JOURNAL OF ALGEBRA 126, 264–292 (1989)

V-Valuations of a Commutative Ring, I

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Communicated by Richard G. Swan

Received October 16, 1987

1. INTRODUCTION

We propose a definition of a valuation of a commutative ring which includes, to the best of our knowledge, all existing definitions. Those are: the Artin definition of an absolute value introduced in order to yield the product formula for a global field, the Samuel notion of a subring whose complement is multiplicatively closed, the Manis notion of a valuation mapping onto an extended group, and, most fundamental, the Krull notion of a valuation of a field (see [2], [5], and [6]).

We introduce the notion of a V-monoid and define what we mean by a V-valuation of a commutative ring. This terminology is, at the insistence of the first author, to honor the second author, who originally brought to his attention the basic idea. We define the notion of a co-multiplicatively closed subset of a ring (CMC subset, for short) and establish a natural bijective correspondence between isomorphism classes of V-valuations of a commutative ring R and the CMC subsets of R. We call certain V-valuations formally finite and show that the above correspondence induces a bijective correspondence between the formally finite V-valuations and the CMC subrings of R.

We investigate the structure of V-monoids and introduce several operations on V-monoids. We determine the structure of all finite V-monoids and show that each such occurs as the target of a V-valuation of a commutative ring. We show that the V-monoids in several other classes are realizable as the targets of V-valuations of commutative rings. We conclude with a determination of all V-valuations of a number field.

In a forthcoming paper entitled "Complex-Valued Places and CMC Subsets of a Field," the authors characterize the nonring CMC subsets of a field F. Given a complex-valued place $\varphi: B \to \mathbf{C}$ on F, i.e., φ is a ring homomorphism defined on a valuation subring B of F and $\varphi(B)$ is a subfield of $\mathbf{C}$, let $A = \{x \in B \mid |\varphi x| \leqslant 1\}$. Then A is a nonring CMC subset of F

0021-8693/89 $3.00

(use $t = \frac{1}{2}$ in Definition 2.13). Moreover, every nonring CMC subset of F is defined in this manner by a complex-valued place φ on F and φ is essentially unique. In that same paper it is shown that the intersection of all CMC subsets of a field consists of zero together with the roots of unity in the field.

By a "ring" we mean a commutative ring with identity. By a "ring homomorphism" we mean a ring homomorphism which preserves the identity. We write $X \subset Y$ if X is a proper subset of Y and $X \setminus Y$ for $\{x \mid x \in X, x \notin Y\}$. By "order-homomorphism" we mean a map which is both order-preserving and sum-preserving.

2. *V*-VALUATIONS

If G is a nontrivial totally ordered abelian group, then

$$G_\infty = G \cup \{\infty\}$$

is an example of the next concept.

DEFINTION 2.1. By a *V-monoid* we mean a triple $(\Gamma, +, \leqslant)$ such that:

(i) $(\Gamma, +)$ is a commutative monoid.

(ii) $(\Gamma, \leqslant)$ is a totally ordered set with a maximum which we will denote by ∞.

(iii) For all α, β, γ in Γ,

$$\alpha \leqslant \beta \Rightarrow \alpha + \gamma \leqslant \beta + \gamma.$$

(iv) For all $\gamma \in \Gamma$,

$$\gamma + \infty = \infty.$$

(v) For all $\alpha, \beta \in \Gamma$,

$$\alpha < \beta \Rightarrow \exists \gamma \in \Gamma \qquad \text{such that } \alpha + \gamma < 0 \leqslant \beta + \gamma.$$

Notation 2.2. For a *V*-monoid Γ we write

$$P(\Gamma) = \{\gamma \in \Gamma \mid 0 \leqslant \gamma\}, \qquad P^+(\Gamma) = \{\gamma \in \Gamma \mid 0 < \gamma\},$$
$$N(\Gamma) = \{\gamma \in \Gamma \mid \gamma \leqslant 0\}, \qquad N^-(\Gamma) = \{\gamma \in \Gamma \mid \gamma < 0\},$$
$$\Gamma' = \Gamma \setminus \{\infty\}.$$

EXAMPLE 2.3. Let $\Gamma_0 = \{0\}$. Γ_0 is essentially the only *V*-monoid in which $0=\infty$; we refer to Γ_0 as the trivial V-monoid. Let Γ_1 denote the V-monoid $(\{\alpha, 0, \beta\}, +, \leqslant)$, where 0 is the zero, β is the infinity, $\alpha+\alpha=\alpha$, and $\alpha<0<\beta$.

