to define analogous topological objects, replacing rings by monoids, modules by topological spaces, and weakening the gluing condition to a homotopy invariant condition. This program has been carried out by the author for projective spaces in [7] where it was shown that the algebraic K -theory of the resulting category of “non-linear sheaves” splits into $n + 1$ copies of WALDHAUSEN’S K -theory space $A(*)$. The aim of the present paper is to generalise this splitting result to arbitrary projective toric varieties, thereby revealing much of the combinatorial content of the earlier result explicitly.

This paper can also be understood as an attempt to describe toric varieties over “brave new rings”, replacing (commutative) rings by ring spectra. The combinatorial structure of toric varieties is rigid enough to allow a treatment with techniques from unstable homotopy theory (using spaces, not spectra). It is not clear what a toric variety should be in that context, but we can nevertheless define quasi-coherent sheaves on such a variety, called non-linear sheaves. This category carries enough structure to define, for example, algebraic K -theory, just as the K -theory of a ring R can be defined in terms of a category of R -modules.

An n -dimensional polytope with integral vertices defines a projective toric variety X_P (its construction is reviewed in §2.2), equipped with an ample (equivariant) line bundle $\mathcal{O}(1)$. (It can be shown that any projective toric variety over \mathbb{C} equipped with an

ample (equivariant) line bundle arises in this way.) We denote the algebraic K -theory of non-linear sheaves on X_P by $K^{\text{nl}}(X_P)$. The following is the main theorem of the paper:

Theorem 3.3.6. *Let $P \subset \mathbb{R}^n$ be a polytope with integral vertices, and assume that P has non-empty interior. Let k denote the number of integral roots of its EHRHART polynomial (cf. Theorem 2.5.1); possibly $k = 0$. Then there is a homotopy equivalence*

$$\underbrace{A(*) \times \cdots \times A(*)}_{(k+1) \text{ factors}} \times K^{\text{nl}}(X_P)^{[k]} \xrightarrow{\sim} K^{\text{nl}}(X_P)$$

where $K^{\text{nl}}(X_P)^{[k]}$ is the algebraic K -theory of the category of those non-linear sheaves Y on X_P which have $\Gamma(Y(i)) \simeq *$ for all $0 \leq i \leq k$.

In fact, $k \geq 0$ is minimal among integers $j \geq 0$ such that the dilated polytope $(j+1)P$ has a lattice point in its interior. The map in the theorem is induced by the assignment

$$(K_0, K_1, \dots, K_k) \mapsto \bigvee_{i=0}^k K_i \wedge \mathcal{O}_P(-i).$$

Here the K_i are pointed topological spaces, \mathcal{O}_P is the non-linear analogue of the structure sheaf on X_P , and $\mathcal{O}_P(i)$ is its i th twist (§2.3). The functor Γ is the total cofibre functor (§2.4), a substitute for the global sections functor and its derived functors in algebraic geometry.

A similar splitting result should hold for the algebraic K -theory of projective toric varieties over \mathbb{C} . To explain the passage to the “linear” world, note that by taking free \mathbb{C} -vector spaces the non-linear sheaves $\mathcal{O}_P(j)$ give rise to the usual twisting sheaves $\mathcal{O}(j) = \mathcal{O}(1)^{\otimes j}$ on the \mathbb{C} -scheme X_P . The meaning of the number k in the theorem is that $H^i(X_P, \mathcal{O}(-k)) = 0$ for all $i \geq 0$, but $H^n(X_P, \mathcal{O}(-k-1)) \neq 0$. It turns out that the obstruction to a further splitting of $K(X_P)$ is the non-vanishing of the cohomology of $\mathcal{O}(-k-1)$, or equivalently, the presence of a lattice point in the interior of $(k+1)P$.

The total cofibres of the non-linear sheaves $\mathcal{O}_P(j)$ exhibit the same behaviour as their linear counterparts $\mathcal{O}(j)$ as is demonstrated by the explicit calculations in §2.5. In particular, the obstruction for splitting off a further copy of $A(*)$ in Theorem 3.3.6 is the non-triviality of the total cofibre of $\mathcal{O}_P(-k-1)$. The similarity between total cofibres and sheaf cohomology is not coincidental as is explained in [9].

If P is an n -dimensional standard simplex (i.e., if P is a lattice simplex with volume $1/n!$), then $k = n$, and the variety X_P is the n -dimensional projective space equipped with the usual twisting sheaf $\mathcal{O}(1)$. In this case, it can be shown that $K^{\text{nl}}(X_P)^{[n]} \simeq *$, so the above theorem reduces to the known splitting of [7].

The proof of the splitting result relies on explicit computations of certain homotopy colimits of “geometrically defined” diagrams. We review polytopal complexes

in §1.1, and prove a result relating the nerve of a subset of a polytopal complex to its underlying space. Results of this type, often well-known for simplicial complexes, are part of the general toolkit for dealing with homotopy types of nerves of posets [14]; the given version is slightly more general than needed for this paper. Other geometrical issues are discussed in §1.2: Links, stars and visibility subcomplexes of the boundary complex of a polytope are introduced and examined in detail. Although quite elementary, it will be important for the sequel to provide explicit descriptions throughout.

In §2 we review the description of quasi-coherent sheaves on projective toric varieties by diagrams of modules. By analogy, a notion of “non-linear sheaves” is introduced (§2.3). Twisting sheaves and tensor products are also defined by analogy. To obtain a homotopically meaningful analogy of global sections, we use the “total cofibre” construction of §2.4. The vanishing criterion for total cofibres makes use of the combinatorial results from §1. Next, we calculate total cofibres of twisting sheaves (§2.5); here the material from §1 is used heavily again. We show that the same techniques also lead to an elementary computation of the cohomology groups $H^r(X; \mathcal{O}(k))$ over any commutative ring R where $X = X_P$ is the toric variety defined by P . (Standard references for toric geometry seem to miss an explicit combinatorial treatment of negative twists. Note also that the given treatment does not use SERRE duality to deal with the case of non-ample line bundles but yields a direct identification of a basis of the unique non-trivial cohomology module. See Remark 2.5.4 for pointers to an algebro-geometric approach.) Comparison with §2.5 shows that the total cofibre construction captures not only global sections, but higher cohomology groups as well.

Finally, §3 is concerned with K -theoretical issues. Following a brief discussion of finiteness conditions for non-linear sheaves (which are also the subject of the paper [8]) we define their algebraic K -theory and prove the splitting result.

1 On polytopal complexes

1.1 Complexes and order filters

A polytope P is the convex hull of a finite set of points in \mathbb{R}^n . We write $F \leq P$ if F is a face of P ; this includes the case of improper faces $F = P$ and $F = \emptyset$.

1.1.1. Definition. A non-empty finite collection K of non-empty polytopes in some \mathbb{R}^n is called a *polytopal complex* if the following conditions are satisfied:

- (1) If $F \in K$ and $\emptyset \neq G \leq F$, then $G \in K$.
- (2) For all $F, G \in K$, the intersection $F \cap G$ is a (possibly empty) face of F and G .

A subset $L \subseteq K$ of a polytopal complex is called an *order filter* if for all $F \in L$ and $G \in K$ with $F \leq G$, we have $G \in L$. A subset $L \subseteq K$ of a polytopal complex is called a *subcomplex* of K if L is a polytopal complex.

Important examples of polytopal complexes are the complex $F(P)_0$ of non-empty faces of a polytope P , and its subcomplex $F(P)_0^1$ of non-empty proper faces of P (sometimes called boundary complex of P).

The intersection of two subcomplexes, if non-empty, is a subcomplex. The (set-theoretic) complement of a subcomplex is an order filter.

1.1.2. Definition. Suppose K is a polytopal complex, and L is a non-empty subset of K . We call $|L| := \bigcup_{F \in L} F$ the *realisation* or the *underlying space* of L . The *combinatorial closure* of L in K is the set of all polytopes in L and their non-empty faces:

$$\bar{L} := \{F \in K \mid \exists G \in L : F \leq G\}.$$

The combinatorial closure of L in K is a complex, and we have $|L| = |\bar{L}|$. If P is an n -dimensional polytope, we have PL -homeomorphisms $|F(P)_0| = P \cong_{PL} B^n$ and $|F(P)_0^1| = \partial P \cong_{PL} S^{n-1}$.

Note that a complex K is naturally a partially ordered set with order given by inclusion of faces. Hence we can view any non-empty subset $L \subseteq K$ as a category with morphisms corresponding to inclusion of faces. Its nerve NL is an abstract simplicial complex; a k -simplex is a strictly increasing sequence $[F_0 < F_1 < \dots < F_k]$ of polytopes in L . For each polytope $F \in L$ there is a corresponding vertex $[F]$ of NL . We denote the geometric realisation of NL by $|NL|$; this space is called the *classifying space* of L .

For $F \in K$ let \hat{F} denote its barycentre. Define a map $\alpha : |NL| \rightarrow |K|$ by sending the zero-simplex $[F] \in NL$ to the point $\hat{F} \in |K|$ and extending linearly over simplices. This map is an embedding and thus allows us to view the abstract simplicial complex NL as a simplicial complex, *i.e.*, a polytopal complex consisting of simplices.

1.1.3. Lemma. *Suppose K is a polytopal complex. The simplicial complex NK is the barycentric subdivision of K . The map $\alpha : |NK| \rightarrow |K|$ is a PL -homeomorphism, and the pair (NK, α) is a triangulation of $|K|$ ([12], p. 17). \square*

1.1.4. Definition. Let K denote a polytopal complex, and fix $A \in K$.

- (1) The (*open*) *star* of A is $st(A) := \{F \in K \mid A \leq F\}$.
- (2) The (*closed*) *star* of A is defined as the combinatorial closure of $st(A)$. Explicitly, $\overline{st}(A) = \{F \in K \mid \exists G \in st(A) : F \leq G\}$.
- (3) The (*closed*) *antistar* of A is $\overline{ast}(A) := K \setminus st(A)$.
- (4) The (*open*) *antistar* of A is $ast(A) := K \setminus \overline{st}(A)$.
- (5) The *link* of A is $lk(A) := \overline{st}(A) \cap \overline{ast}(A)$.

If we have to emphasise the complex K we write $lk_K(A)$ instead of $lk(A)$, and similar for the other expressions.

The sets just defined are combinatorial rather than geometric in nature; for example, $|st(A)| = |\overline{st}(A)|$, but as sets, $st(A)$ and $\overline{st}(A)$ usually differ.

The open star and open antistar are order filters. The closed antistar, if non-empty, is a subcomplex of K . The link $lk(A)$ can also be described by $lk(A) = \overline{st}(A) \setminus st(A) = \overline{ast}(A) \setminus ast(A)$. Geometrically the link of A consists of those polytopes that are “visible” inside $|K|$ from the barycentre \hat{A} of A but do not contain \hat{A} .

Note that the open antistar may be empty while the closed antistar is not. Thus in general $\overline{ast}(A)$ is not the combinatorial closure of $ast(A)$, but see below for the case of manifolds. (For example, consider the complex $K = F(P)_0$. Then $\overline{ast}(P) = F(P)_0^1 = lk(P)$ while $ast(P) = \emptyset$.)

Let L denote the stellar subdivision of K at \hat{A} [3, Definition III.2.1]. By definition, L is obtained from K by removing $st_K(A)$ and adding the cones from \hat{A} on polytopes in $lk_K(A)$. Then it is easy to see that $|lk_K(A)| = |lk_L(\hat{A})|$ and $|\overline{st}_K(A)| = |\overline{st}_L(\hat{A})|$, and this agrees with the definition of link and star in [12], p. 20. Thus we see that $|st(A)|$ is a (topological) neighbourhood of \hat{A} in $|K|$; it is the cone from \hat{A} on $|lk(A)|$. If K is an m -dimensional PL manifold (possibly with boundary), $|st(A)| \cong_{PL} B^m$, while $|lk(A)|$ is PL homeomorphic to S^{m-1} if $\hat{A} \in \text{int}|K|$ and to B^{m-1} if $\hat{A} \in \partial|K|$ by [12, Exercise 2.21 (1)]. Moreover, if $\hat{A} \in \text{int}|K|$ we know that $|lk(A)| = \partial|st(A)|$ is the boundary sphere of the ball $|st(A)|$.

1.1.5. Lemma. *Suppose $|K|$ is an m -dimensional PL manifold without boundary. Then $ast(A) \neq \emptyset$, and the closed antistar of A is a combinatorial closure of $ast(A)$. Moreover, $|ast(A)|$ is the closure of the complement $|K| \setminus |st(A)|$, and $|lk(A)| \cong_{PL} S^{m-1}$ is the boundary of both $|st(A)|$ and $|ast(A)|$.*

Proof. Note first that $ast(A) = \emptyset$ implies $\overline{st}(A) = K$. But $|\overline{st}(A)|$ is a ball since $|K|$ is a PL manifold [12, Exercise 2.21 (1)]. This contradicts the assumption that K has no boundary. Thus necessarily $ast(A) \neq \emptyset$.

Since $\overline{ast}(A)$ is a complex, the combinatorial closure of $ast(A)$ is contained in the closed antistar. Conversely, given an element $F \in \overline{ast}(A) \setminus ast(A)$ we have to show that F is the face of some $G \in ast(A)$. Suppose such G does not exist. Then $st(F) \subseteq \overline{st}(A)$. Moreover, $|st(F)|$ is known to be a ball with \hat{F} in its interior, which shows that \hat{F} is an interior point of the ball $|\overline{st}(A)|$. But since $F \in lk(A)$ we know $\hat{F} \in \partial|\overline{st}(A)|$, a contradiction.—The other assertions are obvious. \square

A similar argument shows more generally:

1.1.6. Lemma. *Let L be a subcomplex of K . Suppose $|K|$ is an m -dimensional PL manifold without boundary, and $|L|$ is an m -dimensional PL manifold with boundary. Then $C := K \setminus L$ is non-empty, and $|C|$ is the closure of the complement $|K| \setminus |L|$. Let B denote the intersection of L and \overline{C} . Then $|B|$ is the boundary of both $|L|$ and $|C|$. \square*

The following lemma shows how to connect data useful for homotopy theory (the nerve of a certain category) with geometric data (subspaces of a PL manifold). Similar results are well-documented for simplicial complexes; extension to polytopal complexes can be achieved by passage to barycentric subdivisions. Our proof utilises the “simplicial neighbourhood theorem” of ROURKE and SANDERSON [12, Theorem 3.11].

