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Geometry in preduals of spaces of 2-homogeneous polynomials on Hilbert spaces

Domingo García · Bogdan C. Grecu · Manuel Maestre

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Abstract Let H be a (real or complex) Hilbert space. Using spectral theory and properties of the Schatten–Von Neumann operators, we prove that every symmetric tensor of unit norm in $H\hat{\otimes}_{s,\pi_s}H$ is an infinite absolute convex combination of points of the form $x \otimes x$ with x in the unit sphere of the Hilbert space. We use this to obtain explicit characterizations of the smooth points of the unit ball of $H\hat{\otimes}_{s,\pi_s}H$.

Keywords Symmetric tensor · Homogeneous polynomial · Smooth point

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D. García (✉) · B. C. Grecu · M. Maestre
Departamento de Análisis Matemático, Universidad de Valencia,
Doctor Moliner 50, 46100 Burjasot (Valencia), Spain
e-mail: domingo.garcia@uv.es

M. Maestre
e-mail: manuel.maestre@uv.es

B. C. Grecu
Department of Pure Mathematics, Queens University Belfast,
Belfast BT7 1NN, UK
e-mail: bogdan.grecu@uv.es; b.grecu@qub.ac.uk

1 Introduction and preliminary results

The existence of a predual of a space of nonlinear functions allows to "linearize" these functions. This linearization process has proved quite often to be a very useful tool in functional analysis (e.g., see [5,8,17]). Grothendieck [15] obtained the following important isometric preduals of spaces of multilinear forms. Given n (real or complex) Banach spaces $X_1, \ldots, X_n$ we denote by $X_1 \otimes \cdots \otimes X_n$ their tensor product and by π the projective norm. If $X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n$ is the completion of $X_1 \otimes \cdots \otimes X_n$ under the projective norm, then we have $(X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n)^* = \mathcal{L}(^n X_1, \ldots, X_n)$, the space of continuous n-linear forms on $X_1 \times \cdots \times X_n$ endowed with the supremum norm. However, as remarked in [7], most of the times the multilinear theory is far from being just a simple translation of the linear one, either when it comes to algebraic or analytical properties (see, for instance [1,14]).

Let X be a Banach space over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We denote by $\mathcal{P}(^n X)$ the space of all scalar valued continuous n-homogeneous polynomials on X endowed with the natural supremum norm. For every n-homogeneous polynomial there exists a unique symmetric n-linear form A such that $P(x) = A(x, \ldots, x)$. The general polarization formula gives $\|P\| \leq \|A\| \leq \frac{n^n}{n!}\|P\|$. There are cases in which the constant $n^n/n!$ can be improved. For instance, for Hilbert spaces we have $\|P\| = \|A\|$ (see Propositions 1.8, p. 10, and 1.44, p. 52 in [9]).

Let $\otimes_{n,s} X$ be the n-fold symmetric tensor product of X, that is the subspace of $X \otimes \cdots \otimes X$ generated by the diagonal tensors $x \otimes \cdots \otimes x$. We endow it with the topology inherited from $X \otimes_\pi \cdots \otimes_\pi X$ and denote its completion by $\hat{\otimes}_{n,s,\pi} X$. Note that the dual of $\hat{\otimes}_{n,s,\pi} X$ is isometrically the space of symmetric n-linear forms on X, endowed with the supremum norm, $\mathcal{L}_s(^n X)$. Ryan showed in [19] that the space $\otimes_{n,s,\pi} X$ can be renormed such that $\mathcal{P}(^n X)$ becomes its isometric dual. Indeed, every element u of $\otimes_{n,s} X$ can be expressed as a finite sum $\sum_{j=1}^k \lambda_j x_j \otimes \cdots \otimes x_j$. Define the *symmetric projective norm* of u by

$$||u||_{\pi_s} = \inf \left\{ \sum_{j=1}^k |\lambda_j| \, ||x_j||^n : u = \sum_{j=1}^k \lambda_j x_j \otimes \cdots \otimes x_j \right\}.$$

We endow $\otimes_{n,s} X$ with this norm and we denote its completion by $\hat{\otimes}_{n,s,\pi_s} X$. Then, if we put $\langle u, P \rangle = \sum_{j=1}^k \lambda_j P(x_j)$ then we have $(\hat{\otimes}_{n,s,\pi_s} X)^* = \mathcal{P}(^n X)$.

In general the projective norm does not respect subspaces, but there exist cases (for instance, for Hilbert spaces) in which $Y_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi Y_n$ is an isometric subspace of $X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n$ whenever Y_i are subspaces of X_i $(i = 1, \ldots, n)$ and the same is true for the symmetric projective norm. See [9,20] for more details on homogeneous polynomials and tensor products.

In [2] the notion of n-smoothness was introduced. A point x_0 in the unit sphere of a Banach space X is called a smooth point of order n (or n-smooth for short) and we write $x_0 \in sm^{(n)}(X)$ if there is a unique n-homogeneous polynomial P on X such that $||P|| = 1 = P(x_0)$. Because of the duality between $\mathcal{P}(^n X)$ and $\hat{\otimes}_{n,s,\pi_s} X$, for a norm one $x_0 \in X$ the n-smoothness is equivalent to $x_0 \otimes \cdots \otimes x_0$ being a smooth point in the unit sphere of $\hat{\otimes}_{n,s,\pi_s} X$.

In this paper we give explicit characterizations of the smooth points in the unit ball of $H\hat{\otimes}_{s,\pi_s}H$, with H a Hilbert space. A key tool for achieving this (in Theorem 3.3) is the fact that every unit tensor in $H\hat{\otimes}_{s,\pi_s}H$ can be written as an infinite convex combination of elementary tensors. Why do we concentrate on the Hilbert space case? The answer is that, in general, finding smooth points in spaces of tensors is made difficult by the fact that tensors do not have unique representations as linear combinations of elementary tensors. However, since in general the linear functional which exposes a smooth point is an extreme point in the unit sphere of the dual, it is reasonable to start exploring the smooth points in $\hat{\otimes}_{n,s,\pi_s}X$ for those Banach spaces X for which the extreme points of $B_{\mathcal{P}(^nX)}$ are known. For instance, this has been done for the two-dimensional ℓ_p's with $1 < p < \infty$ and $n = 2$ [12]. For infinite dimensional spaces, results concerning extreme points of $B_{\mathcal{P}(^nX)}$ are sparse. Nevertheless, a characterization exists for 2 -homogeneous polynomials on (real or complex) Hilbert spaces [13]. In the sequel, the inner product in a Hilbert space will be denoted by $(\cdot,\cdot)$.