EXAMPLE 2.4. For V-monoids Λ and Δ we write

$$(\Lambda \# \Delta, \oplus, \leqq) = (\Lambda' \cup \Delta' \cup \{\infty\}, \oplus, \leqq),$$

where the indicated union is disjoint and with $\oplus$ and $\leqq$ defined as follows:

$\lambda_1 \oplus \lambda_2 = \lambda_1 + \lambda_2$ for all λ_1, λ_2 in Λ',

$\delta_1 \oplus \delta_2 = \delta_1 + \delta_2$ for all δ_1, δ_2 in Δ',

$\lambda + \delta = \delta$ for all $\lambda \in \Lambda'$ and all $\delta \in \Delta'$,

$\delta < \lambda < \varepsilon$ for all $\delta \in N^-(\Delta)$, all $\lambda \in \Lambda'$, and all $\varepsilon \in P(\Delta')$,

∞ is an absorbent maximum and we preserve the given orders within Λ' and Δ'.

One verifies that $\Lambda \# \Delta$ is a V-monoid, in particular that Λ' and Δ' are closed under addition, using the lemma that follows. We call $\Lambda \# \Delta$ the *sharp product* of Λ and Δ.

LEMMA 2.5. *Let Γ be a V-monoid. Then,*

$$\alpha < \beta \quad \text{and} \quad \gamma < \delta \Rightarrow \alpha + \gamma < \beta + \delta.$$

In particular,

$$\alpha, \beta < \infty \Rightarrow \alpha + \beta < \infty.$$

Proof. Suppose $\alpha < \beta$ and $\gamma < \delta$. Let $\rho, \eta \in \Gamma$ be such that

$$\alpha + \rho < 0 \leqslant \beta + \rho \quad \text{and} \quad \gamma + \eta < 0 \leqslant \delta + \eta.$$

Then,

$$\alpha + \gamma + \rho + \eta < 0 \leqslant \beta + \delta + \rho + \eta,$$

which implies that

$$\alpha + \gamma < \beta + \delta.$$

Remark 2.6. One easily checks that the sharp product is associative with identity the trivial V-monoid. Let I be a totally ordered set with minimum, which we denote by 0. Let $\{\Lambda_i\}_{i\in I}$ be a family of V-monoids. Let $\Gamma = (\bigcup_{i\in I} \Lambda_i') \cup \{\infty\}$, where the indicated unions are disjoint. Defining addition and an order as in (2.4), one checks that Γ is a V-monoid, called the sharp product of the family $\{\Lambda_i\}_{i \in I}$.

We next describe the internal version of this construction.

DEFINITION 2.7. For Γ a V-monoid we call an element $e \in \Gamma$ a *sharp idempotent* if $0 \leqslant e$, $e = e + e$, and $\Lambda_e \cup \Delta_e = \Gamma$, where

$$\Lambda_e = \{\gamma \in \Gamma \mid \gamma + e = e\},$$

and

$$\Delta_e = \{\gamma \in \Gamma \mid \gamma + e = \gamma\}.$$

Remark 2.8. If e is a sharp idempotent in a V-monoid Γ, then Λ_e and Δ_e are V-monoids and $\Gamma \cong \Lambda_e \# \Delta_e$, where isomorphism has the obvious meaning.

EXAMPLE 2.9. For n a positive integer we let

$$\Gamma_n = \Gamma_1 \# \cdots \# \Gamma_1 \qquad (n \text{ copies}).$$

Recall that $\Gamma_0 = \{0\}$.