1.1.7. Lemma. *Let K denote a polytopal complex, and let $C \subseteq K$ be an order filter. Assume $|K|$ is an m -dimensional PL manifold without boundary and $|K \setminus C|$ is an m -dimensional PL manifold with boundary. Then $|C|$ is a regular neighbourhood of $|NC|$ in $|K|$, hence $|NC|$ is a deformation retract of $|C|$.*

Proof. The map α from 1.1.3 allows us to consider $|NC|$ as a subspace of $|C|$. Given that $|C|$ is a regular neighbourhood of $|NC|$, the collapsing criterion [12, Corollary 3.30] shows that $|C|$ collapses to $|NC|$, thereby proving the proposition.

Define $B := \bar{C} \setminus C$. Using Lemma 1.1.3 we see that $|C| = |\bar{C}| \cong_{PL} |N\bar{C}|$. Moreover, NC and NB are simplicial subcomplexes of $N\bar{C}$. By the ‘‘simplicial neighbourhood theorem’’ [12, Theorem 3.11], it thus suffices to prove the following assertions:

- (1) $|N\bar{C}|$ is a compact PL manifold with boundary, and NB is a triangulation of $\partial|N\bar{C}|$
- (2) NC is a full subcomplex of $N\bar{C}$
- (3) NB is the simplicial complement of NC inside $N\bar{C}$
- (4) $N\bar{C}$ is the simplicial neighbourhood of NC in $N\bar{C}$
- (5) $|NC|$ lies in the interior of $|N\bar{C}|$

(1): Lemma 1.1.6 implies that $|C|$ is a PL manifold with boundary $|B|$ (compactness is automatic since all our complexes are finite). By 1.1.3 there are homeomorphisms $|C| \cong_{PL} |N\bar{C}|$ and $|B| \cong_{PL} |NB|$. Thus NB is a triangulation of $\partial|C| = |B|$.

(2): A k -simplex F of $N\bar{C}$ is a chain of polytopes

$$F = [A_0 < A_1 < \cdots < A_k]$$

with $A_i \in \bar{C}$. Assume the boundary of F is contained in NC . Then in particular all its vertices $[A_i]$ are in NC , i.e., $A_i \in C$, hence F is an element of NC by definition of the nerve. By [12], Exercise 3.2 this implies assertion 2.

(3): The simplicial complement of NC inside $N\bar{C}$ is, by definition, the set

$$\{F \in N\bar{C} \mid F \cap |NC| = \emptyset\}.$$

Let $F = [A_0 < \cdots < A_k]$ be a k -simplex in $N\bar{C}$. Then $F \cap |NC| \neq \emptyset$ if and only if there is a simplex $G = [B_0 < \cdots < B_l] \in NC$ with $F \cap G \neq \emptyset$. But $F \cap G$ is also a face of F and G . In particular, F and G have non-empty intersection if and only if they have a common vertex $A_i = B_j \in C$. This shows that $F \cap |NC| = \emptyset$ if and only if no A_i is in C , i.e., if and only if $F \in NB = N(\bar{C} \setminus C)$.

(4): The simplicial neighbourhood of NC in $N\bar{C}$ is, by definition, the set

$$T := \{F \in N\bar{C} \mid \exists G \in NC : F \leq G \text{ and } G \cap |NC| \neq \emptyset\}.$$

Let $F = [A_0 < \cdots < A_k]$ be a k -simplex of $N\bar{C}$. If $A_k \in C$ we have $F \cap |NC| \neq \emptyset$ by the arguments in (3), hence $F \in T$. Otherwise there exists $A_{k+1} \in C$ with $A_k < A_{k+1}$ by definition of the combinatorial closure. Then F is a face of $G := [A_0 < \cdots < A_k < A_{k+1}]$ and $G \cap |NC| \neq \emptyset$ by construction, thus $F \in T$.

(5): We show that $|NC| \cap \partial|N\bar{C}| = \emptyset$. Recall that $\partial|N\bar{C}| = |B|$. Let $F \in N\bar{C}$ be given. By arguments similar to those of (3), applied to the complex NB , we see that $F \cap |B|$ is non-empty if and only if the chain $A_0 < \dots < A_k$ representing F satisfies $A_0 \in B$. But that cannot happen for $F \in NC$. \square

1.1.8. Corollary. *Let K denote a polytopal complex, and let $C \subseteq K$ be an order filter. Suppose $|K|$ is an m -dimensional PL sphere, and $|K \setminus C|$ is an m -dimensional ball. Then $|NC|$ is contractible.*

Proof. From Lemma 1.1.6 and [12, Corollary 3.13_m], we know that $|C|$ is a PL ball, hence is contractible. Consequently its deformation retract $|NC|$ is contractible as well. \square

1.2 The boundary complex of a polytope

We restrict attention to the special case of the boundary complex of an n -dimensional polytope $P \subseteq \mathbb{R}^n$. Its realisation is ∂P , thus it is a PL sphere of dimension $n - 1$. In order to apply Corollary 1.1.8 we need to construct “interesting” $(n - 1)$ -balls inside ∂P . One class of examples is given by the closed stars which can be characterised by purely combinatorial means. We also discuss examples given by subsets of faces satisfying certain geometric conditions.

Links, stars and antistars

For a polytope P , the set $F(P)_0^1$ of non-empty proper faces of P is a polytopal complex, called *boundary complex* of P . Links and antistars admit convenient combinatorial descriptions in this case.

The set $F(P)$ of all faces of P (including P and \emptyset) is known to be a finite graded lattice [15, Theorem 2.7]. We write $F \vee G$ for the join of F and G in $F(P)$; it is the smallest face of P containing $F \cup G$. Links, stars and antistars are computed in the complex $K = F(P)_0^1$ unless indicated otherwise; in particular, the star of a proper face of P will not contain P itself.

1.2.1. Lemma (Combinatorial description of star, link and antistar.). *Let A denote a proper non-empty face of P .*

- (1) $\overline{st}(A) = \{F \in F(P)_0^1 \mid F \vee A \neq P\}$
- (2) $lk(A) = \{F \in F(P)_0^1 \mid F \vee A \neq P \ \& \ A \not\subseteq F\}$
- (3) $ast(A) = \{F \in F(P)_0^1 \mid F \vee A = P\}$

Proof. To prove (1), suppose $F \in F(P)_0^1$ satisfies $F \vee A \neq P$. Then $F \vee A \in F(P)_0^1$, and from $F \leq F \vee A \geq A$ we get $F \in \overline{st}(A)$. Conversely, if F is an element of the closed star of A , we find a proper face G of P with $F \leq G \geq A$. But then $F \vee A \leq G \vee A = G \neq P$.

Assertions (2) and (3) follow immediately from (1). \square

1.2.2. Corollary. *Let A be a proper non-empty face of P . If $B \in st(A) \setminus \{A\}$, we have $A \in lk(B)$ and*

$$\overline{st}_{lk(B)}(A) = \{F \in lk(B) \mid B \not\subseteq F \vee A\}$$

where $\overline{st}_{lk(B)}(A)$ denotes the closed star of A in the polytopal complex $lk(B)$.

Proof. By hypothesis $B \supset A$, hence $A \in lk(B)$. Suppose we have an element $F \in \overline{st}_{lk(B)}(A)$. By definition of the closed star, there is a $G \in lk(B)$ with $F \leq G \geq A$. But then

$$F \vee A \subseteq G \vee A = G \in lk(B).$$

Since $B \not\subseteq G$ by definition of the link, this implies $B \not\subseteq F \vee A$.

Conversely, given $F \in lk(B)$ with $B \not\subseteq F \vee A$, we know that $F \vee A \subseteq F \vee A \vee B = F \vee B \neq P$, thus $F \vee A \in lk(B)$. From $F \leq F \vee A \geq A$ we conclude $F \in \overline{st}_{lk(B)}(A)$. \square

Visible and invisible faces

1.2.3. Definition. A face $F \in F(P)_0^1$ is called *visible* from the point $x \in \mathbb{R}^n \setminus P$ if $[p, x] \cap P = \{p\}$ for all $p \in F$. (Here $[p, x]$ denotes the line segment between p and x .) Equivalently, F is visible if $p + \lambda(x - p) \notin P$ for all points $p \in F$ and real numbers $\lambda > 0$. We denote the set of visible faces by $Vis(x)$; its complement $Inv(x) := F(P)_0^1 \setminus Vis(x)$ is the set of *invisible faces*. Let $\overline{Inv(x)}$ denote the combinatorial closure of $Inv(x)$, and define $\partial Inv(x) := \overline{Inv(x)} \setminus Inv(x) = \overline{Inv(x)} \cap Vis(x)$.

1.2.4. Lemma. *A facet F of P is visible from x if and only if x and $\text{int } P$ are on different sides of the affine hyperplane spanned by F . A proper non-empty face of P is visible if and only if it is contained in a visible facet of P .* \square

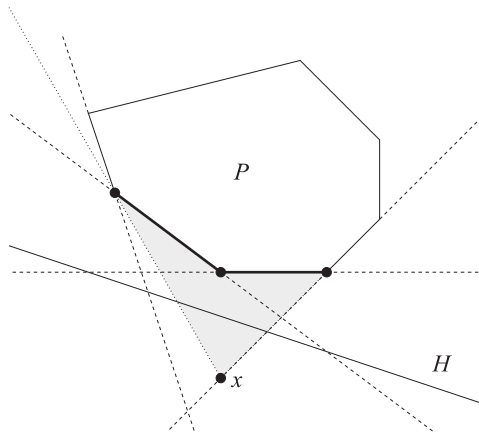


Fig. 1. Visible faces

In particular, the sets $Vis(x)$ and $Inv(x)$ are non-empty. Since a face of a visible face is visible itself, $Vis(x)$ and $\partial Inv(x)$ are subcomplexes while $Inv(x)$ is an order filter. If x is beyond F in the sense of [15], p. 78, the set of visible faces $Vis(x)$ coincides with the closed star of F .

1.2.5. Proposition. (1) *There is a PL homeomorphism $|Vis(x)| \cong B^{n-1}$.*

(2) *There is a PL homeomorphism $|Inv(x)| \cong B^{n-1}$.*

(3) *$|Vis(x)| \cap |Inv(x)| = |\partial Inv(x)|$ is the common boundary of both $|Vis(x)|$ and $|Inv(x)|$, hence is PL-homeomorphic to S^{n-2} .*

Proof. Applying a translation if necessary we may assume $x = 0$. For statement (1), let H be any hyperplane separating 0 and P (Fig. 1). Let C denote the cone (with apex 0) on P . Then C is a pointed polyhedral cone, hence $C \cap H$ is a PL ball [3, Theorem V.1.1]. Projection along C provides a homeomorphism $|Vis(x)| \cong C \cap H$. By the “pseudo radial projection” technique ([12], proof of Lemma 2.19) this can be modified to give a PL homeomorphism.

Statements (2) and (3) follow from Lemma 1.1.6 and the fact that the closure of the complement of a (full dimensional) PL ball inside a PL sphere is a PL ball itself ([12], Corollary 3.13_n). □

1.2.6. Corollary. *The classifying space of $Inv(x)$ is a deformation retract of $|Inv(x)|$. In particular, $|N Inv(x)|$ is contractible.*

Proof. This follows from Corollary 1.1.8 applied to $K = F(P)_0^1$ and $C = Inv(x)$, using Proposition 1.2.5 (1). □

Front and back faces

1.2.7. Definition. A face $F \in F(P)_0^1$ is called a *back face* with respect to the point $x \in \mathbb{R}^n \setminus \text{int } P$ if for all points $p \in F$ and all real numbers $\lambda > 0$ we have $p + \lambda(p - x) \notin P$. The set of back faces is denoted by $Back(x)$; its complement $Front(x) := F(P)_0^1 \setminus Back(x)$ is the set of *front faces*. Let $\overline{Front(x)}$ denote the combinatorial closure of $Front(x)$, and define $\partial Front(x) := Back(x) \cap \overline{Front(x)}$.

1.2.8. Lemma. *Suppose F is a facet of P . Then F is a back face with respect to x if and only if x and $\text{int } P$ are on the same side of the affine hyperplane spanned by F . A proper non-empty face F of P is a back face if and only if it is contained in a facet of P which is a back face.* □

In particular, the sets $Back(x)$ and $Front(x)$ are non-empty. Since a face of a back face is a back face itself, $Back(x)$ and $\partial Front(x)$ are subcomplexes while $Front(x)$ is an order filter.