Proposition 1.1 [13]

(i) *Let H be a real Hilbert space. A 2-homogeneous polynomial $P\colon H \to \mathbb{R}$ is an extreme point of $B_{\mathcal{P}(^2H)}$ if and only if there exists an orthogonal decomposition of $H = H_1 \oplus H_2$ such that $P(x) = \|\pi_1 x\|^2 - \|\pi_2 x\|^2$, where π_1 and π_2 are the orthogonal projections on H_1 and H_2, respectively.*

(ii) *Let H be a complex Hilbert space. A 2-homogeneous polynomial $P\colon H \to \mathbb{C}$ is an extreme point of $B_{\mathcal{P}(^2H)}$ if and only if there exists an orthonormal basis $\{e_\alpha\}_\alpha$ of H such that $P(x) = \sum_\alpha (x, e_\alpha)^2$.*

We will also use a description of the extreme points of the unit ball of $H\hat{\otimes}_{s,\pi_s}H$, both for real and complex H. The approximation property and the Radon–Nikodým property for H yield an isometric identification of $H\hat{\otimes}_{s,\pi_s}H$ with the space of nuclear 2-homogeneous polynomials on H, $\mathcal{P}_N(^2H)$ which, in turn, is the same as the space of integral 2-homogeneous polynomials on H, $\mathcal{P}_I(^2H)$. Using this, Boyd and Ryan [6] and Dineen [10] obtained the following result.

Proposition 1.2 *The extreme points of the unit ball of $H\hat{\otimes}_{s,\pi}H$ are the tensors $\pm x\otimes x$ with x in the unit ball of H.*

2 Symmetric tensors of unit norm as infinite convex combinations of elementary tensors

In general, for every symmetric tensor u of unit norm in $\hat{\otimes}_{n,s,\pi_s}X$ and for any $\eta > 0$, there exists a sequence (λ_i) of scalars with $\sum_{i=1}^{\infty}|\lambda_i| < 1+\eta$ and a sequence (x_i) in the unit sphere of X such that $u = \sum_{i=1}^{\infty}\lambda_i x_i\otimes\cdots\otimes x_i$. For the study of geometrical properties like norm attainment and smoothness, we would like $\sum_{i=1}^{\infty}|\lambda_i| = 1 = ||u||_{\pi_s}$. This does not happen in general. Indeed, such a fact would imply that if P is an n-homogeneous polynomial on X and $\langle u, P\rangle = 1$ then $|P(x_i)| = 1$ whenever $\lambda_i \neq 0$ and so, each time P attains its norm as a linear functional on $\hat{\otimes}_{n,s,\pi_s}X$, it would also attain its norm as a polynomial on X. Then, by the linear Bishop–Phelps Theorem applied to the space $\hat{\otimes}_{n,s,\pi_s}X$ and its dual, $\mathcal{P}(^nX)$, this would imply that the

n-homogeneous polynomials that attain their norm on the unit ball of X are always a dense set in $\mathcal{P}(^n X)$, a fact which is not true [16].

We restrict our attention to $H\hat{\otimes}_{s,\pi_s}H$, where H is a (real or complex) Hilbert space. When H is finite dimensional, by the Caratheodory and Krein–Milman Theorems and by the description of extreme points given in Proposition 1.2, it follows that every unit tensor in $H\hat{\otimes}_{s,\pi_s}H$ can be written as a finite sum $u = \sum \lambda_i x_i \otimes x_i$ with x_i unit vectors and $\sum |\lambda_i| = 1$. When H is complex, we can even take $0 \leq \lambda_i \leq 1$. We will prove, in Theorem 2.2, for the real case, and Theorem 2.4, for the complex case, that if we allow the sum to become infinite then the result is also true for infinite dimensional spaces. In order to obtain it we have to use strong tools as the Schmidt representation for operators in the Schatten–von Neumann classes.

We will work first with *real Hilbert spaces*. The following result holds for both finite and infinite dimensional Hilbert spaces.

Proposition 2.1 *If $u = \sum_{i=1}^{\infty} \lambda_i x_i \otimes x_i$ with x_i unit vectors and $\sum_{i=1}^{\infty} |\lambda_i| = 1$ then $||u||_{\pi_s} = 1$ if and only if any two elements x_i and x_j for which $\lambda_i > 0$ and $\lambda_j < 0$ are orthogonal.*

Proof The easy part is the sufficiency. Indeed, if we put $A = \{i : \lambda_i > 0\}$ and $B = \{j : \lambda_j < 0\}$ then for $H_1 = \overline{span}\{x_i : i \in A\}$ and $H_2 = \overline{span}\{x_j : j \in B\}$ and $P(x) = ||\pi_1 x||^2 - ||\pi_2 x||^2$ we have $\langle u, P\rangle = 1 = ||P||$ and so $||u||_{\pi_s} \geq 1$.