DEFINITION 2.10. For a ring R by a *V-valuation* of R we mean a pair (v, Γ), where Γ is a V-monoid and $v: R \to \Gamma$ is a surjective map such that:

(a) $v(rs) = v(r) + v(s)$ for all $r, s \in R$, and

(b) $\exists$ a unit $t \in R$ such that:

   (i) $\min\{v(r), v(s)\} \leqslant v(r+s) + v(t)\ \forall r, s \in R$, and

   (ii) $v(s) < 0 \Rightarrow \exists n \in \mathbf{N}$ such that $nv(s) + v(t) < 0$.

If $t = 1$ satisfies (b) we say (v, Γ) is *formally finite*. Otherwise, we say (v, Γ) is *formally infinite*.

Remark 2.11. Let $v: R \to \Gamma$ be a V-valuation. Since v is surjective, $v(1) = 0$ and $v(0) = \infty$. One checks that $v: \mathbf{C} \to \mathbf{R}_\infty$ given by $v(r) = -\log(|r|)$ is a V-valuation of $\mathbf{C}$ with $t = \frac{1}{2}$.

DEFINITION 2.12. For V-valuations (v, Γ) and (w, Δ) of a ring R, we say (v, Γ) is *isomorphic* to (w, Δ) if there exists an isomorphism $\sigma: \Gamma \to \Delta$ such that $\sigma \circ v = w$.

DEFINITION 2.13. For a ring R we call a subset A of R a *CMC subset* if:

(a) $AA \subseteq A$, $(R\backslash A)(R\backslash A) \subseteq R\backslash A$, $0, 1 \in A$, and

(b) $\exists$ a unit $t \in R$ such that:

(i) $a, b \in A \Rightarrow t(a+b) \in A$, and

(ii) $s \notin A \Rightarrow \exists n \in \mathbf{N}$ such that $s^n t \notin A$.

A CMC subset A of R that is also a subring of R will be called a *CMC subring* of R.

Remark 2.14. A ring R can be viewed as a CMC subset of itself. Let A be a CMC subset of a ring R. Notice that $A = -A$. For $r \in R$, let

$$(A\colon r) = \{s \in R \mid rs \in A\}.$$

One checks that the set of all such is linearly ordered by inclusion. With this one checks that the CMC subset A is a subring if and only if $t = 1$ satisfies (b) above.

THEOREM 2.15. *There exists a natural bijective correspondence between the class of all isomorphism classes of V-valuations of a ring R and the set of all CMC subsets of R. This correspondence is given by*

$$(v, \Gamma) \mapsto A_v = \{r \in R \mid 0 \leqslant v(r)\}.$$

Furthermore, this induces a natural bijective correspondence between the class of all isomorphism classes of formally finite V-valuations of R and the set of all CMC subrings of R.

Proof. Let (v, Γ) be a V-valuation of R and let $A_v = \{r \in R \mid 0 \leqslant v(r)\}$. One easily verifies that both A_v and $R\backslash A_v$ are closed under multiplication. Let t be a unit of R satisfying (2.10b). One checks that t verifies (2.13b). It is obvious that A_v depends only on the isomorphism class of (v, Γ).

Now suppose A is a CMC subset of R. Define an equivalence relation on R by declaring

$$r \sim s \qquad \text{if} \quad A\colon r = A\colon s.$$

Write $v(r)$ for the equivalence class of r and let Γ_A denote the set of all such equivalence classes. Define addition by

$$v(r) + v(s) = v(rs);$$

one checks this is well-defined, commutative, and associative and that Γ_A has $v(1)$ as its zero and $v(0)$ as ∞. Define

$$v(r) \leqslant v(s) \qquad \text{if} \quad A\colon r \subseteq A\colon s;$$

one observes this is well-defined and that $(\Gamma_A, \leqslant)$ is a totally ordered set. One further checks that $(\Gamma_A, +, \leqslant)$ is a V-monoid and that $v: R \to \Gamma_A$ is a V-valuation. We refer to Γ_A and $v: R \to \Gamma_A$ as the *standard V-monoid* and *standard V-valuation* associated with A.