By arguments similar to the ones used for the case of visible faces, we can show:

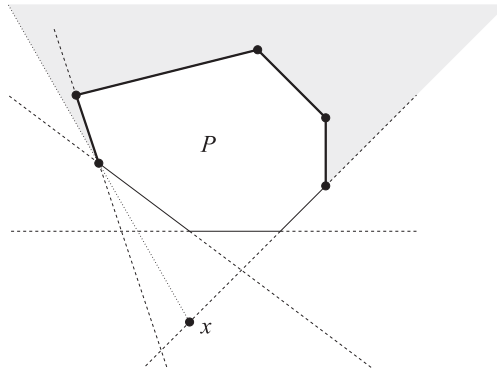


Fig. 2. Back faces

1.2.9. Proposition. (1) *There is a PL homeomorphism $|Back(x)| \cong B^{n-1}$.*

(2) *There is a PL homeomorphism $|Front(x)| \cong B^{n-1}$.*

(3) *$|Front(x)| \cap |Back(x)| = |\partial Front(x)|$ is the boundary of both $|Front(x)|$ and $|Back(x)|$, hence is PL-homeomorphic to S^{n-2} . \square*

1.2.10. Corollary. *The classifying space of $Front(x)$ is a deformation retract of $|Front(x)|$. In particular, $|N Front(x)|$ is contractible. \square*

Upper and lower faces

1.2.11. Definition. A face $F \in F(P)_0^1$ is called a *lower face* with respect to the direction $x \in \mathbb{R}^n \setminus \{0\}$ if for all points $p \in F$ and all real numbers $\lambda > 0$ we have $p - \lambda x \notin P$. The set of lower faces is denoted by $Low(x)$; its complement $Up(x) := F(P)_0^1 \setminus Low(x)$ is the set of *upper faces*. Let $\overline{Up(x)}$ denote the combinatorial closure of $Up(x)$, and define $\partial Up(x) := Low(x) \cap \overline{Up(x)}$.

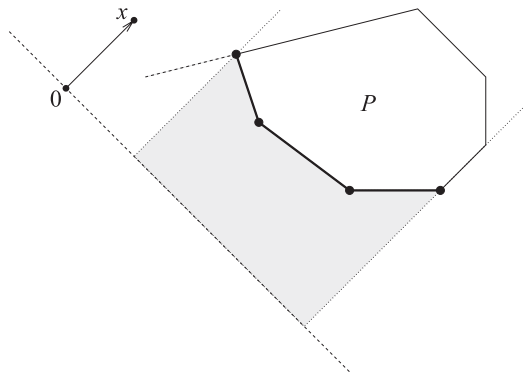


Fig. 3. Lower faces

1.2.12. Lemma. *Suppose F is a facet of P with inward pointing normal vector v . Then F is a lower face with respect to x if and only if $\langle x, v \rangle > 0$. A proper non-empty face of P is a lower face if and only if it is contained in a facet of P which is a lower face. \square*

In particular, the sets $Low(x)$ and $Up(x)$ are non-empty. Since a face of a lower face is a lower face itself, $Low(x)$ and $\partial Up(x)$ are subcomplexes while $Up(x)$ is an order filter.

By arguments similar to the ones used for the case of visible faces, we can show:

1.2.13. Proposition. (1) *There is a PL homeomorphism $|Low(x)| \cong B^{n-1}$.*

(2) *There is a PL homeomorphism $|Up(x)| \cong B^{n-1}$.*

(3) *$|Low(x)| \cap |Up(x)| = |\partial Up(x)|$ is the common boundary of both $|Low(x)|$ and $|Up(x)|$, hence is PL-homeomorphic to S^{n-2} . \square*

1.2.14. Corollary. *The classifying space of $Up(x)$ is a deformation retract of $|Up(x)|$. In particular, $|N Up(x)|$ is contractible. \square*

2 Non-linear sheaves and total cofibres

2.1 Equivariant spaces

Before describing quasi-coherent sheaves on projective toric varieties we introduce some terminology concerning topological spaces. Let M denote an abelian pointed monoid (*i.e.*, we have elements $*, 0 \in M$ such that $0 + m = m$ and $* + m = *$ for all $m \in M$). Any abelian monoid can be made into a pointed monoid by artificially adding a disjoint basepoint $*$. We consider M as a discrete topological space with $*$ as base point. The category of pointed topological spaces with a right (base point preserving) action of M will be denoted $M\text{-Top}_*$. The M -equivariant n -cell is the space $D_+^n \wedge M$, its boundary is $\partial D_+^n \wedge M$. Let K be an object of $M\text{-Top}_*$.

(1) We call K *cellular* if K can be obtained from a point by attaching (possibly infinitely many) cells, not necessarily in order of increasing dimension.

(2) We call K *cofibrant* if K is a retract of a cellular space. The full subcategory of $M\text{-Top}_*$ consisting of cofibrant spaces is denoted $\mathbf{C}(M)$. If $M = S^0$ is the initial pointed monoid, we abbreviate this to \mathbf{C} .

(3) The space K is called *finite* if K can be obtained from a point by attaching finitely many cells, not necessarily in order of increasing dimension. The full subcategory of $M\text{-Top}_*$ consisting of finite spaces is denoted $\mathbf{C}_f(M)$. If $M = S^0$ is the initial pointed monoid, we abbreviate this to \mathbf{C}_f .

(4) The space K is called *homotopy finite* if there is a chain (or zigzag) of weak equivalences connecting K to an object of $\mathbf{C}_f(M)$. The full subcategory of $M\text{-Top}_*$ consisting of cofibrant, homotopy finite spaces is denoted $\mathbf{C}_{\text{hf}}(M)$. If $M = S^0$ is the initial pointed monoid, we abbreviate this to \mathbf{C}_{hf} .

In (4) a *weak equivalence* is an equivariant map which is a weak homotopy equivalence on underlying topological spaces.

2.1.1. Remark. The category $M\text{-Top}_*$ admits a QUILLEN model structure with weak equivalences as above, and fibrations those maps which are SERRE fibrations on underlying topological spaces. The resulting notion of a cofibrant object coincides with the one given above.

General arguments from model category theory, or a variation on the WHITEHEAD theorem, imply that a map $X \rightarrow Y$ of cofibrant spaces is a weak equivalence if and only if it is a homotopy equivalence. In particular, a cofibrant space is weakly contractible if and only if it is contractible. Since M is discrete, the forgetful functor restricts to a functor $\mathbf{C}(M) \rightarrow \mathbf{C}$. In particular, objects of $\mathbf{C}(M)$ are *well-pointed* in the sense that the inclusion of the base point has the homotopy extension property.

We will have occasion to use the following standard fact frequently in the remainder of the paper:

2.1.2. Lemma. *Let $f : X \rightarrow Y$ be a map in \mathbf{Top}_* such that its homotopy cofibre (reduced mapping cone) is contractible. Then the reduced suspension Σf of f is a homotopy equivalence.*

Proof. For any space V we have an exact sequence of pointed sets

$$[X, V] \xleftarrow{f^*} [Y, V] \longleftarrow [\text{hocofibre } f, V] \longleftarrow [\Sigma X, V] \xleftarrow{\Sigma f^*} [\Sigma Y, V],$$

cf. [11, Satz 6], where $[A, B]$ denotes the set of (pointed) homotopy classes of maps $A \rightarrow B$ in \mathbf{Top}_* . Since $\text{hocofibre } f \simeq *$ we have $[\text{hocofibre } f, V] = 0$. Hence f^* is monomorphic in the sense of PUPPE [11, Footnote 1], and Σf^* is surjective. It follows that Σf has both a left homotopy inverse [11, §3.1] and a right homotopy inverse [11, §3.3]. \square

2.2 Barrier cones and projective toric varieties

We will now recall the construction of toric varieties from polytopes. Standard references are FULTON's book [4] and DANILOV's article [2] which contain a wealth of information on the general theory of toric varieties. More specifically, to construct varieties from polytopes see [4, §1.5, §3.4] and [2, §5.8, §11.12].

Let $P \subset \mathbb{R}^n$ be a lattice polytope (the convex hull of a finite set of points in \mathbb{Z}^n) with non-empty interior. Given a non-empty face F of P we define the *barrier cone* C_F of P at F as the set of finite linear combination with non-negative real coefficients spanned by the set $P - F := \{p - f \mid p \in P \text{ and } f \in F\}$. Since C_F is a cone, the integral points in C_F form a monoid (with respect to the usual vector sum). By adding a disjoint basepoint, we thus obtain an abelian pointed monoid $S_F := (C_F \cap \mathbb{Z}^n)_+$.

For a commutative ring R , let $\tilde{R}[S_F] = R[S_F]/R[*]$ denote the reduced monoid ring. If $\emptyset \neq G \subseteq F$ are faces of P , it can be shown that S_F is obtained from S_G by inverting a single element. Thus $\text{Spec } \tilde{R}[S_F]$ is a principal open subset of $\text{Spec } \tilde{R}[S_G]$. By gluing the affine schemes $U_F := \text{Spec } \tilde{R}[S_F]$ for all non-empty faces of P we obtain an R -scheme X_P , called the toric variety associated to P .

It can be shown that X_P is projective. For $R = \mathbb{C}$ this follows from [4, p. 72]. However, the result remains true for arbitrary commutative rings R . First of all, instead of P we may consider the dilated polytope $P_D := nP$ without changing the toric variety (note that the barrier cones of corresponding faces of P and P_D are the same). Next, the polytope P_D defines a Cartier divisor, hence a line bundle, on X_P as explained in [4, page 72] and [2, §11.12]; the construction works over any ring, and in fact the resulting line bundle can explicitly be described as the linearisation of the objects $\mathcal{O}_P(n)$ to be introduced in Definition 2.3.4 below. Finally, this line bundle determines a map from X_P to some projective space which can be shown to be an embedding using Proposition II.7.2 of [6]. It remains to see that the hypotheses of that Proposition are verified, the main point being the surjectivity of ring homomorphisms from certain polynomial rings to rings of the form $\tilde{R}[S_v]$ for v a vertex of P_D . For this it is enough to verify that for each v the monoid S_v is generated by the set of difference vectors $\{p - v \mid p \in P_D \cap \mathbb{Z}^n\}$. But this is true since P_D is the n th dilation of an n -dimensional polytope, cf. Lemma VII.3.8 of [3].

A quasi-coherent sheaf \mathcal{F} of \mathcal{O}_{X_P} -modules gives rise, by evaluation on the open affine sets U_F , to a collection of modules $\mathcal{F}(U_F)$ over the various rings $\tilde{R}[S_F]$, together with “restriction maps”. Moreover this data completely determines the sheaf \mathcal{F} . So we can define a quasi-coherent sheaf as a functor with values in R -modules

$$M : F(P)_0 \rightarrow R\text{-Mod}, \quad F \mapsto M^F$$

(where $F(P)_0$ is the poset of non-empty faces of P) together with the following data:

- (1) For each $F \in F(P)_0$, the module M^F is equipped with the structure of a $\tilde{R}[S_F]$ -module;
- (2) For each inclusion $G \subseteq F$ in $F(P)_0$, the associated map $M^G \rightarrow M^F$ is $\tilde{R}[S_G]$ -linear;
- (3) The adjoint $M^G \otimes_{\tilde{R}[S_G]} \tilde{R}[S_F] \rightarrow M^F$ of the map above is an isomorphism of $\tilde{R}[S_F]$ -modules.

2.3 Non-linear sheaves

2.3.1. Definition. A *non-linear sheaf* on X_P is a functor

$$Y : F(P)_0 \rightarrow \mathbf{Top}_*, \quad F \mapsto Y^F$$

together with the following data:

- (1) For each $F \in F(P)_0$, the space Y^F is equipped with a base point preserving (right) action of the pointed monoid S_F ;
- (2) For each inclusion $G \subseteq F$ in $F(P)_0$, the associated map $Y^G \rightarrow Y^F$ is S_G -equivariant;
- (3) Using the notation of (2), let $Y_c^G \xrightarrow{g} Y^G$ be a cofibrant replacement of Y^G , i.e., $Y_c^G \in \mathbf{C}(S_G)$ and g is an S_G -equivariant weak homotopy equivalence. Then the map

$$Y_c^G \wedge_{S_G} S_F \rightarrow Y^F,$$

adjoint to the composition $Y_c^G \xrightarrow{\simeq} Y^G \rightarrow Y^F$, is a weak equivalence.

Existence of cofibrant replacements is a direct consequence of the model category structure mentioned in Remark 2.1.1. Using a cofibrant replacement ensures that the “gluing condition” (3) is weakly homotopy invariant. Moreover, standard model category arguments show that we could equivalently have worked with a fixed cofibrant replacement, or we could have asked for the gluing condition to be satisfied for all cofibrant replacements. In particular, we obtain:

2.3.2. Lemma. *Suppose $Y : F(P)_0 \rightarrow \mathbf{Top}_*$ is a diagram satisfying conditions (1) and (2) above. Suppose moreover that Y is locally cofibrant in the sense that $Y^F \in \mathbf{C}(S_F)$ for all $F \in F(P)_0$. Then Y satisfies the gluing condition (3) if and only if for all inclusions $G \subseteq F$ in $F(P)_0$, the map*

$$Y^G \wedge_{S_G} S_F \rightarrow Y^F,$$

adjoint to the structure map $Y^G \rightarrow Y^F$, is a weak equivalence. □

2.3.3. Definition. (1) A non-linear sheaf Y on X_P is called *weakly cofibrant* if for all $F \in F(P)_0$ the component Y^F is cofibrant as a pointed topological space, i.e., $Y^F \in \mathbf{C}$. The category of weakly cofibrant non-linear sheaves on X_P is denoted $\overline{\mathfrak{h}\mathfrak{C}\mathfrak{oh}}(P)$.

(2) A non-linear sheaf Y on X_P is called *locally cofibrant* if $Y^F \in \mathbf{C}(S_F)$ for all $F \in F(P)_0$. The category of locally cofibrant non-linear sheaves on X_P is denoted $\mathfrak{h}\mathfrak{C}\mathfrak{oh}(P)$.

(3) A map of non-linear sheaves is called a *weak equivalence* if all its components are weak homotopy equivalences of spaces.

The notation $\mathfrak{h}\mathfrak{C}\mathfrak{oh}(P)$ is intended to suggest that a non-linear sheaf is a homotopy-theoretic version of a quasi-coherent sheaf. Every locally cofibrant non-linear sheaf is weakly cofibrant.

The most important examples of non-linear sheaves are the “twisting sheaves”, formed by using translates of the monoids S_F .