Conversely, it is enough to prove the statement for an absolute convex combination of two elementary tensors. Indeed, $||u||_{\pi_s} = 1$ forces $\|\lambda_i x_i \otimes x_i + \lambda_j x_j \otimes x_j\|_{\pi_s} = |\lambda_i| + |\lambda_j|$ whenever $i \neq j$. Thus, we need to show that if the tensor $u = \lambda_1 x_1 \otimes x_1 + \lambda_2 x_2 \otimes x_2$ with $\lambda_1 > 0$ and $\lambda_2 < 0$ has unit norm, then x_1 and x_2 are orthogonal. Since $\|u\|_{\pi_s} = 1$, the elements x_1 and x_2 are linearly independent. Let us put $E = span\{x_1, x_2\}$. We can work in $E \otimes_{s,\pi_s} E$, since it is an isometric subspace of $H\hat{\otimes}_{s,\pi_s}H$. Let P be the extreme polynomial on E which norms u. It is clear that $P \neq \pm\|\cdot\|^2$. Thus, there exists an orthonormal basis $\{e_1, e_2\}$ for E such that $P(x) = (x, e_1)^2 - (x, e_2)^2$. Consequently, $\langle u, P\rangle = \lambda_1 P(x_1) + \lambda_2 P(x_2) = 1$, which yields $P(x_1) = 1$ and $P(x_2) = -1$. Thus we must necessarily have $x_1 = e_1$ and $x_2 = e_2$, hence the orthogonality. □

In general, for two normed spaces X and Y, a tensor $v = \sum_{i=1}^{\infty} \lambda_i x_i \otimes y_i$ in $X\hat{\otimes}_\pi Y$ can be viewed as a nuclear (therefore compact) operator from X^* to Y through the identification J between $X\hat{\otimes}_\pi Y$ and the space of nuclear operators $\mathcal{N}(X^*, Y)$, that associates to v the operator T_v defined as $T_v(\phi) = \sum_{i=1}^{\infty} \lambda_i \phi(x_i) y_i$.

In the case that H is a real Hilbert space, $x \mapsto (\cdot, x)$ is an isometric isomorphism between H and H^* and the approximation property for H insures that J is an isometric isomorphism between $H\hat{\otimes}_\pi H$ and $\mathcal{N}(H, H)$.

For $u = \sum_{i=1}^{\infty} \lambda_i x_i \otimes x_i$, the operator $Ju = T_u$ is self adjoint. Indeed,

$$(T_u x, y) = \sum_{i=1}^{\infty} \lambda_i (x_i, x)(x_i, y) = \sum_{i=1}^{\infty} \lambda_i (x, x_i)(x_i, y) = (x, T_u y).$$

Since the polarization constant for Hilbert spaces is 1, the spaces $H\hat{\otimes}_{s,\pi}H$ and $H\hat{\otimes}_{s,\pi_s}H$ are isometrically isomorphic and so $\|u\|_{\pi_s} = \|u\|_\pi$ and both are the same as the nuclear norm $\|T_u\|_N$.

According to the spectral theorem for self adjoint operators (Proposition 16.2, p. 148 in [18]), T_u admits a representation

$$T_u(x) = \sum_{n=1}^{\infty} \tau_n(x, e_n)e_n = \sum_{n=1}^{\infty} \tau_n(e_n, x)e_n$$

with $(e_n)_n$ an orthonormal sequence in H and $(\tau_n)_n$ the null sequence of eigenvalues of T_u. Thus we obtain a representation for u

$$u = \sum_{n=1}^{\infty} \tau_n e_n \otimes e_n.$$

As in the proof of the previous proposition, $\|u\|_{\pi_s} = \sum_{n=1}^{\infty} |\tau_n|$, and so we obtain the following result.

Theorem 2.2 *Let H be a real Hilbert space. Every element u of unit norm of $H\hat{\otimes}_{s,\pi_s}H$ can be written $u = \sum_{i=1}^{\infty} \lambda_i x_i \otimes x_i$ with $\|x_i\| = 1$ and $\sum_{i=1}^{\infty} |\lambda_i| = 1$. Furthermore $(x_i)_i$ can be chosen to be an orthonormal sequence.*

Now let us turn our attention to *complex Hilbert spaces*. We begin with a nondifficult characterization of symmetric tensor for which the symmetric tensor norm is attained by one particular representation.

Proposition 2.3 *Let H be a complex Hilbert space. If u is an element of $H\hat{\otimes}_{s,\pi_s}H$ such that $u = \sum_{i=1}^{\infty} \lambda_i x_i \otimes x_i$ with x_i unit vectors and $\sum_{i=1}^{\infty} \lambda_i = 1$ then $||u||_{\pi_s} = 1$ if and only if there exists an orthonormal basis $\{e_\alpha\}_\alpha$ with respect to which the coordinates of all the x_i's are real.*

Proof If $u = \sum_{i=1}^{\infty} \lambda_i x_i \otimes x_i$ with x_i unit vectors and $\sum_{i=1}^{\infty} \lambda_i = 1$ and $\|u\|_{\pi_s} = 1$ then there exists an extreme polynomial P such that $1 = \langle u, P\rangle = \sum_{i=1}^{\infty} \lambda_i P(x_i)$. Thus $P(x_i) = 1$ and there exists an orthonormal basis $\{e_\alpha\}_\alpha$ for H such that $P(x) = \sum_\alpha (x, e_\alpha)^2$. This yields $1 = \sum_\alpha (x_i, e_\alpha)^2 = \sum_\alpha |(x_i, e_\alpha)|^2$ for every x_i, which means that the coordinates of all the x_i's with respect to the basis $\{e_\alpha\}_\alpha$ are real. The converse is an immediate consequence from the fact that if $\{e_\alpha\}_\alpha$ is an orthonormal basis and $P(x) = \sum_\alpha (x, e_\alpha)^2$ then $P(x) = \|x\|^2$ for every $x \in H$ with $(x, e_\alpha) \in \mathbb{R}$ for all α. □