Using the lemma that follows, one verifies that these associations are inverse to each other. Since a CMC subset A is a subring of R if and only if A is closed under addition, this correspondence induces a bijective correspondence between the class of isomorphism classes of formally finite V-valuations of R and the set of CMC subrings of R.

LEMMA 2.16. *For a V-monoid Γ and $\alpha, \beta \in \Gamma$,*

$$\alpha < \beta \Leftrightarrow P(\Gamma): \alpha \subset P(\Gamma): \beta,$$

where for an arbitrary element $\delta \in \Gamma$,

$$P(\Gamma): \delta = \{\gamma \in \Gamma \mid \gamma + \delta \in P(\Gamma)\}.$$

Proof. This follows from (2.1v).

The construction of the standard V-monoid associated with a CMC subring of a ring was mentioned in an unpublished paper of Griffin [4]. However, the idea was not further developed in that preprint.

3. SOME STRUCTURE AND REALIZATION QUESTIONS

DEFINITIONS 3.1. We say a V-monoid Γ is *complete* if

(i) Every subset of Γ has an infimum in Γ, and

(ii) $\inf(X+Y) = \inf X + \inf Y$, where

$$X+Y = \{x+y \mid x \in X, y \in Y\}.$$

By a *completion* of a V-monoid Γ we mean a pair (φ, Δ), where:

(i) Δ is a complete V-monoid.

(ii) $\varphi: \Gamma \to \Delta$ is an injective order-homomorphism such that $\varphi(0) = 0$ and $\varphi(\infty) = \infty$.

(iii) For all $\delta \in \Delta$, $\exists X \subseteq \Gamma$ such that $\delta = \inf \varphi(X)$.

THEOREM 3.2. *Let Γ be a V-monoid. There exists an essentially unique completion $(\varphi, \Gamma^{\wedge})$ of Γ, where essentially unique means that if (ψ, Δ) is another completion of Γ, then there exists a unique isomorphism $\sigma: \Gamma^{\wedge} \to \Delta$* such that $\sigma \circ \varphi = \psi$.

Proof. Introduce an equivalence relation on the power set of Γ by declaring

$$X \sim Y \Leftrightarrow P(\Gamma): X = P(\Gamma): Y.$$

Here, for an arbitrary subset Z of Γ,

$$P(\Gamma): Z = \{\gamma \in \Gamma \mid \gamma + z \in P(\Gamma)\, \forall z \in Z\}.$$

Write $[X]$ for the equivalence class of X and let $\Gamma^\wedge$ denote the set of all such equivalence classes.

Define

$$[X] + [Y] = [X + Y];$$

one checks this is well-defined, associative, commutative, and has $[\{0\}]$ as its zero.

Define

$$[X] \leqslant [Y] \qquad \text{if} \quad P(\Gamma): X \subseteq P(\Gamma): Y;$$

one checks this is well-defined, $[\varnothing]$ is an absorbent maximum, and that $(\Gamma^\wedge, \leqslant)$ is a totally ordered set. One further checks that $\Gamma^\wedge$ is a V-monoid.

One checks that

$$\inf\{[X_i]\}_{i \in I} = \left[\bigcup_{i \in I} X_i\right]$$

and verifies that $\Gamma^\wedge$ is a complete V-monoid.

Define $\varphi: \Gamma \to \Gamma^\wedge$ by $\varphi(\gamma) = [\{\gamma\}]$. Clearly, φ is a sum-preserving. By (2.16), φ is injective and order-preserving. One observes $[X] = \inf \varphi(X)$ for each $X \subseteq \Gamma$ and hence $(\varphi, \Gamma^\wedge)$ is a completion of Γ.