2.3.4. Definition. For all $k \in \mathbb{Z}$, we define the k th *twisting sheaf*, denoted $\mathcal{O}_P(k)$, as the non-linear sheaf

$$\mathcal{O}_P(k) : F(P)_0 \rightarrow \mathbf{Top}_*, \quad F \mapsto (\mathbb{Z}^n \cap (C_F + kF))_+$$

where C_F is the barrier cone of P at F , and

$$C_F + kF = \{x + kf \mid x \in C_F \text{ and } f \in F\}.$$

Note that $\mathcal{O}_P(0)^F = S_F$ and $\mathcal{O}_P(k)^F \cong S_F$ (not canonically). Moreover, $\mathcal{O}_P(k)$ is a non-linear sheaf by Lemma 2.3.2. See Figure 4 for a picture of $\mathcal{O}_P(1)^F$ and $\mathcal{O}_P(-1)^F$ (the shaded areas) for F a vertex of P .

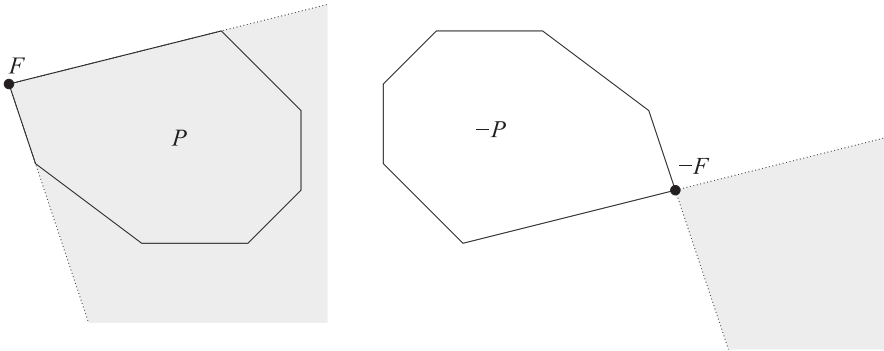


Fig. 4. The construction of $\mathcal{O}_P(k)$

By passage to reduced free modules, we obtain a diagram $F \mapsto \tilde{\mathcal{R}}[\mathcal{O}_P(k)^F]$ which is a quasi-coherent sheaf in the sense of §2.2; this is the algebraic geometers’ k th twisting sheaf on X_P .

2.3.5. Definition. For $Y, Z \in \mathfrak{h}\mathcal{Coh}(P)$ we define their *tensor product* $Y \otimes Z$ by

$$Y \otimes Z : F(P)_0 \rightarrow \mathbf{Top}_*, \quad F \mapsto Y^F \wedge_{S_F} Z^F.$$

Here $Y^F \wedge_{S_F} Z^F$ is the co-equaliser of the two maps $Y^F \wedge S_F \wedge Z^F \rightarrow Y^F \wedge Z^F$ given by the action of S_F on Y^F and Z^F , respectively.

For $j \in \mathbb{Z}$ we define the j th *twist* of Y as $Y(j) := Y \otimes \mathcal{O}_P(j)$. Both $Y \otimes Z$ and $Y(j)$ are objects of $\mathfrak{h}\mathcal{Coh}(P)$ again. The twisting functor $Y \mapsto Y(j)$ will also be denoted $\theta_j : \mathfrak{h}\mathcal{Coh}(P) \rightarrow \mathfrak{h}\mathcal{Coh}(P)$.

Note that the isomorphism $\mathcal{O}_P(k)^F \cong \mathcal{S}_F$ induces a non-canonical isomorphism $Y(j)^F \cong Y^F$. It is easy to check that $\mathcal{O}_P(j) \otimes \mathcal{O}_P(k) \cong \mathcal{O}_P(j+k)$ and thus $Y(j)(k) \cong Y(j+k)$. Moreover, $Y(0) \cong Y$, so twisting defines a self-equivalence of the category $\mathfrak{h}\mathfrak{Cob}(P)$ which maps weak equivalences to weak equivalences.

2.3.6. Definition. Given a space $K \in \mathbf{Top}_*$ we define the diagram $K \wedge \mathcal{O}_P(k)$ by $(K \wedge \mathcal{O}_P(k))^F := K \wedge \mathcal{O}_P(k)^F$. The functor

$$\psi_k : \mathbf{C} \rightarrow \mathfrak{h}\mathfrak{Cob}(P), \quad K \mapsto K \wedge \mathcal{O}_P(k)$$

is called the k th *canonical sheaf* functor.

From the remarks above we have isomorphisms $\theta_j \circ \psi_k(K) \cong \psi_{k+j}(K)$ which are natural in K .

2.4 Total cofibres

Let $P \subset \mathbb{R}^n$ be a polytope. The set $F(P)$ of all faces of P is partially ordered by inclusion and can thus be considered as a category with initial object \emptyset and terminal object P . We define $F(P)^1 = F(P) \setminus \{P\}$.

2.4.1. Definition. Given a functor $Y : F(P) \rightarrow \mathbf{C}$, $F \mapsto Y^F$ we define the *total cofibre* of Y , denoted $\Gamma(Y)$, as the cofibre of the canonical cofibration

$$(*) \quad \text{hocolim } Y|_{F(P)^1} \rightarrow \text{hocolim } Y.$$

2.4.2. Remark. (1) Since $F(P)$ has the terminal object P the space $\text{hocolim } Y$ is weakly homotopy equivalent to Y^P , and the total cofibre of P is weakly equivalent to the homotopy cofibre of the map $\text{hocolim } Y|_{F(P)^1} \rightarrow Y^P$.

(2) The definition of $\Gamma(Y)$ depends on the combinatorial type of P only, not on its actual geometry. If $P = \Delta^{n-1}$ is a simplex, this definition coincides with the usual definition of the total cofibre of an n -cubical diagram as given in [5, Definition 1.4].

A diagram $Y : F(P)_0 \rightarrow \mathbf{C}$, e.g., a weakly cofibrant non-linear sheaf, can be considered as a diagram defined on all of $F(P)$ by setting $Y^\emptyset = *$, so $\Gamma(Y)$ is defined in this case as well.

Iterating the total cofibre construction

2.4.3. Lemma. *Suppose P and Q are polytopes, and suppose Z is a diagram $Z : F(P) \times F(Q) \rightarrow \mathbf{C}$, $(F, G) \mapsto Z_F^G$. There is a natural homeomorphism*

$$\Gamma(F \mapsto \Gamma(Z_F^?)) \cong \Gamma(G \mapsto \Gamma(Z_?^G)).$$

Proof. The proof is encoded into the following diagram:

$$\begin{array}{ccccc}
 \text{hocolim}_{F \in F(P)^1} \text{hocolim}_{G \in F(Q)^1} Z_F^G & \longrightarrow & \text{hocolim}_{F \in F(P)} \text{hocolim}_{G \in F(Q)^1} Z_F^G & \longrightarrow & \text{hocolim}_{G \in F(Q)^1} \Gamma(Z_?^G) \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{hocolim}_{F \in F(P)^1} \text{hocolim}_{G \in F(Q)} Z_F^G & \longrightarrow & \text{hocolim}_{F \in F(P)} \text{hocolim}_{G \in F(Q)} Z_F^G & \longrightarrow & \text{hocolim}_{G \in F(Q)} \Gamma(Z_?^G) \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{hocolim}_{F \in F(P)^1} \Gamma(Z_F^?) & \longrightarrow & \text{hocolim}_{F \in F(P)} \Gamma(Z_F^?) & \longrightarrow & \Gamma(Z_?^?)
 \end{array}$$

All rows and columns are cofibre sequences. For example, the first column is obtained by applying the functor $\text{hocolim}_{F \in F(P)^1}$ to the cofibre sequence defining $\Gamma(Z_F^?)$, and the first row is obtained by applying the functor $\text{hocolim}_{G \in F(Q)^1}$ to the cofibre sequence defining $\Gamma(Z_?^G)$; note that homotopy colimits commute among themselves as well as with taking cofibres. \square

A vanishing theorem for total cofibres

2.4.4. Observation. An $(n + 1)$ -cubical diagram $Y : F(\Delta^n) \rightarrow \mathbf{C}$ can be written as a map $f : Z_1 \rightarrow Z_2$ of n -cubical diagrams: If v is a vertex of Δ^n , then Z_1 is the restriction of Y to the poset of all faces of Δ^n not containing v , and Z_2 is the restriction of Y to the poset of all faces of Δ^n containing v . The components of f are the structure maps $Y^F \rightarrow Y^{F \vee \{v\}}$ of Y for $v \notin F \in F(\Delta^n)$. (For $n = 1$, the diagram Y is a square, and f is the map from the top to the bottom arrow, or the map from the left to the right arrow.) If f consists of weak equivalences, the diagram Y is homotopy cocartesian (see the remarks preceding Definition 1.4 in [5]), and its total cofibre is weakly contractible.

The point is that $\Gamma(Y)$ is homeomorphic to the total cofibre of the n -cubical diagram hoco-fibre f . If f consists of weak equivalences, hoco-fibre f consists of weakly contractible spaces only, so its total cofibre is homotopically trivial.

We will prove the following generalisation of this simple vanishing criterion (an essential ingredient for the proofs of Lemma 2.6.2 and Proposition 2.6.3):

2.4.5. Theorem. *Suppose the functor $Y : F(P) \rightarrow \mathbf{C}$ has the property that for some non-empty face A of P “all structure maps in A -direction are weak equivalences”, i.e., for all $F \in F(P)$ the map $Y^F \rightarrow Y^{F \vee A}$ is a weak equivalence. Then the total cofibre of Y is weakly contractible.*

The total cofibre $\Gamma(Y)$ measures the deviation of Y from being a homotopy colimit diagram. If $Y^P \simeq \text{hocolim}(Y|_{F(P)^1})$, i.e., if the canonical map $(*)$ of Definition 2.4.1

is a (weak) homotopy equivalence, then $\Gamma(Y)$ is (weakly) contractible. Conversely, if $\Gamma(Y)$ is contractible, the canonical map $(*)$ suspends to a weak equivalence (Lemma 2.1.2 and Remark 2.4.2 (1)). In this sense, a vanishing result for total cofibres is nothing but a “computation” of a homotopy colimit, up to suspension.

We begin with some technical preliminaries. To simplify the notation, we define three subcategories of $F(P)^1$:

$$\mathcal{C}_0 := \{\emptyset\} \cup lk(A)$$

$$\mathcal{C}_1 := \{\emptyset\} \cup \overline{st}(A)$$

$$\mathcal{C}_2 := \{\emptyset\} \cup \overline{ast}(A)$$

Links, stars and antistars are computed in the complex $F(P)_0^1$ unless indicated otherwise. Note that $\mathcal{C}_1 \cap \mathcal{C}_2 = \mathcal{C}_0$.

Let $\iota : st(A) \rightarrow \mathcal{C}_1$ denote the inclusion, and define

$$\Phi : \mathcal{C}_1 \rightarrow st(A), \quad F \mapsto F \vee A$$

(this is well defined since $F \vee A \neq P$ by 1.2.1 (2)). Then $\Phi \circ \iota = \text{id}_{st(A)}$, and there is a natural transformation of functors $\theta : \text{id} \rightarrow \iota \circ \Phi$ with F -component given by the inclusion $F \rightarrow F \vee A$.

2.4.6. Lemma. *The inclusion $\iota : st(A) \rightarrow \mathcal{C}_1$ induces a homotopy equivalence $\iota_* : \text{hocolim } Y|_{st(A)} \rightarrow \text{hocolim } Y|_{\mathcal{C}_1}$ with homotopy inverse γ induced by Φ and θ .*

Proof. This follows from [14, Corollary 3.14]. We provide a proof for the reader’s convenience. The map γ induced by Φ and θ factors as

$$\text{hocolim } Y|_{\mathcal{C}_1} \xrightarrow{\theta_*} \text{hocolim}(\Phi^*(Y|_{st(A)})) \xrightarrow{\Phi_*} \text{hocolim } Y|_{st(A)}$$

where the first map is induced by the natural transformation θ , and the second map is induced by Φ . The composition $\Phi_* \circ \theta_* \circ \iota_* = \gamma \circ \iota_*$ is the identity map of $\text{hocolim } Y|_{st(A)}$ since $\Phi \circ \iota = \text{id}$.

We are left to show that $\iota_* \circ \gamma$ is homotopic to the identity map of $\text{hocolim } Y|_{\mathcal{C}_1}$. The natural transformation $\theta : \text{id} \rightarrow \iota \circ \Phi$ can be encoded as a single functor

$$\Upsilon : \mathcal{C}_1 \times [1] \rightarrow \mathcal{C}_1$$

(where $[1] = \{0 \rightarrow 1\}$ is the category with two objects and a single non-trivial morphism) such that $\Upsilon|_{\mathcal{C}_1 \times \{0\}} = \text{id}$ and $\Upsilon|_{\mathcal{C}_1 \times \{1\}} = \iota \circ \Phi$. Now $\text{hocolim } \Upsilon^*(Y|_{\mathcal{C}_1})$ is homeomorphic to the mapping cylinder Z_γ of the map γ , and the functor Υ induces a map

$$Z_\gamma \cong \text{hocolim } \Upsilon^*(Y|_{\mathcal{C}_1}) \rightarrow \text{hocolim } Y|_{\mathcal{C}_1}.$$

Pre-composition with the map $(\text{hocolim } Y|_{\mathcal{C}_1}) \times [0, 1] \rightarrow Z_\gamma$ yields the desired homotopy. \square

2.4.7. Lemma. *Let Ψ denote the composition $\mathcal{C}_0 \rightarrow \mathcal{C}_1 \xrightarrow{\Phi} \text{st}(A)$. Then Ψ induces a homotopy equivalence $\alpha : \text{hocolim } \Psi^*(Y|_{\text{st}(A)}) \rightarrow \text{hocolim } Y|_{\text{st}(A)}$.*

Proof. It suffices to show that Ψ is right cofinal [1, dual of Theorem XI.9.2] [14, Proposition 3.10], i.e., for all elements $B \in \text{st}(A)$ the category $B \downarrow \Psi$ is contractible.