Now we are going to show that an infinite convex combination can always be found for every norm one symmetric tensor even in the case in which H is infinite dimensional. But, if we try to follow the same line of reasoning as in the real case, we come across several problems, since $x \mapsto (\cdot, x)$ is no longer an isometric isomorphism between H and H^*, due to the fact that the inner product is linear only in the first argument. We can try to mend this by associating to the tensor $u = \sum_{i=1}^{\infty} \lambda_i x_i \otimes x_i$ the linear operator $S_u(x) = \sum_{i=1}^{\infty} \lambda_i (x, x_i) x_i$, but this association is not well defined in general. For example, if x_1 and x_2 are two orthogonal elements of unit norm in H, let $u = (x_1 + ix_2) \otimes (x_1 + ix_2) = x_1 \otimes x_1 + ix_1 \otimes x_2 + ix_2 \otimes x_1 - x_2 \otimes x_2$. Then we have simultaneously $S_u(x) = ((x, x_1) - i(x, x_2))x_1 + ((x, x_2) + i(x, x_1))x_2$ and $S_u(x) = ((x, x_1) - i(x, x_2))x_1 + (-(x, x_2) - i(x, x_1))x_2$, which is not possible. However, if we fix an orthonormal basis $\{e_\alpha\}_\alpha$ for H and define for every $x = \sum_\alpha (x, e_\alpha) e_\alpha$ its conjugate as $\bar{x} = \sum_\alpha \overline{(x, e_\alpha)} e_\alpha$, then $x \mapsto (\cdot, \bar{x})$ is an isometric isomorphism between H and H^*, a fact which makes the natural association J_e between a tensor $v = \sum_{i=1}^{\infty} \lambda_i x_i \otimes y_i$ in $H \hat{\otimes}_\pi H$ and the nuclear operator $T_{v,e}(x) = \sum_{i=1}^{\infty} \lambda_i (x_i, \bar{x}) y_i$ an isometric isomorphism between $H \hat{\otimes}_\pi H$ and $\mathcal{N}(H, H)$. Note that this isomorphism *does* depend on the orthonormal basis that we have fixed.

Since H is a complex space, every symmetric tensor can be written as $u = \sum_{i=1}^{\infty} \lambda_i x_i \otimes x_i$ with the scalars λ_i real and nonnegative. Unlike in the real case, the operator $J_e u = T_{u,e}$ is not self adjoint for every u. Indeed, $(T_{u,e}(x), y) = \sum_{i=1}^{\infty} \lambda_i (x_i, \bar{x})(x_i, y)$ and $(x, T_{u,e}(y)) = \sum_{i=1}^{\infty} \lambda_i (\bar{y}, x_i)(x, x_i)$ are, in general, different. For example, if $u = (e_{\alpha_1} + ie_{\alpha_2}) \otimes (e_{\alpha_1} + ie_{\alpha_2})$ and $x = e_{\alpha_1}$ and $y = e_{\alpha_1} + ie_{\alpha_2}$ then $(T_{u,e}(x), y) = 2$ and $(x, T_{u,e}(y)) = 0$.

Theorem 2.4 *Let H be a complex Hilbert space. Every element u of unit norm of $H \hat{\otimes}_{s,\pi_s} H$ is an infinite convex combination of tensors of the form $\{x \otimes x : \|x\| = 1\}$ which can be chosen so that there exists an orthonormal basis $\{e_\alpha\}_\alpha$ with respect to which the coordinates of all the x's are real.*

Proof We will regard every tensor u in $H \hat{\otimes}_{s,\pi} H \subset H \hat{\otimes}_\pi H$ as a compact operator from H to H. Since $\|u\|_{\pi_s} = 1$, there exists an extreme polynomial P such that $1 = \langle u, P \rangle$. Thus there exists an orthonormal basis $\{e_\alpha\}_\alpha$ for H such that $P(x) = \sum_\alpha (x, e_\alpha)^2$. We define the conjugation in H with respect to this basis and we associate to the tensor $u = \sum_{i=1}^{\infty} \lambda_i x_i \otimes x_i$ the nuclear operator $T_{u,e}(x) = \sum_{i=1}^{\infty} \lambda_i (x_i, \bar{x}) x_i$. According to the general spectral theorem (Proposition 16.3, p. 149 in [18] or 4.1, p. 76 in [11]), $T_{u,e}$ admits a Schmidt representation

$$T_{u,e}(x) = \sum_{n=1}^{\infty} \tau_n (x, g_n) f_n = \sum_{n=1}^{\infty} \tau_n (\overline{g_n}, \bar{x}) f_n$$

with $(g_n)_n$ and $(f_n)_n$ orthonormal sequences in H and $(\tau_n)_n$ a nonnegative decreasing sequence. So we get $u = \sum_{n=1}^{\infty} \tau_n \overline{g_n} \otimes f_n$. Moreover, the scalar sequence is uniquely determined, namely

$$\tau_n = \inf\{\|v - u\| : v \in B(H), \operatorname{rank}(v) < n\}.$$

Thus, if u is not of finite rank then all τ_n are positive.

We recall that the pth Schatten–von Neumann class $\mathcal{S}_p(H)$ consist of all compact operators from H to H which admit a Schmidt representation

$$T(x) = \sum_{n=1}^{\infty} \alpha_n (x, e_n) h_n$$

such that $\sum_{n=1}^{\infty} |\alpha_n|^p < \infty$, in which case $\nu_p(T) := (\sum_{n=1}^{\infty} |\alpha_n|^p)^{1/p}$ does not depend on the orthonormal sequences $(e_n)_n$ and $(h_n)_n$ and defines a norm with respect to which $\mathcal{S}_p(H)$ is a Banach space.

Using the characterization of $\mathcal{S}_p(H)$ (Theorem 4.6, p. 81 in [11]), it follows that $H\hat{\otimes}_\pi H = \mathcal{N}(H, H) = \mathcal{S}_1(H)$ isometrically. Let us put $e_n = \overline{g_n}$. Thus, we write $u = \sum_{n=1}^{\infty} \tau_n e_n \otimes f_n$ and $\|u\|_\pi = \sum_{n=1}^{\infty} \tau_n$.