Let (θ, Σ) and (ψ, Δ) be completions of Γ. Define $\tau: \Sigma \to \Delta$ by

$$\tau(\sigma) = \tau(\inf \theta(X)) = \inf \psi(X).$$

One checks this is well-defined and is an isomorphism using the two lemmas which follow. The uniqueness of τ is easily checked.

LEMMA 3.3. *In a complete V-monoid Λ, for any subset X* of Λ,

$$\lambda = \inf X \Leftrightarrow P(\Lambda): \lambda = P(\Lambda): X.$$

Proof. Let $\lambda = \inf X$. Clearly, $P(\Lambda): \lambda \subseteq P(\Lambda): X$. Suppose

$$0 \leqslant x + \rho \qquad \forall x \in X.$$

Then,

$$0 \leqslant \inf(X + \{\rho\}) = \inf X + \rho = \lambda + \rho,$$

which says that $\rho \in P(\Lambda): \lambda$. Hence $P(\Lambda): \lambda = P(\Lambda): X$.

Now suppose $P(\Lambda): \lambda = P(\Lambda): X$. Just suppose $x < \lambda$ for some $x \in X$. Then, there exists $\rho \in \Lambda$ such that

$$x + \rho < 0 \leqslant \lambda + \rho,$$

contradicting the assumption that $P(\Lambda): \lambda = P(\Lambda): X$. Hence $\lambda \leqslant \inf X$. Just suppose $\lambda < \inf X$. Then there exists $\rho \in \Lambda$ such that

$$\lambda + \rho < 0 \leqslant \inf X + \rho \leqslant x + \rho \qquad \forall x \in X,$$

again contradicting our hypothesis. Thus $\lambda = \inf X$.

LEMMA 3.4. *If (θ, Σ) is a completion of the V-monoid Γ and $X, Y \subseteq \Gamma$, then*

$$P(\Gamma): X \subseteq P(\Gamma): Y \Leftrightarrow P(\Sigma): \theta(X) \subseteq P(\Sigma): \theta(Y).$$

Proof. Assume that $P(\Gamma): X \subseteq P(\Gamma): Y$. Suppose

$$0 \leqslant \theta(x) + \sigma \qquad \forall x \in X.$$

Write $\sigma = \inf \theta(Z)$ for some $Z \subseteq X$. Then

$$0 \leqslant \theta(x) + \theta(z) = \theta(x + z) \qquad \forall x \in X, z \in Z.$$

Hence

$$0 \leqslant x + z \qquad \forall x \in X, z \in Z$$

as θ is injective and order-preserving. Thus

$$Z \subseteq P(\Gamma): X \subseteq P(\Gamma): Y$$

so that

$$0 \leqslant y + z \qquad \forall y \in Y, z \in Z.$$

Thus

$$0 \leqslant \theta(y) + \theta(z) \qquad \forall y \in Y, z \in Z$$

and hence

$$\begin{aligned} 0 \leqslant \inf(\theta(Y)+\theta(Z)) &= \inf \theta(Y) + \inf \theta(Z) \\ &= \inf \theta(Y) + \sigma \\ &\leqslant \theta(y) + \sigma \qquad \forall y \in Y. \end{aligned}$$

Therefore $\sigma \in P(\Sigma)\colon \theta(Y)$.

Conversely, assume that

$$P(\Sigma)\colon \theta(X) \subseteq P(\Sigma)\colon \theta(Y).$$

Suppose

$$0 \leqslant x+\gamma \qquad \forall x \in X.$$

Then

$$0 \leqslant \theta(x)+\theta(\gamma) \qquad \forall x \in X,$$

which implies

$$0 \leqslant \theta(y)+\theta(\gamma)=\theta(y+\gamma) \qquad \forall y \in Y.$$

Since θ is injective and order-preserving

$$0 \leqslant y+\gamma \qquad \forall y \in Y.$$

Thus $\gamma \in P(\Gamma)\colon Y$.