Case 1: $B = A$. Then $B \downarrow \Psi = \mathcal{C}_0$ has the initial object \emptyset , hence has contractible classifying space.

Case 2: $B \supset A$. Then $B \downarrow \Psi = \{F \in \text{lk}(A) \mid B \subseteq F \vee A\}$ by definition of Ψ . We also have the equality

$$B \downarrow \Psi = \{F \in \text{lk}(B) \mid B \subseteq F \vee A\}.$$

Indeed, using 1.2.1 (2), we conclude that for every $F \in B \downarrow \Psi$ we have $B \not\subseteq F$ since F does not contain A , and

$$F \vee B = F \vee A \vee B = F \vee A \neq P$$

(since $B \subseteq F \vee A$ by definition of $B \downarrow \Psi$), thus $F \in \text{lk}(B)$ by Lemma 1.2.1 (2). Conversely, if $F \in \text{lk}(B)$ satisfies $B \subseteq F \vee A$, we have $A \not\subseteq F$ since otherwise $B \subseteq F \vee A = F$ which contradicts $F \in \text{lk}(B)$. Moreover, $F \vee A \subseteq F \vee B \neq P$, and we conclude $F \in B \downarrow \Psi$.

By Corollary 1.2.2 we know

$$\text{lk}(B) \setminus (B \downarrow \Psi) = \{F \in \text{lk}(B) \mid B \not\subseteq F \vee A\} = \overline{\text{st}}_{\text{lk}(B)}(A)$$

and consequently $B \downarrow \Psi = \text{ast}_{\text{lk}(B)}(A)$. Now $|\text{lk}(B)| \cong_{PL} S^{n-2}$ since the boundary of P is PL -homeomorphic to an $(n-1)$ -sphere. Thus $|\overline{\text{st}}_{\text{lk}(B)}(A)|$ is an $(n-2)$ -dimensional ball. We can now apply Corollary 1.1.8 to show that the classifying space of $B \downarrow \Psi$ is contractible. \square

2.4.8. Lemma. *The inclusion $\Xi : \text{ast}(A) \rightarrow \mathcal{C}_2$ induces a homotopy equivalence $\delta : \text{hocolim } \Xi^*(Y|_{\mathcal{C}_2}) \rightarrow \text{hocolim } Y|_{\mathcal{C}_2}$.*

Proof. It suffices to show that Ξ is right cofinal [1, Theorem XI.9.2] [14, Proposition 3.10], i.e., for all elements $G \in \mathcal{C}_2$ the category $G \downarrow \Xi$ is contractible. Fix an object $G \in \mathcal{C}_2$.

Case 1: $G \in \text{ast}(A)$. Then $G \downarrow \Xi$ contains G as an initial object and hence is contractible.

Case 2: $G = \emptyset$. Then $G \downarrow \Xi = \text{ast}(A)$. Its classifying space is contractible by Corollary 1.1.8, applied to $K = F(P)_0^1$ and $C = \text{ast}(A)$.

Case 3: $G \in lk(A)$. Then $G \downarrow \Xi = \{F \in ast(A) \mid G \leq F\} = st(G) \cap ast(A)$; this is an order filter in $F(P)_0^1$. Its complement is $Z := \overline{ast}(G) \cup \overline{st}(A)$. Now

$$\overline{ast}(G) \cap \overline{st}(A) = \overline{st}(A) \setminus st(G) = \overline{st}(A) \setminus st_{\overline{st}(A)}(G).$$

Consequently, we can write

$$Z = \overline{ast}(G) \cup \overline{st}(A) = \overline{ast}(G) \amalg st_{\overline{st}(A)}(G) = \overline{ast}(G) \cup \overline{st}_{\overline{st}(A)}(G)$$

(where the last equality holds since Z is a complex anyway, thus using the closed star instead of the open star does not make any difference). Thus $|Z|$ is the union of the two $(n-1)$ -dimensional balls $|\overline{ast}(G)|$ and $|\overline{st}_{\overline{st}(A)}(G)|$; their intersection is $lk_{\overline{st}(A)}(G)$ which is an $(n-2)$ -dimensional ball since $G \in lk(A)$ (whence $\hat{G} \in \partial|\overline{st}(A)|$). We conclude that $|Z|$ is an $(n-1)$ -dimensional ball [12, Corollary 3.16]. Now Corollary 1.1.8, applied to $K = F(P)_0^1$ and $C = st(G) \cap ast(A)$, shows that $|N(G \downarrow \Xi)| = |N(st(G) \cap ast(A))| \simeq *$ as claimed. \square

Proof of Theorem 2.4.5. Since the categories \mathcal{C}_1 and \mathcal{C}_2 form a convex cover [5, §0] of $F(P)^1$ with intersection \mathcal{C}_0 , Proposition 0.2, *op. cit.*, shows that the following square is homotopy cocartesian:

$$\begin{array}{ccc} \text{hocolim } Y|_{\mathcal{C}_0} & \longrightarrow & \text{hocolim } Y|_{\mathcal{C}_1} \\ \downarrow & & \downarrow \\ \text{hocolim } Y|_{\mathcal{C}_2} & \longrightarrow & \text{hocolim } Y|_{F(P)^1} \end{array}$$

In particular, the space $\Gamma(Y)$ is weakly homotopy equivalent to the total cofibre of the following square (we have used Remark 2.4.2 (1) to replace $\text{hocolim } Y$ by the weakly equivalent space Y^P in the terminal entry):

$$(*) \quad \begin{array}{ccc} \text{hocolim } Y|_{\mathcal{C}_0} & \xrightarrow{f} & \text{hocolim } Y|_{\mathcal{C}_1} \\ \downarrow & & \downarrow \\ \text{hocolim } Y|_{\mathcal{C}_2} & \xrightarrow{g} & Y^P \end{array}$$

We will show that f and g are weak homotopy equivalences. Then their homotopy cofibres are weakly contractible, and since

$$\Gamma(Y) \cong \text{hocofibre}(\text{hocofibre}(f)) \rightarrow \text{hocofibre}(g)$$

this proves the assertion of the theorem.

We can embed the square $(*)$ into the bigger commutative diagram shown in Fig. 5. (Here $Y(P)^{ast(A)}$ denotes the constant diagram on $ast(A)$ with value Y^P .)

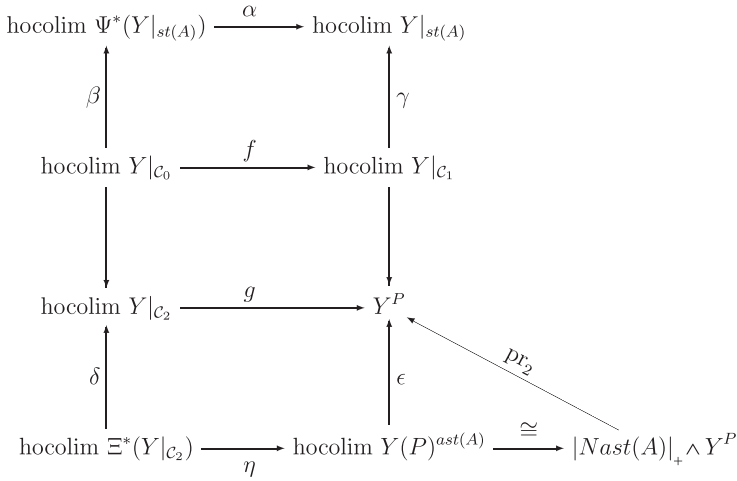


Fig. 5. The diagram used in the proof of Theorem 2.4.5

The map α is induced by Ψ ; it is a weak equivalence by Lemma 2.4.7. Similarly, Φ induces the weak equivalence γ by Lemma 2.4.6. The map β is induced by the natural transformation $Y|_{\mathcal{C}_0} \rightarrow \Psi^*(Y|_{st(A)})$ with F -components given by $Y^F \rightarrow Y^{F \vee A}$. But the latter are weak homotopy equivalences by hypothesis on Y . Hence β is a weak equivalence by the Homotopy Lemma [1, Lemma XII.4.2] [14, Lemma 4.6]. This proves that f is a weak equivalence as well.

Since $|F(P)_0^1| \cong_{PL} S^{n-1}$ application of Corollary 1.1.8 yields $|Nast(A)| \simeq *$. It follows that ε is a weak equivalence. The map δ , induced by Ξ , is a weak equivalence by Lemma 2.4.8. Finally, the map η is induced by the natural transformation $\Xi^*(Y|_{\mathcal{C}_2}) \rightarrow Y(P)^{ast(A)}$ with F -components given by the weak homotopy equivalences $Y^F \rightarrow Y^{F \vee A}$ (recall from 1.2.1 (3) that $F \vee A = P$ for all $F \in ast(A)$). Hence η is a weak equivalence itself by the Homotopy Lemma [1, Lemma XII.4.2] [14, Lemma 4.6]. This proves that g is a weak homotopy equivalence as claimed. \square

2.5 Total cofibres of canonical sheaves

Let $P \subset \mathbb{R}^n$ be a lattice polytope with non-empty interior. For any integer $k \in \mathbb{Z}$ we define $kP := \{kp \mid p \in P\}$.

2.5.1. Theorem [3, §IV.6]. *There is a polynomial $E_P(T) \in \mathbb{Q}[T]$ of degree n with the following properties:*

- (1) *If $k \geq 0$ is an integer, then $E_P(k) = \#(kP \cap \mathbb{Z}^n)$. In particular, $E_P(0) = 1$.*
- (2) *If $k < 0$ is an integer, then $(-1)^n E_P(k) = \#(\text{int}(kP) \cap \mathbb{Z}^n)$.* \square

The polynomial $E_P(T)$ of the theorem is called the *EHRHART polynomial* of P .

For a non-empty proper face F of P let T_F denote the supporting cone of F ; it is the intersection of all supporting half-spaces containing F in their boundary. (Of course it is enough to restrict to facet-defining half-spaces.) By convention $T_P = \mathbb{R}^n$.

Let C_F denote the barrier cone (§2.2) of P at F ; it is the set of linear combinations with non-negative real coefficients spanned by $P - F = \{p - f \mid p \in P, f \in F\}$. Using FARKAS' lemma ([15, §1.4] or [3, Lemma I.3.5]) it can be shown that $F + C_F = T_F$. Moreover, every polytope is the intersection of all its supporting half-spaces, thus $P = \bigcap_{F \in F(P)_0^!} T_F$.

Recall that for $k \in \mathbb{Z}$ the twisting sheaf $\mathcal{O}_P(k)$ is defined as

$$\mathcal{O}_P(k) : F(P)_0 \rightarrow \mathbf{Top}_*, \quad F \mapsto ((kF + C_F) \cap \mathbb{Z}^n)_+,$$

and for $K \in \mathbf{Top}_*$ the k th canonical sheaf is defined as $\psi_k(K) = K \wedge \mathcal{O}_P(k)$. Note that $\mathcal{O}_P(1)^F = (T_F \cap \mathbb{Z}^n)_+$ and $\mathcal{O}_P(0)^F = (C_F \cap \mathbb{Z}^n)_+ = S_F$.

The following theorem generalises [7], Corollaries 3.7.4–5 (the case $P = \Delta^n = \text{conv}\{0, e_1, \dots, e_n\}$).

2.5.2. Theorem. *Suppose $P \subseteq \mathbb{R}^n$ is a lattice polytope with non-empty interior. Let $K \in \mathbf{C}$ be a cofibrant pointed topological space.*

(1) *For every integer $k \geq 0$ there is a natural homotopy equivalence*

$$\Gamma(\psi_k(K)) \simeq (kP \cap \mathbb{Z}^n)_+ \wedge S^n \wedge K = \bigvee_{E_P(k)} \Sigma^n K.$$

(2) *For every integer $k < 0$ there is a natural homotopy equivalence*

$$\Gamma(\psi_k(K)) \simeq ((\text{int } kP) \cap \mathbb{Z}^n)_+ \wedge K = \bigvee_{(-1)^n E_P(k)} K.$$

*In particular, $\Gamma(\psi_k(K)) \simeq *$ if the interior of kP does not contain any lattice point (i.e., if $E_P(k) = 0$).*

Proof. Since homotopy colimits commute with smash products there is a canonical isomorphism $\Gamma(\psi_k(K)) = \Gamma(\psi_k(S^0 \wedge K)) \cong \Gamma(\psi_k(S^0)) \wedge K$. It is thus enough to consider the case $K = S^0$ only. Note that $\psi_k(S^0) \cong \mathcal{O}_P(k)$. The space $\Gamma(\mathcal{O}_P(k))$ is homeomorphic to the homotopy cofibre of the natural map

$$\kappa : \text{hocolim } \mathcal{O}_P(k)|_{F(P)_0^!} \rightarrow (\mathbb{Z}^n)_+$$

which is induced by the inclusions $kF + C_F \subseteq kP + C_P = \mathbb{R}^n$. Define, for fixed $x \in \mathbb{Z}^n$, the functor with values in (unpointed) topological spaces

$$T(k)^x : F(P)_0^! \rightarrow \mathbf{Top}, \quad F \mapsto \{x\} \cap \mathcal{O}_P(k)^F.$$

There is a natural isomorphism of functors $\mathcal{O}_P(k)|_{F(P)_0^!} \cong (\coprod_{x \in \mathbb{Z}^n} T(k)^x)_+$. Consequently, there is a homeomorphism

$$\text{hocolim}(\mathcal{O}_P(k)|_{F(P)_0^1}) \cong (\coprod_{x \in \mathbb{Z}^n} \text{hocolim}' T(k)^x)_+$$

where $\text{hocolim}'$ denotes the *unpointed* homotopy colimit. This homeomorphism induces, for any point $x \in \mathbb{Z}^n$, a homeomorphism $\kappa^{-1}(x) \cong \text{hocolim} T(k)^x$.