The symmetrization operator $s : H \otimes_\pi H \to H \otimes_{s,\pi} H$, is defined by $s(x \otimes y) = \frac{1}{2}(x \otimes y + y \otimes x) = \frac{1}{4}((x + y) \otimes (x + y) - (x - y) \otimes (x - y))$ and extended by linearity and continuity to $H\hat{\otimes}_\pi H$. Since u is symmetric, we have

$$u = s(u) = \sum_{n=1}^{\infty} \tau_n s(e_n \otimes f_n).$$

Since $H\hat{\otimes}_{s,\pi} H$ and $H\hat{\otimes}_{s,\pi_s} H$ are isometrically isomorphic, we have

$$1 = \|u\|_{\pi_s} = \sum_{n=1}^{\infty} \tau_n.$$

Thus $1 = \langle u, P\rangle = \sum_{n=1}^{\infty} \tau_n \check{P}(e_n, f_n)$, where $\check{P}$ is the symmetric bilinear form associated with P. From here, we necessarily have $\check{P}(e_n, f_n) = 1$ for all n, and so $\sum_\alpha (e_n, e_\alpha)(f_n, e_\alpha) = 1$. It follows that $(f_n, e_\alpha) = \overline{(e_n, e_\alpha)}$. Let us put $(e_n, e_\alpha) = a_{\alpha n} + i b_{\alpha n}$. Then

$$e_n \otimes f_n = \sum_\alpha \sum_\gamma (a_{\alpha n} a_{\gamma n} + b_{\alpha n} b_{\gamma n} + i b_{\alpha n} a_{\gamma n} - i a_{\alpha n} b_{\gamma n}) e_\alpha \otimes e_\gamma$$

and thus

$$\begin{aligned} s(e_n \otimes f_n) &= \sum_\alpha \sum_\gamma (a_{\alpha n} a_{\gamma n} + b_{\alpha n} b_{\gamma n}) e_\alpha \otimes e_\gamma \\ &= \sum_\alpha (a_{\alpha n} e_\alpha) \otimes \sum_\gamma (a_{\gamma n} e_\gamma) + \sum_\alpha (b_{\alpha n} e_\alpha) \otimes \sum_\gamma (b_{\gamma n} e_\gamma) \\ &= a_n \otimes a_n + b_n \otimes b_n \end{aligned}$$

where $a_n = \sum_\alpha a_{\alpha n} e_\alpha$ and $b_n = \sum_\alpha b_{\alpha n} e_\alpha$.

Since $\|a_n\|^2 + \|b_n\|^2 = 1$, the conclusion follows. □

Definition 2.5 [3] Let X be a normed space and $x \in B_X$. If e is an extreme point of B_X, $\|y\| \leq 1$, $0 < \lambda < 1$ and $x = \lambda e + (1 - \lambda) y$, we say that the ordered triple (e, y, λ) is amenable to x. In this case we define

$$\lambda(x) = \sup\{\lambda : (e, y, \lambda) \text{ is amenable to } x\}.$$

The space X is said to have the λ-property if each $x \in B_X$ admits an amenable triple.

It is proved in [4] that a normed space X has the λ-property if and only if every element of B_X is an infinite convex combination of extreme points of B_X.

Therefore the previous results yield the following corollary.

Corollary 2.6 *Let H be a (real or complex) Hilbert space. The space $H\hat{\otimes}_{s,\pi_s}H$ has the λ-property.*

We have just showed that every tensor u in $H\hat{\otimes}_{s,\pi}H$ can be written $u=\sum_{i=1}^{\infty}\lambda_i x_i\otimes x_i$ with the scalars λ_i real and all the x_i having real coordinates with respect to an orthonormal basis $\{e_\alpha\}_\alpha$. Due to these additional conditions, $T_{u,e}$ *is* self adjoint. Indeed,

$$(x,T_{u,e}(y))=\sum_{i=1}^{\infty}\lambda_i(\bar{y},x_i)(x,x_i)=\sum_{i=1}^{\infty}\lambda_i(x_i,y)(x_i,\bar{x})=(T_{u,e}(x),y)$$

since, in general we have $(z,w)=\sum_\alpha(z,e_\alpha)\overline{(w,e_\alpha)}=(\overline{w},\bar{z})$. This hints at the fact that the x_i's might be taken to be orthogonal.

Corollary 2.7 *Let H be a complex Hilbert space. Then for every tensor u of unit norm in $H\hat{\otimes}_{s,\pi}H$ there exists an orthonormal sequence $(f_n)_n$ such that u is an infinite convex combination of the elementary tensors $f_n\otimes f_n$.*

Proof By Theorem 2.4, there exist an orthonormal basis $\{e_\alpha\}_\alpha$, a sequence (x_n) with real coordinates with respect to the basis and (λ_n) with $\lambda_n\geq 0$ and $\sum_{n=1}^{\infty}\lambda_n=1$ such that $u=\sum_{n=1}^{\infty}\lambda_n x_n\otimes x_n$. Let us put $K=\overline{span}_{\mathbb{R}}\{e_\alpha\}_\alpha$. Since $x_n\in K$ for all n, in that real Hilbert space we can define $v=\sum_{n=1}^{\infty}\lambda_n x_n\otimes x_n$. Again the tensor v is a norm one element in $K\hat{\otimes}_{s,\pi_s}K$. According to Proposition 2.2, there exists an orthonormal sequence $(g_n)_n$ in K such that $v=\sum_n\mu_n g_n\otimes g_n$ and $\sum_n|\mu_n|=1$. Consider $Q=Q_1+iQ_2$ a complex 2-homogeneous continuous polynomial on H. Since Q_1 and Q_2 are two real 2-homogeneous continuous polynomials on K, we have

$$\begin{aligned}\langle u,Q\rangle&=\sum_{n=1}^{\infty}\lambda_n Q_1(x_n)+i\left(\sum_{n=1}^{\infty}\lambda_n Q_2(x_n)\right)=\langle v,Q_1\rangle+i\,\langle v,Q_2\rangle\\&=\sum_{n=1}^{\infty}\mu_n Q_1(g_n)+i\left(\sum_{n=1}^{\infty}\mu_n Q_2(g_n)\right)=\left\langle\sum_{n=1}^{\infty}\mu_n g_n\otimes g_n,Q\right\rangle,\end{aligned}$$

for all $Q\in\mathcal{P}(^2H)$. Hence, if working in H we put $f_n=g_n$ if $\mu_n>0$, and $f_n=ig_n$ if $\mu_n<0$ then $u=\sum_n|\mu_n|f_n\otimes f_n$ in $H\hat{\otimes}_{s,\pi}H$. □