To prove (1) it is thus sufficient to show that $\text{hocolim}' T(k)^x \cong S^{n-1}$ for $x \in \mathbb{Z}^n \cap kP$ and $\text{hocolim}' T(k)^x \simeq *$ for $x \notin \mathbb{Z}^n \cap kP$.

Assume $k > 0$ first. It is enough to consider the case $k = 1$ since the functors $\mathcal{O}_P(k)$ and $\mathcal{O}_{kP}(1)$ are isomorphic, and we have an equality $E_P(kT) = E_{kP}(T)$. So assume $k = 1$. Then $\mathcal{O}_P(1)^F = (T_F \cap \mathbb{Z}^n)_+$ is the intersection of the supporting cone of F with \mathbb{Z}^n (plus a disjoint base point). Fix a point $x \in \mathbb{Z}^n$.

If $x \in P$, the functor $T(k)^x$ is the constant functor with a one point space as value, hence $\text{hocolim}' T(k)^x \cong |NF(P)_0^1| \cong S^{n-1}$ by Lemma 1.1.3.

Now assume $x \notin P$. Let F denote a proper non-empty face of P . From Lemma 1.2.4 and the definition of supporting cones we conclude that $x \in T_F$ if and only if F is invisible from x . In particular, $T(k)^x(F) = \{x\}$ if $F \in \text{Inv}(x)$, and $T(k)^x(F) = \emptyset$ if $F \notin \text{Inv}(x)$. By definition of homotopy colimits, this implies

$$\text{hocolim}' T(k)^x \cong |N \text{Inv}(x)|.$$

But by Corollary 1.2.6, this space is contractible.

Now assume $k = 0$. Then $\mathcal{O}_P(0)^F = (C_F \cap \mathbb{Z}^n)_+$ is the intersection of the barrier cone of P at F with \mathbb{Z}^n (plus a disjoint base point). Fix a point $x \in \mathbb{Z}^n$.

If $x = 0$ we see

$$\text{hocolim}' T(0)^0 \cong |NF(P)_0^1| \cong_{pL} S^{n-1}$$

since by their definition all barrier cones contain the origin, *i.e.*, $T(0)^0$ is the constant functor with value $\{0\}$ in this case.

If $x \neq 0$, let $N_F := C_F^\vee$ denote the dual cone of C_F ; it is given by

$$N_F = \{v \in \mathbb{R}^n \mid \forall p \in C_F : \langle p, v \rangle \geq 0\}.$$

It can be shown that N_F is the cone of inward pointing normal vectors of F , and that the dual of N_F , given by

$$N_F^\vee := \{p \in \mathbb{R}^n \mid \forall v \in N_F : \langle v, p \rangle \geq 0\},$$

is the barrier cone C_F [3, §I.4 and §V.2].

Let $U(x)$ denote the poset of all non-empty proper faces F of P satisfying $x \in \mathcal{O}_P(0)^F = (C_F \cap \mathbb{Z}^n)_+$. Then $\text{hocolim}' T(0)^x \cong |NU(x)|$. By the above we have equivalences

$$F \in U(x) \Leftrightarrow x \in C_F = N_F^\vee \Leftrightarrow \forall v \in N_F : \langle -x, v \rangle \leq 0.$$

This means that $U(x) = Up(-x)$ is the set of upper faces of P (with respect to $-x$) in the sense of Definition 1.2.11. By Corollary 1.2.14 we can conclude $|NU(x)| \simeq *$ as required.

To prove (2) it is enough to show that $\text{hocolim}' T(k)^x = \emptyset$ for $x \in \mathbb{Z}^n \cap \text{int} kP$ and $\text{hocolim}' T(k)^x \simeq *$ otherwise. Since $\mathcal{O}_P(k)$ is the same functor as $\mathcal{O}_{-kP}(-1)$, it suffices to consider $k = -1$.

So assume $k = -1$. Fix a point $x \in \mathbb{Z}^n$ and a face $F \in F(P)_0^1$. Then $x \notin T(-1)^x(F)$ if and only if there is a facet $G \supseteq F$ of P such that x and $\text{int}(-P)$ are on the same side of the affine hyperplane spanned by $-G$. Such a facet certainly exists if $x \in \text{int}(-P)$. Hence $\text{hocolim}' T(k)^x = \emptyset$ in this case.

If x is not in the interior of $-P$, Lemma 1.2.8, applied to the polytope $-P$, shows that $x \in T(-1)^x(F)$ if and only if $-F$ is a front face of $-P$ in the sense of Definition 1.2.7. It follows from Corollary 1.2.10 that $\text{hocolim}' T(k)^x \simeq |N \text{Front}(x)| \simeq *$. \square

Appendix: Cohomology of X_P

The techniques from the computation of the space $\Gamma(\mathcal{O}_P(k))$ are applicable in the context of algebraic geometry: They can be used to give a complete calculation of the cohomology groups of X_P with coefficients in a twist of the structure sheaf. Let R be a commutative ring, and let $\mathcal{F}(k)$ denote the twisting sheaf $F \mapsto \tilde{R}[\mathcal{O}_P(k)] = R[(C_F + kF) \cap \mathbb{Z}^n]$. After choosing orientations for the faces of P , we can define a cochain complex C^\bullet of R -modules by setting $C^j := \bigoplus_{\dim F=j} \mathcal{F}(k)^F$. The coboundary map is induced by

$$\mathcal{F}(k)^F = R[(C_F + kF) \cap \mathbb{Z}^n] \xrightarrow{[F:G]} R[(C_G + kG) \cap \mathbb{Z}^n] = \mathcal{F}(k)^G$$

(for faces F, G of P with $\dim G = 1 + \dim F$) where $[F:G]$ is the incidence number of F and G . The cohomology groups of C^\bullet are the cohomology groups of X_P with coefficients in $\mathcal{F}(k)$ [8, §2]: $H^r(X_P; \mathcal{F}(k)) \cong h^r(C^\bullet)$.

Now all the terms in C^\bullet carry a natural \mathbb{Z}^n -grading, and the coboundary maps are homogeneous of degree 0. Consequently, C^\bullet splits into a direct sum of chain complexes $C^\bullet = \bigoplus_{x \in \mathbb{Z}^n} C_x^\bullet$, and $h^k(C^\bullet) = \bigoplus_{x \in \mathbb{Z}^n} h^k(C_x^\bullet)$. The cochain complex C_x^\bullet is given by

$$C_x^j = \bigoplus_{\substack{\dim F=j \\ x \in C_F + kF}} R$$

with coboundary maps given by incidence numbers as before.

Let D^\bullet be the cochain complex defined by $D^j = \bigoplus_{\dim F=j} R$ with coboundaries given by incidence numbers. Then D^\bullet is the cochain complex computing the (cellular) cohomology $H^*(P; R)$ of the polytope P . Hence $h^0(D^\bullet) = R$, and $h^j(D^\bullet) = 0$ for $j \neq 0$. Note that there is an inclusion map $C_x^\bullet \rightarrow D^\bullet$.

Now consider the case $k < 0$. If $x \in \text{int}(kP)$, then the proof of Theorem 2.5.2 shows that $x \notin C_F + kF$ for all proper faces F of P , so $C_x^n = R$ and $C_x^j = 0$ for $j \neq n$. Conse-

quently, the only non-vanishing cohomology group of C_x^\bullet is $h^n(C_x^\bullet) = R$.—If however $x \notin \text{int}(kP)$, then the proof of Theorem 2.5.2 shows that $x \in C_F + kF$ if and only if either $F = P$ or F is a front face of P with respect to x . We can thus identify the quotient cochain complex D^\bullet/C_x^\bullet with the cochain complex computing the (cellular) cohomology of the space $|\text{Back}(x)| \cong B^{n-1}$, cf. Lemma 1.2.9. By the long exact sequence of cohomology groups associated to the short exact sequence of cochain complexes

$$0 \rightarrow C_x^\bullet \rightarrow D^\bullet \rightarrow D^\bullet/C_x^\bullet \rightarrow 0$$

we infer that all cohomology groups of C_x^\bullet are trivial.

Similar arguments apply to the cases $k = 0$ and $k > 0$. By summation over all $x \in \mathbb{Z}^n$, we obtain:

2.5.3. Theorem. *Let $P \subseteq \mathbb{R}^n$ be a lattice polytope with non-empty interior, let R be a commutative ring, and denote by $\mathcal{F}(k)$ the quasi-coherent sheaf on X_P determined by kP . Let $k \in \mathbb{Z}$ and $r \in \mathbb{N}$.*

(1) *If $k \geq 0$, then $H^r(X_P; \mathcal{F}(k)) = 0$ for $r \neq 0$, and there is an isomorphism $H^0(X_P; \mathcal{F}(k)) \cong R[kP \cap \mathbb{Z}^n]$. In particular $H^0(X_P; \mathcal{F}(0)) \cong R$.*

(2) *If $k < 0$, then $H^r(X_P; \mathcal{F}(k)) = 0$ for $r \neq n$, and there is an isomorphism $H^n(X_P; \mathcal{F}(k)) = R[\text{int}(kP) \cap \mathbb{Z}^n]$. In particular $H^n(X_P; \mathcal{F}(k)) = 0$ if $E_P(k) = 0$.*

Thus the total cohomology $H^(X_P; \mathcal{F}(k))$ is a free R -module of rank $|E_P(k)|$. \square*

2.5.4. Remark. As mentioned in the introduction of the paper, the interesting feature of this calculation is that it avoids the use of SERRE duality [2, §7.7] in favour of a topological argument. The reader might be interested in having a reference for the algebro-geometric version as well. For R a field, it follows from classical results in toric geometry [2, Corollary 7.3] that $H^r(X_P; \mathcal{F}(k)) = 0$ for $r > 0$ and $k \geq 0$, and that $H^0(X_P; \mathcal{F}(k))$ has a canonical vector space basis given by the set $P \cap \mathbb{Z}^n$, cf. [2, 11.12]. SERRE duality implies that for negative k we have $H^r(X_P; \mathcal{F}(k)) = 0$ if $r \neq n = \dim(P)$. Replacing X_P by a non-singular variety and invoking SERRE duality again, the argument given in [2, §11.12.4] provides an alternative proof of the above theorem.

2.6 Computing $\psi_0 \circ \Gamma$

We have calculated the composition $\Gamma \circ \psi_k$ in Theorem 2.5.2 above. For the splitting result in K -theory we also need to examine the composition $\psi_0 \circ \Gamma$: It is connected by a chain of natural transformations to the functor Σ^n . We begin by constructing two models ε and σ for the suspension functor on the category of non-linear sheaves; the functor ε is naturally isomorphic to Σ^n , and $\Sigma\sigma$ is naturally weakly equivalent to $\Sigma\varepsilon$.

2.6.1. Construction. Fix $Y \in \mathfrak{h}\mathcal{C}\text{ob}(P)$. For a non-empty face A of P , let $\text{spr}_A Y$ denote the diagram

$$\text{spr}_A Y : F(P)_0 \rightarrow \mathbf{Top}_*, \quad F \mapsto Y^{A \vee F}.$$

(In the language of algebraic geometry, $\text{spr}_A Y$ describes the sheaf $f_*(Y|_{U_A})$ where $f : U_A = \text{Spec } \mathbf{R}[S_A] \subseteq X_P$ is the inclusion of an open affine. We retain the notation spr from [7, §3.8] where $\text{spr}_A Y$ is called a ‘‘spread sheaf’’.) Since A is a face of $A \vee F$ there is an inclusion of monoids $S_A \subseteq S_{A \vee F}$. Consequently, all spaces in the diagram $\text{spr}_A Y$ have an S_A -action, so $\Gamma(\text{spr}_A Y)$ has an S_A -action. This construction is natural in A : If B is a face of P containing A , the structure maps of Y define a natural transformation $\text{spr}_A Y \rightarrow \text{spr}_B Y$ with S_A -equivariant components. Consequently, application of Γ yields a diagram σY , defined as

$$\sigma Y : F(P)_0 \rightarrow \mathbf{Top}_*, \quad A \mapsto \Gamma(\text{spr}_A Y).$$

For a space $K \in \mathbf{Top}_*$ we define the constant diagram

$$\text{con } K : F(P)_0 \rightarrow \mathbf{Top}_*, \quad F \mapsto K.$$

Given $A \in F(P)_0$, the structure maps of Y define a natural transformation $\text{con } Y^A \rightarrow \text{spr}_A Y$. By naturality in A we obtain the diagram

$$\varepsilon Y : F(P)_0 \rightarrow \mathbf{Top}_*, \quad F \mapsto \Gamma(\text{con } Y^F)$$

and a map of diagrams $\varepsilon Y \rightarrow \sigma Y$.

2.6.2. Lemma. (1) *The diagram εY is naturally isomorphic to $\Sigma^n Y$. In particular, $\varepsilon Y \in \mathfrak{h}\mathfrak{Coh}(P)$.*

(2) *The components of the map $\varepsilon Y \rightarrow \sigma Y$ have contractible homotopy cofibres. In particular, the diagram $\Sigma(\sigma Y)$ is weakly equivalent to $\Sigma^{n+1} Y$ and thus is a non-linear sheaf.*

(3) *The functor σ defines a functor $\Sigma\sigma : \mathfrak{h}\mathfrak{Coh}(P) \rightarrow \overline{\mathfrak{h}\mathfrak{Coh}}(P)$.*

Proof. (1) For any space $K \in \mathbf{C}$ there are natural isomorphisms $\Gamma \text{con } K \cong \Gamma \text{con}(S^0 \wedge K) \cong (\Gamma \text{con } S^0) \wedge K$. By definition of the total cofibre, the space $\Gamma \text{con } S^0$ is the homotopy cofibre of the map $NF(P)_0^1 \rightarrow NF(P)_0$. Now $NF(P)_0^1$ is the barycentric subdivision of $\partial P \cong S^{n-1}$, and $NF(P)_0$ is the barycentric subdivision of $P \cong B^n$. Consequently, $\Gamma \text{con } S^0 \cong S^n$, proving the claim.