3 Smooth symmetric tensors

In [2], where the notion of n-smoothness was introduced, characterization of the n-smooth points of a series of classical Banach spaces was given. In particular, it was proved in [2, Theorem 2.3] that $sm^{(n)}(\ell_p)=\{\lambda e_j:|\lambda|=1,\ j\in\mathbb{N}\}$ if $2\leq n<p$

and $sm^{(n)}(\ell_p) = \emptyset$ if $p \leq n$ whenever $1 \leq p < \infty$ and $n \geq 2$. We can improve these results as follows: if $w = (w_n)$ is a nonincreasing sequence of positive real numbers satisfying $w_1 = 1$ and $w \notin \ell_1$, recall that the Lorentz sequence space $d(w, p)$ is given by

$$d(w, p) = \left\{ y \in \mathbb{C}^{\mathbb{N}} : \sup_{\sigma} \sum_{n=1}^{\infty} |y_{\sigma(n)}|^p w_n < +\infty, \sigma : \mathbb{N} \to \mathbb{N} \text{ permutation} \right\},$$

which, when endowed with the natural norm $\|y\|_{w,p} = \max_{\sigma} (\sum_{n=1}^{\infty} |y_{\sigma(n)}|^p w_n)^{1/p}$, is a Banach space. The condition $w_1 = 1$ is equivalent to the fact that $\|e_j\|_{w,p} = 1$ for every element of the canonical basis $\{e_n\}_n$.

Proposition 3.1 *Let $1 \leq p < \infty$ and $n \geq 2$ be a positive integer. Then*

$$sm^{(n)}(d(\omega, p)) = \begin{cases} \{\lambda e_j : |\lambda| = 1, j \in \mathbb{N}\} & \text{if } 2 \leq n < p \\ \emptyset & \text{if } p \leq n \end{cases}$$

Proof Given $y \in d(w, p)$ such that $\|y\|_{w,p} = 1$ and a permutation $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$1 = \|y\|_{w,p} = \left(\sum_{n=1}^{\infty} |y_{\sigma(n)}|^p w_n \right)^{1/p},$$

we define $T : d(w, p) \longrightarrow \ell_p$ as $T(x) = (w_n^{1/p} x_{\sigma(n)})$ for $x \in d(w, p)$. Clearly T is linear and continuous and $\|T\| \leq 1$. But obviously $\|T(y)\|_p = 1$. Now, if $T(y) \notin sm^{(n)}(\ell_p)$ then there exist two n-homogeneous continuous polynomials $P, Q : \ell_p \longrightarrow \mathbb{K}$ of norm one such that $P \neq Q$ and $P(T(y)) = 1 = Q(T(y))$. We have that $P \circ T$ and $Q \circ T$ are n-homogeneous continuous polynomials on $d(w, p)$ and, since $\|P \circ T\| \leq \|P\| \|T\|^n$ and $P(T(y)) = 1 = Q(T(y))$, we have that

$$\|P \circ T\| = 1 = \|Q \circ T\| = P(T(y)) = Q(T(y)).$$

Finally as $\mathbb{K}^{(\mathbb{N})}$ is dense in $d(w, p)$ and $T(\mathbb{K}^{(\mathbb{N})}) = \mathbb{K}^{(\mathbb{N})}$ is dense in ℓ_p, we obtain that $P \circ T \neq Q \circ T$, i.e., y is not n-smooth in $d(w, p)$. This shows that $sm^{(n)}(d(w, p)) \subset \{\lambda e_j : |\lambda| = 1, j \in \mathbb{N}\}$ if $2 \leq n < p$ and $sm^{(n)}(d(w, p)) = \emptyset$ if $p \leq n$. On the other hand, if for $2 \leq n < p$ there exists e_j such that e_j is not n-smooth in $d(w, p)$, then there exist two n-homogeneous continuous polynomials $R, S : d(w, p) \longrightarrow \mathbb{K}$ of norm one such that $R \neq S$ and $R(e_j) = 1 = Q(e_j)$. The canonical injection i of ℓ_p into $d(w, p)$ has norm one, hence $R \circ i$ and $S \circ i$ are n-homogeneous continuous polynomials on ℓ_p with

$$1 \geq \|R \circ i\| \geq R(e_j) = 1 = S(e_j) \leq \|S \circ i\| \leq 1.$$

The fact that $i(\mathbb{K}^{(\mathbb{N})}) = \mathbb{K}^{(\mathbb{N})}$ implies that $R \circ i \neq S \circ i$. Hence $e_j \notin sm^{(n)}(\ell_p)$, a contradiction. □

Since for a unit vector $x_0 \in X$ the n-smoothness is equivalent to $x_0 \otimes \cdots \otimes x_0$ being a smooth point in the unit sphere of $\hat{\otimes}_{n,s,\pi_s} X$, Proposition 3.1 shows that are many spaces (ℓ_2 is such an example) whose unit spheres do not contain smooth points of higher order than 1. However, the unit sphere of the projective tensor product $\ell_2 \hat{\otimes}_{s,\pi_s} \ell_2$ does contain smooth points.

Proposition 3.2 *Let $\{e_n\}_n$ be the canonical basis of ℓ_2. The tensor $u = \sum_{n=1}^{\infty} \lambda_n e_n \otimes e_n$ with $\sum_{n=1}^{\infty} |\lambda_n| = 1$ and $\lambda_n \neq 0$ for all n is a smooth point of the unit sphere of $\ell_2 \hat{\otimes}_{s,\pi_s} \ell_2$.*

Proof Let $P(x) = \sum_{n=1}^{\infty} sign\lambda_n x_n^2$. Since $\langle u, P\rangle = 1$, it follows that $\|u\|_{\pi_s} = 1$. Suppose that there is another 2-homogeneous polynomial Q such that $\langle u, Q\rangle = 1 = \|Q\|$. We necessarily have $Q(e_n) = sign\lambda_n$ for all n. Let us define $T : \ell_2 \to \ell_2$ by $(x, Ty) = A(x, y)$ where A is the symmetric bilinear form associated with Q. The operator T is linear if we work with real scalars and conjugate linear in the complex case. Since in general $\|Tx\| \leq \|x\|$ we obtain $Te_n = sign\lambda_n e_n$ for all n and so $A(e_n, e_m) = 0$ if $n \neq m$. This yields $Q(x) = \sum_{n=1}^{\infty} sign\lambda_n x_n^2 = P(x)$. □

The goal of this section is to give an explicit description of the smooth points in $H \hat{\otimes}_{s,\pi_s} H$, with H a Hilbert space. We will make full use of the results of the previous section. Theorem 3.3, among other equivalent characterizations, shows that a reciprocal of the above proposition is true.

In the setting of Theorem 2.4, the associated operator $J_e u = T_{u,e}$ is self adjoint and so we can write $H = \ker T_{u,e} \oplus L$, with L the norm closure in H of $T_{u,e}(H)$. Then u is, in fact, an element of $L \hat{\otimes}_{\pi_s} L$. Indeed, let P_1 and P_2 denote the orthogonal projections from H onto $\ker T_{u,e}$ and L, respectively. Then

$$T_{u,e}(x) = P_2 T_{u,e}(P_2 x) = \sum_{i=1}^{\infty} \lambda_i (x_i, \overline{P_2 x}) P_2 x_i = \sum_{i=1}^{\infty} \lambda_i (P_2 x, x_i) P_2 x_i$$
$$= \sum_{i=1}^{\infty} \lambda_i (x, P_2 x_i) P_2 x_i$$

and so $u = \sum_{i=1}^{\infty} \lambda_i P_2 x_i \otimes P_2 x_i \in L \hat{\otimes}_{\pi_s} L$.

The same conclusion holds when working with a real Hilbert space with L the norm closure in H of $T_u(H)$, since T_u is self adjoint.

We are ready now for the characterization of smooth points of the unit ball of $H \hat{\otimes}_{s,\pi_s} H$ when H is infinite dimensional. When working with a real space, the conjugation is obviously the identity and "$J_e u$" means simply "$Ju = T_u$".

Theorem 3.3 *Let H be a (real or complex) Hilbert space. The following are equivalent for an element u of unit norm of $H \hat{\otimes}_{s,\pi_s} H$.*

(a) *The real case*:
   (i) *u is a smooth point of $B_{H \hat{\otimes}_{s,\pi_s} H}$,*
   (ii) *there exists a representation $u = \sum_{i=1}^{\infty} \lambda_i x_i \otimes x_i$ with $\|x_i\| = 1$ and $\lambda_i \neq 0$ for all i satisfying $\sum_{i=1}^{\infty} |\lambda_i| = 1$ and for every such representation we have $\overline{span}\{x_i : i = 1, 2, \ldots\} = H$,*

(iii) *there exists an orthonormal basis* $\{e_n\}_n$ *for* H *such that* $u = \sum_{n=1}^{\infty} \lambda_n e_n \otimes e_n$ *with* $\lambda_n \neq 0$ *for all* n *satisfying* $\sum_{n=1}^{\infty} |\lambda_n| = 1$,

(iv) *the operator* $T_u : H \to H$ *is injective.*

(b) *The complex case*:

(i) u *is a smooth point of* $B_{H\hat{\otimes}_{s,\pi_s}H}$,

(ii) *there exists a representation* $u = \sum_{i=1}^{\infty} \lambda_i x_i \otimes x_i$ *with* $\|x_i\| = 1$ *and* $0 < \lambda_i \leq 1$ *for all* i *satisfying* $\sum_{i=1}^{\infty} \lambda_i = 1$ *and for every such representation we have* $\overline{span}\{x_i : i = 1, 2, \ldots\} = H$,

(iii) *there exists an orthonormal basis* $\{e_n\}_n$ *for* H *such that* $u = \sum_{n=1}^{\infty} \lambda_n e_n \otimes e_n$ *with* $0 < \lambda_n \leq 1$ *for all* n *satisfying* $\sum_{n=1}^{\infty} \lambda_n = 1$,

(iv) *if* $u = \sum_{i=1}^{\infty} \lambda_i x_i \otimes x_i$ *with* $\|x_i\| = 1$ *and* $0 < \lambda_i \leq 1$ *for all* i *satisfying* $\sum_{i=1}^{\infty} \lambda_i = 1$ *and* $\{e_\alpha\}_\alpha$ *is an orthonormal basis for* H *with respect to which the coordinates of all the* x_i*'s are real, then the operator* $J_e u : H \to H$ *is injective.*

Proof (i)⇒(iv) Let $K = \ker T_u$ when H is real and $K = \ker J_e u$ when H is complex. Write $H = K \oplus L$. Thus u is a unit norm element of $L\hat{\otimes}_{s,\pi_s}L$ and there exists a polynomial Q of unit norm on L such that $\langle u, Q\rangle = 1$. Now, for every polynomial R of unit norm on K, by decomposing every x in H as $x_1 + x_2$ with x_1 in L and x_2 in K and defining $P(x) = Q(x_1) + R(x_2)$, we have $\|P\| = 1$ and $\langle u, P\rangle = 1$. Since u is smooth, it follows that we must have $R = 0$ and consequently $K = \{0\}$, so $J_e u$ is injective.

(iv)⇒ (ii) The existence is given by Theorems 2.2 and 2.4. Take y an element of H orthogonal to $\overline{span}\{x_i : i = 1, \ldots, \infty\}$. Then $T_u(y) = \sum_{i=1}^{\infty} \lambda_i (x_i, \bar{y})x_i = 0$ and so $\bar{y} = y = 0$.

(ii) ⇒ (iii) Theorem 2.2 and Corollary 2.7 give the existence of an orthonormal sequence $\{e_n\}_n$ in H such that the tensor u has the required representation. But by (ii) we have $\overline{span}\{e_n : n = 1, \ldots, \infty\} = H$ and so $\{e_n\}_n$ is an orthonormal basis.