(2) The A -component of the map $\varepsilon Y \rightarrow \sigma Y$ is given by applying Γ to the natural transformation $\nu : \text{con } Y^A \rightarrow \text{spr}_A Y$. We want to show that the homotopy cofibre of $\Gamma(\nu)$ is contractible. Since Γ commutes with taking homotopy cofibres, it is enough to show that the componentwise homotopy cofibre $Z = \text{hocofibre}(\nu)$ of the natural transformation ν has contractible total cofibre. The diagram $Z : F(P)_0 \rightarrow \mathbf{C}$ is given as follows:

$$Z^F = \text{hocofibre}(Y^A = (\text{con } Y^A)^F \rightarrow (\text{spr}_A Y)^F = Y^{A \vee F})$$

We consider Z as a diagram defined on $F(P)$ by setting $Z^\emptyset = *$, and want to show $\Gamma(Z) \simeq *$.

We claim that all the structure maps $Z^F \rightarrow Z^{A \vee F}$ are homotopy equivalences. In fact, for $F = \emptyset$ the source is a single point and the target is the homotopy cofibre of the identity on Y^A which is contractible. If $F \neq \emptyset$, the definition of Z shows that the structure map is the identity of the homotopy cofibre of $Y^A \rightarrow Y^{A \vee F}$.—By Theorem 2.4.5 this finishes the proof.

(3) This follows from (2) and the fact that all constructions involved in the definition of σ , when applied to locally (or even weakly) cofibrant objects, produce weakly cofibrant objects. Note that in general σY will not be locally cofibrant; this happens, for example, if $Y = \mathcal{O}_P$. \square

Recall that the structure maps of Y define a natural transformation of diagrams $Y \rightarrow \text{spr}_A Y$. The construction $\text{spr}_A Y$ is natural in A , and taking total cofibres gives a map $\Gamma(Y) \rightarrow \lim_{A \in F(P)_0} \Gamma(\text{spr}_A Y)$. The space $\Gamma(\text{spr}_A Y)$ has an S_A -action, and by passage to the adjoint (forcing equivariance), we obtain a natural transformation

$$\tau : \psi_0 \circ \Gamma(Y) \rightarrow \sigma Y$$

(where ψ_0 is the canonical sheaf functor of Definition 2.3.6).

2.6.3. Proposition. *The map of spaces $\Sigma\Gamma(\tau) : \Sigma\Gamma \circ \psi_0 \circ \Gamma(Y) \rightarrow \Sigma\Gamma(\sigma Y)$ is a homotopy equivalence.*

Proof. For this proof, we consider diagrams defined on $F(P)$, with $Y^\emptyset = *$. Let $\text{spr}_\emptyset(Y)$ denote the trivial diagram with value $*$ everywhere. Then $(\text{spr}_F Y)^G = (\text{spr}_G Y)^F$ for all $F, G \in F(P)$. It is also convenient to define the pointed monoid $S_\emptyset = *$. We can now rewrite the map $\Gamma(\tau)$ as follows:

$$\begin{array}{ccc} \Gamma(F \mapsto \Gamma(Y) \wedge S_F) & \xrightarrow{\Gamma(\tau)} & \Gamma(F \mapsto \Gamma(\text{spr}_F Y)) \\ \cong \downarrow & & \downarrow = \\ \Gamma(F \mapsto \Gamma(G \mapsto Y^G \wedge S_F)) & \xrightarrow{\Gamma(\Gamma(f))} & \Gamma(F \mapsto \Gamma(G \mapsto (\text{spr}_F Y)^G)) \\ \cong \downarrow \text{(Lemma 2.4.3)} & & \cong \downarrow \text{(Lemma 2.4.3)} \\ \Gamma(G \mapsto \Gamma(F \mapsto Y^G \wedge S_F)) & \xrightarrow{\Gamma(\Gamma(g))} & \Gamma(G \mapsto \Gamma(F \mapsto (\text{spr}_G Y)^F)) \end{array}$$

Here the map f is induced by the composition

$$Y^G \wedge S_F \longrightarrow Y^{F \vee G} \wedge S_F \xrightarrow{\text{action}} Y^{F \vee G} = (\text{spr}_F Y)^G,$$

and g is the map $\psi_0(Y^G) \rightarrow \text{spr}_G Y$ with F -component (for the non-trivial case $G, F \neq \emptyset$)

$$g^F : (\psi_0(Y^G))^F = Y^G \wedge S_F \longrightarrow Y^{G \vee F} \wedge S_F \xrightarrow{\text{action}} Y^{G \vee F} = (\text{spr}_G Y)^F.$$

We will show that $\Gamma(g)$ suspends to a weak equivalence; then the same is true for $\Gamma(\Gamma(g)) \cong \Gamma(\tau)$ as suspension commutes with taking total cofibres.

We define two diagrams $R, Q : F(P) \rightarrow \mathbf{Top}_*$, depending on $G \in F(P)$, by setting

$$R^F := \begin{cases} Y^G \wedge S^0 & \text{if } F = \emptyset \\ (\psi_0(Y^G))^F & \text{if } F \neq \emptyset \end{cases} \quad \text{and} \quad Q^F := \begin{cases} * & \text{if } G = \emptyset \\ Y^{G \vee F} & \text{if } G \neq \emptyset \end{cases}$$

with structure map of R given by mapping the non-basepoint element of S^0 to $0 \in S_F$ (the neutral element, not the base point). Structure maps in Q are the obvious ones induced by the structure maps of Y . Except for their value at \emptyset the diagrams R and Q coincide with source and target of g , respectively.

Observe that $\Gamma(R)$ and $\Gamma(Q)$ are both contractible. For the latter space this follows from Theorem 2.4.5 since all structure maps in G -direction are identities. For the former it follows from a slight modification of the proof of Theorem 2.5.2. Note that $R = Y^G \wedge \overline{\mathcal{C}}_P(0)$ where $\overline{\mathcal{C}}_P(0)^F = \mathcal{C}_P(0)^F$ for all non-empty faces F , and $\overline{\mathcal{C}}_P(0)^\emptyset = S^0 = \{0\}_+$, the initial pointed monoid. Thus it suffices to show $\Gamma(\overline{\mathcal{C}}_P(0)) \simeq *$. In the notation used in Theorem 2.5.2 (the case $k = 0$ and $x = 0$), this means considering $T(0)^0$ as a functor on $F(P)^1$ with $T(0)^0(\emptyset) = \{0\}$ whose (unpointed) homotopy colimit is $\text{hocolim}' T(0)^0 \cong |NF(P)^1| \simeq *$.

For any space $K \in \mathbf{Top}_*$ let $\delta(K)$ denote the diagram which is trivial (with value $*$) everywhere except that $\delta(K)^\emptyset = K$. We can build a commutative diagram

$$\begin{array}{ccccc} \psi_0(Y^G) & \longrightarrow & R & \longrightarrow & \delta(Y^G \wedge S^0) \cong \delta(Y^G) \\ g \downarrow & & \downarrow & & \downarrow \\ \text{spr}_G Y & \longrightarrow & Q & \longrightarrow & \delta(Y^G) \end{array} \quad \swarrow \delta(\text{id}_{Y^G})$$

where both rows are (componentwise) homotopy cofibre sequences. Indeed, the left horizontal maps are identities everywhere except possibly at \emptyset in which case the source is a single point. So the natural maps from the homotopy cofibres to the diagrams on the right are weak equivalences.

Applying Γ to this diagram then gives a map of two cofibre sequences of topological spaces. By construction, the map on cofibres is the identity. The map on middle terms is a homotopy equivalence since $\Gamma(R) \simeq * \simeq \Gamma(Q)$ as remarked above. By considering the next step in the PUPPE sequence of both rows we obtain a diagram of homotopy cofibre sequences

$$\begin{array}{ccccc} \Gamma(R) & \longrightarrow & \Gamma(\delta(Y^G)) & \longrightarrow & \Sigma\Gamma(\psi_0(Y^G)) \\ \simeq \downarrow & & \text{id} \downarrow & & \downarrow \Sigma(\Gamma(g)) \\ \Gamma(Q) & \longrightarrow & \Gamma(\delta(Y^G)) & \longrightarrow & \Sigma\Gamma(\text{spr}_G Y) \end{array}$$

which proves that $\Sigma\Gamma(g)$ is a weak equivalence. \square

3 Algebraic K -theory of non-linear sheaves

For all of §3, let $P \subset \mathbb{R}^n$ be a lattice polytope with non-empty interior.

3.1 Finiteness conditions

3.1.1. Definition. Let Y be a non-linear sheaf on X_P (Definition 2.3.1).

(1) The object Y is called *locally finite* if $Y^F \in \mathbf{C}_f(S_F)$, cf. §2.1, for all $F \in F(P)_0$. The full subcategory of $\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}(P)$ (Definition 2.3.3) consisting of locally finite non-linear sheaves is denoted $\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}(P)_f$.

(2) The object Y is called *homotopy finite* if it can be connected by a chain of weak equivalences to a locally finite non-linear sheaf. The full subcategory of $\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}(P)$ (Definition 2.3.3) consisting of homotopy finite, locally cofibrant non-linear sheaves is denoted $\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}(P)_{\text{hf}}$. The full subcategory of $\overline{\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}}(P)$ consisting of homotopy finite, weakly cofibrant (2.3.3) non-linear sheaves is denoted $\overline{\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}}(P)_{\text{hf}}$.

3.1.2. Remark. If a non-linear sheaf Y on X_P is homotopy finite then necessarily $Y^F \in \mathbf{C}_{\text{hf}}(S_F)$ for all $F \in F(P)_0$. This latter condition is sufficient as well; in short, one chooses spaces $\tilde{Z}^F \in \mathbf{C}_f(S_F)$ and weak equivalences $\tilde{Z}^f \rightarrow Y^F$ and constructs, by induction on $\dim F$, a weak equivalence $Z \rightarrow Y$ with $Z \in \mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}(P)_f$. The components of Z will be built from the spaces \tilde{Z}^F by iterated mapping cylinder constructions used to strictify homotopy commutative diagrams. For P a simplex a detailed argument is given in [7, Lemma 4.1.2], the general case is similar.

The canonical sheaf functors ψ_k defined in 2.3.6 preserve finiteness and weak equivalences. Hence they restrict to functors $\psi_k : \mathbf{C}_{\text{hf}} \rightarrow \mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}(P)_{\text{hf}}$.

3.1.3. Proposition. *The total cofibre construction restricts to a functor*

$$\Sigma\Gamma : \overline{\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}}(P)_{\text{hf}} \rightarrow \mathbf{C}_{\text{hf}}.$$

Proof. For locally finite non-linear sheaves this is Theorem 3.9 of [8]. Since both suspension and Γ are weakly homotopy invariant, the general case follows. \square

3.2 Algebraic K -theory and reduced K -theory

To define algebraic K -theory we use WALDHAUSEN's \mathcal{S}_\bullet -construction for categories with cofibrations and weak equivalences [13]. We will work with the category $\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}(P)_{\text{hf}}$ of homotopy finite non-linear sheaves. A map $f : Y \rightarrow Z$ of non-linear sheaves is called a *cofibration* if all its components are cofibrations of equivariant spaces. The map f is called an *h -equivalence* if it is a weak equivalence, *i.e.*, if all its components are weak homotopy equivalences of spaces. With respect to these cofibrations and weak equivalences, we define the algebraic K -theory of the non-linear projective toric variety X_P to be the space

$$K^{\text{nl}}(X_P) := \Omega|h\mathcal{S}\bullet\mathfrak{h}\mathfrak{Cob}(P)_{\text{hf}}|.$$

The functor $\Sigma\Gamma : \mathfrak{h}\mathfrak{Cob}(P)_{\text{hf}} \rightarrow \mathbf{C}_{\text{hf}}$ is exact and thus induces a map of K -theory spaces $K^{\text{nl}}(X_P) \rightarrow \Omega|h\mathcal{S}\bullet\mathbf{C}_{\text{hf}}| = A(*)$. Roughly speaking the functor ψ_0 provides a section up to homotopy of this map; consequently we can split off a copy of $A(*)$ from $K^{\text{nl}}(X_P)$.

On a technical level, we use WALDHAUSEN's fibration theorem [13, Theorem 1.6.4]. We call a map $f : Y \rightarrow Z$ of non-linear sheaves an $h_{[0]}$ -equivalence if $\Sigma^2\Gamma(f)$ is a weak homotopy equivalence of spaces. Note that the double suspension of the total cofibre of a non-linear sheaf is a simply connected cofibrant pointed space, so f is an $h_{[0]}$ -equivalence if and only if $\Gamma(f)$ induces an isomorphism of singular homology groups. It follows that the class of $h_{[0]}$ -equivalences satisfy WALDHAUSEN's extension axiom [13, §1.2]. Since moreover every h -equivalence is an $h_{[0]}$ -equivalence, we can apply the fibration theorem to obtain a fibration sequence

$$(\dagger) \quad \Omega|h\mathcal{S}\bullet\mathfrak{h}\mathfrak{Cob}(P)_{\text{hf}}^{[0]}| \xrightarrow{\iota} K^{\text{nl}}(X_P) \rightarrow \Omega|h_{[0]}\mathcal{S}\bullet\mathfrak{h}\mathfrak{Cob}(P)_{\text{hf}}|$$

where $\mathfrak{h}\mathfrak{Cob}(P)_{\text{hf}}^{[0]}$ is the full subcategory of $\mathfrak{h}\mathfrak{Cob}(P)_{\text{hf}}$ consisting of those objects Y satisfying $\Sigma^2\Gamma(Y) \simeq *$ (i.e., the map $Y \rightarrow *$ is an $h_{[0]}$ -equivalence), and the map ι is induced by inclusion.

We need a lemma first. A map $f : Y \rightarrow Z$ of non-linear sheaves is called a *weak cofibration* if all its components are cofibrations of underlying pointed topological spaces.

3.2.1. Lemma. *The inclusion $\mathfrak{h}\mathfrak{Cob}(P)_{\text{hf}} \subseteq \overline{\mathfrak{h}\mathfrak{Cob}}(P)_{\text{hf}}$ induces a weak equivalence*

$$\Omega|h_{[0]}\mathcal{S}\bullet\mathfrak{h}\mathfrak{Cob}(P)_{\text{hf}}| \simeq \Omega|h_{[0]}\mathcal{S}\bullet\overline{\mathfrak{h}\mathfrak{Cob}}(P)_{\text{hf}}|$$

where both K -theory spaces are defined with respect to $h_{[0]}$ -equivalences, and on the right we use weak cofibrations.

Proof. The category of diagrams $F(P)_0 \rightarrow \mathbf{Top}_*$ which satisfy conditions (1) and (2) of Definition 2.3.1 has a QUILLEN closed model structure with cofibrations and weak equivalences (h -equivalences) as used for the category $\mathfrak{h}\mathfrak{Cob}(P)$. This is a straightforward generalisation of [7, Proposition 3.4.4], and can be considered as a special case of a model structure for “twisted” diagrams [10, Theorem 3.3.5]. Consequently, every map $Y \rightarrow Z$ of a locally cofibrant object to a weakly cofibrant object can be factored as a cofibration $Y \rightarrow W$ (making W locally cofibrant) followed by a weak equivalence $W \rightarrow Z$ (making W homotopy finite). Since a weak equivalence is an $h_{[0]}$ -equivalence, we can now apply the Approximation Theorem [13, 1.6.7]. \square

We are now in the position to identify the base of the fibration sequence (\dagger) with $A(*)$.

3.2.2. Lemma. *The functor $\Sigma^2\Gamma : \mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}(P)_{\text{hf}} \rightarrow \mathbf{C}_{\text{hf}}$ induces a homotopy equivalence $\Omega|h_{[0]}\mathcal{S}\bullet\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}(P)_{\text{hf}}| \simeq A(*)$.*

Proof. By Theorem 2.5.2 (1) the composite $\Sigma^2\Gamma \circ \psi_0$ is weakly equivalent to Σ^{n+2} , hence induces a self homotopy equivalence of $A(*)$. Consequently, the map induced by $\Sigma^2\Gamma$ is surjective on homotopy groups.

We want to show that the composition $\psi_0 \circ \Sigma^2\Gamma \cong \Sigma^2\psi_0 \circ \Gamma$ is weakly equivalent, with respect to $h_{[0]}$ -equivalences, to Σ^{n+2} . By Lemma 3.2.1 it is enough to show that this is the case if both functors are considered as endofunctors on $\overline{\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}}(P)_{\text{hf}}$. By Lemma 2.6.2 and Proposition 2.6.3, the functors $\Sigma^2\psi_0 \circ \Gamma$ and Σ^{n+2} are connected by a chain of $h_{[0]}$ -equivalences

$$\psi_0 \circ \Sigma^2\Gamma \cong \Sigma^2\psi_0 \circ \Gamma \rightarrow \Sigma^2\sigma \leftarrow \Sigma^2\varepsilon \cong \Sigma^{n+2},$$

thus induce self homotopy equivalences on the K -theory space $\Omega|h_{[0]}\mathcal{S}\bullet\overline{\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}}(P)_{\text{hf}}|$. In particular, the map induced by $\Sigma^2\Gamma$ is injective on homotopy groups. (Note that the chain of $h_{[0]}$ -equivalences involves the functor σ which takes values in $\overline{\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}}(P)_{\text{hf}}$; this is the reason why weakly cofibrant objects are needed for the argument.)

3.2.3. Definition. The fibre of the fibration sequence (†) is called the *reduced K -theory* of X_P , written $\tilde{K}^{\text{nl}}(X_P)$.

Thus (†) yields a fibration sequence $\tilde{K}^{\text{nl}}(X_P) \xrightarrow{t} K^{\text{nl}}(X_P) \xrightarrow{\Sigma^2\Gamma} A(*)$. Since $\Sigma^2\Gamma \circ \psi_0$ induces a self homotopy equivalence of $A(*)$ by Theorem 2.5.2 (1), ψ_0 provides a section up to homotopy of the fibration sequence and we obtain a homotopy equivalence

$$\tilde{K}^{\text{nl}}(X_P) \times A(*) \xrightarrow{t \vee \psi_0} K^{\text{nl}}(X_P).$$

3.3 Splitting $\tilde{K}^{\text{nl}}(X_P)$

If the polytope P does not contain lattice points in its interior it is possible to split off further copies of $A(*)$ from $\tilde{K}^{\text{nl}}(X_P)$. As in §3.2, this is done by producing suitable fibration sequences with sections.

3.3.1. Definition. For $k \in \mathbb{Z}$ let $[k] := \{0, 1, \dots, k\}$. A map $f : Y \rightarrow Z$ of non-linear sheaves is called an $h_{[k]}$ -equivalence if for all $j \in [k]$ the map $\Sigma^2\Gamma(\theta_j(f))$ is a weak homotopy equivalence. (Here θ_j denotes the twisting functor of Definition 2.3.5.) We denote by $\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}(P)_{\text{hf}}^{[k]}$ the full subcategory of $\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}(P)_{\text{hf}}$ consisting of non-linear sheaves Y for which the map $Y \rightarrow *$ is an $h_{[k]}$ -equivalence. The category $\overline{\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}}(P)_{\text{hf}}^{[k]}$ is defined similarly as a full subcategory of $\overline{\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}}(P)_{\text{hf}}$.

3.3.2. Lemma. *Let $k \geq 1$, and suppose $E_P(-k) = 0$, i.e., suppose that kP does not contain lattice points in its interior. Then ψ_{-k} can be considered as a functor $\psi_{-k} : \mathbf{C}_{\text{hf}} \rightarrow \mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}(P)_{\text{hf}}^{[k-1]}$.*

Proof. Let $K \in \mathbf{C}_{\text{hf}}$. We have to show that for each $l \in [k-1]$ the space $\Sigma^2 \Gamma(\theta_l(\psi_{-k}(K))) \cong \Sigma^2 \Gamma(\psi_{l-k}(K))$ is contractible. Now from Theorem 2.5.1 (2) it is clear that $E_P(-k) = 0$ implies $E_P(l-k) = 0$ since $-k \leq l-k \leq -1$. Thus the claim follows from Theorem 2.5.2 (2). \square

3.3.3. Lemma. *For any $k \geq 1$, the inclusion $\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}(P)_{\text{hf}} \subseteq \overline{\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}}(P)_{\text{hf}}$ induces a weak equivalence*

$$\Omega|h_{[k]}\mathcal{S}\bullet\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}(P)_{\text{hf}}^{[k-1]}| \simeq \Omega|h_{[k]}\mathcal{S}\bullet\overline{\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}}(P)_{\text{hf}}^{[k-1]}|$$

where both K -theory spaces are defined with respect to $h_{[k]}$ -equivalences, and on the right we use weak cofibrations.

Proof. This is similar to the proof of Lemma 3.2.1. \square

3.3.4. Lemma. *Let $k \geq 1$, and suppose $E_P(-k) = 0$. The functor*

$$\Sigma^2 \Gamma \circ \theta_k : \mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}(P)_{\text{hf}}^{[k-1]} \rightarrow \mathbf{C}_{\text{hf}}$$

induces a homotopy equivalence $\Omega|h_{[k]}\mathcal{S}\bullet\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}(P)_{\text{hf}}^{[k-1]}| \simeq A(*)$.

Proof. By Lemma 3.3.2, the functor ψ_{-k} induces a map backwards. By Theorem 2.5.2 (1), the composition $(\Sigma^2 \Gamma \circ \theta_k) \circ \psi_{-k} \cong \Sigma^2 \Gamma \circ \psi_0$ is weakly equivalent to Σ^{n+2} , hence induces a self homotopy equivalence of $A(*)$. Consequently, the map induced by $\Sigma^2 \Gamma \circ \theta_k$ is surjective on homotopy groups.

As in the proof of Lemma 3.2.2 we see that the composition $\psi_0 \circ \Sigma^2 \Gamma \cong \Sigma^2 \psi_0 \circ \Gamma$ is connected to Σ^{n+2} , both considered as an endofunctors of $\overline{\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}}(P)_{\text{hf}}$, by a chain of $h_{[0]}$ -equivalences. Consequently, the conjugate

$$\theta_{-k} \circ (\psi_0 \circ \Sigma^2 \Gamma) \circ \theta_k \cong \psi_{-k} \circ (\Sigma^2 \Gamma \circ \theta_k)$$

is connected to Σ^{n+2} by a chain of $h_{[k]}$ -equivalences (recall that, by definition, any object $Y \in \overline{\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}}(P)_{\text{hf}}^{[k-1]}$ has the property that $\Sigma^2 \Gamma(Y(l)) \cong \Sigma^2 \Gamma \circ \theta_l(Y) \simeq *$ for all $l \in [k-1]$). Since the inclusion $\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}(P)_{\text{hf}} \subseteq \overline{\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}}(P)_{\text{hf}}$ induces an equivalence on K -theory spaces (Lemma 3.3.3), it follows that $\psi_{-k} \circ (\Sigma^2 \Gamma \circ \theta_k)$ induces a self homotopy equivalence on the K -theory space $\Omega|h_{[k]}\mathcal{S}\bullet\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}(P)_{\text{hf}}^{[k-1]}|$. In particular, the map induced by $\Sigma^2 \Gamma \circ \theta_k$ is injective on homotopy groups. \square

3.3.5. Lemma. *Let $k \geq 1$, and suppose $E_P(-k) = 0$. The functor ψ_{-k} induces a homotopy equivalence $\Omega|h_{\mathcal{S}}\bullet\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}(P)_{\text{hf}}^{[k]}| \times A(*) \xrightarrow{\iota \nu \psi_{-k}} \Omega|h_{\mathcal{S}}\bullet\overline{\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}}(P)_{\text{hf}}^{[k-1]}|$. Here ι denotes the inclusion $\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}(P)_{\text{hf}}^{[k]} \rightarrow \overline{\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}}(P)_{\text{hf}}^{[k-1]}$.*

Proof. By the Fibration Theorem [13, Theorem 1.6.4] there is a fibration sequence

$$\Omega|h_{\mathcal{S}}\bullet\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}(P)_{\text{hf}}^{[k]}| \rightarrow \Omega|h_{\mathcal{S}}\bullet\overline{\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}}(P)_{\text{hf}}^{[k-1]}| \rightarrow \Omega|h_{[k]}\mathcal{S}\bullet\mathfrak{h}\mathfrak{C}\mathfrak{o}\mathfrak{h}(P)_{\text{hf}}^{[k-1]}|.$$

By Lemma 3.3.4 the base space is homotopy equivalent to $A(*)$, so we have a fibration sequence

$$\Omega|h\mathcal{L}\bullet\mathfrak{h}\mathfrak{C}\mathfrak{O}\mathfrak{h}(P)_{\text{hf}}^{[k]}| \longrightarrow \Omega|h\mathcal{L}\bullet\mathfrak{h}\mathfrak{C}\mathfrak{O}\mathfrak{h}(P)_{\text{hf}}^{[k-1]}| \xrightarrow{\Sigma^2\Gamma\circ\theta_k} A(*)$$

The functor ψ_{-k} induces a section up to homotopy by Theorem 2.5.2 (1) since $\Gamma \circ \theta_k \circ \psi_{-k} \cong \Gamma \circ \psi_0$. □

Note that for $k = 1$ the target of this homotopy equivalence is nothing but $\tilde{K}^{\text{nl}}(X_P)$. From these homotopy equivalences for $k = 1, 2, \dots$ we obtain:

3.3.6. Theorem. *Suppose $P \subset \mathbb{R}^n$ is a lattice polytope with non-empty interior. Let k denote the number of integral roots of the EHRHART polynomial $E_P(T)$, and let ι denote the inclusion functor $\mathfrak{h}\mathfrak{C}\mathfrak{O}\mathfrak{h}(P)_{\text{hf}}^{[k]} \rightarrow \mathfrak{h}\mathfrak{C}\mathfrak{O}\mathfrak{h}(P)_{\text{hf}}$. The functor*

$$\mathfrak{h}\mathfrak{C}\mathfrak{O}\mathfrak{h}(P)_{\text{hf}}^{[k]} \times \underbrace{\mathbf{C}_{\text{hf}} \times \dots \times \mathbf{C}_{\text{hf}}}_{(k+1) \text{ factors}} \xrightarrow{\iota \vee \psi_0 \vee \psi_{-1} \vee \dots \vee \psi_{-k}} \mathfrak{h}\mathfrak{C}\mathfrak{O}\mathfrak{h}(P)_{\text{hf}}$$

induces a homotopy equivalence

$$\Omega|h\mathcal{L}\bullet\mathfrak{h}\mathfrak{C}\mathfrak{O}\mathfrak{h}(P)_{\text{hf}}^{[k]}| \times \underbrace{A(*) \times \dots \times A(*)}_{(k+1) \text{ factors}} \xrightarrow{\sim} K^{\text{nl}}(X_P). \quad \square$$

In the case $k = 0$ this reduces to the splitting $\tilde{K}^{\text{nl}}(X_P) \times A(*) \simeq K^{\text{nl}}(X_P)$. Note that the obstruction for splitting off another copy of $A(*)$ is the failure of Lemma 3.3.2: The functor ψ_{-k-1} will not factorise through the category $\mathfrak{h}\mathfrak{C}\mathfrak{O}\mathfrak{h}(P)_{\text{hf}}^{[k]}$. In geometric terms, this happens since the polytope $(k+1)P$ contains lattice points in its interior (but kP does not). In the language of algebraic geometry, this means $H^*(X_P; \mathcal{F}(-k-1)) \neq 0$ while $H^*(X_P; \mathcal{F}(-k)) = 0$, using the notation of the Appendix to §2.5.

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