(iii) ⇒ (i) Here we will have to deal separately with the real and complex spaces.

(a) The real case. Let $A = \{n : \lambda_n > 0\}$ and $B = \{n : \lambda_n < 0\}$. Let P be an extreme 2-homogeneous polynomial on H such that $1 = ||P|| = \langle u, P\rangle$. Let us put $P(x) = \|\pi_{K_1}x\|^2 - \|\pi_{K_2}x\|^2$ with $H = K_1 \oplus K_2$. Then

$$1 = \sum_n \lambda_n P(e_n) \leq \sum_n |\lambda_n| = 1$$

and so $P(e_n) = sign\lambda_n$ for every n. It follows that $e_n \in K_1$ for all $n \in A$ and $e_n \in K_2$ for all $n \in B$. Thus, putting $H_1 = \overline{span}\{e_n : n \in A\}$ and $H_2 = \overline{span}\{e_n : n \in B\}$, we have $H_1 \subset K_1$ and $H_2 \subset K_2$. But $H = H_1 \oplus H_2$ and so $K_1 = H_1$ and $K_2 = H_2$. Thus $P(x) = \left|\left|\pi_{H_1}x\right|\right|^2 - \left|\left|\pi_{H_2}x\right|\right|^2$ is the only extreme polynomial that exposes u. Now, since $\mathcal{P}(^2H)$ has the λ-property [13], every 2-homogeneous polynomial on H is an infinite convex combination of extreme polynomials and so P is the only 2-homogeneous polynomial on H that exposes u.

(b) The complex case. Clearly $P(x) = \sum_n (x, e_n)^2$ exposes u. Since $\mathcal{P}(^2H)$ has the λ-property, it only remains to show that P is the only extreme 2-homogeneous polynomial that exposes u. Let $\{f_m\}_m$ be an orthonormal basis for H such that $Q(x) =$

$\sum_m (x, f_m)^2$ and $\langle u, Q\rangle = 1$ (which, in particular, yields that the coordinates of all e_n's with respect to $\{f_m\}_m$ are real). Put $K = \overline{span}_{\mathbb{R}}\{e_n\}$ and $L = \overline{span}_{\mathbb{R}}\{f_m\}$. Let x be an element of K. Then, since $\overline{span}\{e_n\} = H$, there exist a_n and b_n real numbers such that $x + i0 = \sum_n (a_n + ib_n)e_n$, which means that $x = \sum_n a_n e_n$ and so it has real coordinates with respect to $\{f_m\}_m$. Thus we have $K \subset L$. But since $K \oplus iK = H = L \oplus iL$ (as real spaces), we get $K = L$ and consequently $P = Q$.

Remark 3.4 The same characterizations hold for a finite dimensional space H, with the obvious changes in the statements: in (iii) the sum goes up to the dimension of H, while in (ii) and (iv) we can as well work with finite sums.

Note that while (iii) shows that all smooth tensors are more or less of the form described in Proposition 3.2, (ii) gives more practical ways of constructing and recognizing smooth tensors, in the sense that to obtain a smooth tensor it is not necessary to work with absolute convex combination of elementary tensors of the elements of an orthonormal basis. For instance, $u = 1/6e_1 \otimes e_1 - \sqrt{2}/8(e_2 - e_3) \otimes (e_2 - e_3) - \sqrt{5}/20(e_2 + 2e_3) \otimes (e_2 + 2e_3) + 1/3 \sum_{n=4}^{\infty} e_n \otimes e_n$ is a smooth point of the unit sphere of $\ell_2 \hat{\otimes}_{s,\pi_s} \ell_2$.

Corollary 3.5 *The unit sphere of $H\hat{\otimes}_{s,\pi_s} H$ has smooth points if and only if H is separable.*

Remark 3.6 Now let H be a real or complex separable Hilbert space. Since we have

$$H\hat{\otimes}_{s,\pi_s} H = \mathcal{P}_N(^2H) = \mathcal{P}_I(^2H)$$

with equality of norms, for every nonzero symmetric tensor u in $H\hat{\otimes}_{s,\pi_s} H$, there exists a regular Borel measure μ on B_H with its weak topology such that $||u||_{\pi_s} = ||\mu||$ and

$$\left\langle u, \varphi^2 \right\rangle = \int_{B_H} \varphi(x)^2 d\mu$$

for every φ in H and so

$$\langle u, P\rangle = \int_{B_H} P(x) d\mu$$

for every 2-homogeneous polynomial P of finite type on H. Since H has a Schauder basis, every 2-homogeneous polynomial on H is the pointwise limit of a bounded sequence of finite type 2-homogeneous polynomials and so, by the Dominated Convergence Theorem, the integral formula above extends to all elements of $\mathcal{P}(^2H)$.

Since H is separable, by the Pettis Measurability Theorem (Proposition 2.15, p. 26 in [20]), we obtain a Bochner integral representation for symmetric tensors, $u = \int_{B_H} x \otimes x \, d\mu$.

Theorem 2.2 for H a real Hilbert space and Theorem 2.4 for H a complex Hilbert space imply that if $\|u\|_{\pi_s} = 1$, there exist a sequence (x_n) of norm one elements of

H and a sequence $(\lambda_n) \subset \mathbb{K}$ with $\sum_{n=1}^{\infty} |\lambda_n| = 1$ (and $\lambda_n \geq 0$ if $\mathbb{K} = \mathbb{C}$) such that $u = \sum_{i=1}^{\infty} \lambda_n x_n \otimes x_n$ and so $\mu = \sum_{n=1}^{\infty} \lambda_n \delta_{x_n}$ is a measure that represents u. We do not know if there is another measure ν with $\|u\|_{\pi_s} = \|\nu\|$ and $\langle u, P\rangle = \int_{B_H} P(x) d\nu$ for all $P \in \mathcal{P}(^2H)$ such that ν is not an infinite absolute convex combination of evaluations δ_x $(x \in H, \|x\| = 1)$.

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