DEFINITION 3.5. We call a V-monoid Γ *Boolean* if

$$\gamma+\gamma=\gamma \qquad \forall \gamma \in \Gamma.$$

PROPOSITION 3.6. *For a Boolean V-monoid Γ, the completion $\Gamma^\wedge$ of Γ is again Boolean.*

Proof. Let $\sigma \in \Gamma^\wedge$. Write $\sigma = \inf \varphi(X)$ for some $X \subseteq \Gamma$. As Γ is Boolean, $X+X=X$ by the lemma which follows. Hence

$$\sigma+\sigma=\inf \varphi(X)+\inf \varphi(X)=\inf \varphi(X+X)=\inf \varphi(X)=\sigma.$$

LEMMA 3.7. *For Γ a totally ordered commutative monoid and for e and f idempotents of Γ, either $e+f=e$* or *$e+f=f$.*

Proof. Without loss, assume that $e \leqslant f$. One checks that

$$0 \leqslant e \Rightarrow e+f=f \qquad \text{and} \qquad f \leqslant 0 \Rightarrow e+f=e.$$

If $e<0<f$, then $e+f$ is e or f according to whether the sum is negative or positive.

EXAMPLE 3.8. Let L be a totally ordered set with a maximum, which we will denote by ∞. Let L^* denote the dual totally ordered set. Let

$$\Gamma(L)=L^* \cup \{0\} \cup L,$$

where the indicated union is disjoint. Define a commutative addition on L by

$$x+y=\begin{cases} \max\{x, y\} & \text{if } x, y \in L \\ \min\{x, y\} & \text{if } x, y \in L^* \\ y & \text{if } x=z^* \in L^*,\ y \in L, \text{ and } z \leqslant y \\ x & \text{if } x=z^* \in L^*,\ y \in L, \text{ and } y<z. \end{cases}$$

Define $\leqslant$ on $\Gamma(L)$ so as to preserve the given orderings on L and L^* and so that

$$L^*=N^-(\Gamma(L)) \qquad \text{and} \qquad L=P^+(\Gamma(L)).$$

A tedious, but straightforward, check shows that $(\Gamma(L), +, \leqslant)$ is a Boolean V-monoid.

THEOREM 3.9.. *For Γ a complete Boolean V-monoid,*

$$\Gamma \simeq \Gamma(L),$$

where

$$L=P^+(\Gamma).$$

Proof. The theorem follows from the next two lemmas.

LEMMA 3.10. *For Γ a complete V-monoid, define $\sigma: \Gamma \rightarrow \Gamma$ by*

$$\sigma(\gamma)=\inf(P(\Gamma): \gamma).$$

Then, σ is order-reversing and satisfies $\sigma^2=1$.

Proof. By (3.3), $0 \leqslant \gamma+\sigma(\gamma)\ \forall \gamma \in \Gamma$. Hence

$$0 \leqslant \gamma+\beta \Leftrightarrow \sigma(\gamma) \leqslant \beta \qquad \forall \gamma, \beta \in \Gamma.$$

Suppose $\alpha < \beta$ in Γ. Then there exists $\delta \in \Gamma$ such that

$$\alpha + \delta < 0 \leqslant \beta + \delta.$$

Hence

$$\sigma(\beta) \leqslant \delta < \sigma(\alpha).$$

Thus σ is order-reversing.

Since $0 \leqslant \sigma(\gamma) + \gamma$, $\sigma^2(\gamma) \leqslant \gamma$. Just suppose $\sigma^2(\gamma) < \gamma$. Then there exists $\delta \in \Gamma$ such that

$$\sigma^2(\gamma) + \delta < 0 \leqslant \gamma + \delta.$$

Since $0 \leqslant \gamma + \delta$, $\sigma(\gamma) \leqslant \delta$. Thus $\sigma(\delta) \leqslant \sigma^2(\gamma)$. Hence

$$0 \leqslant \delta + \sigma(\delta) \leqslant \delta + \sigma^2(\gamma),$$

contracting the assumption that $\sigma^2(\gamma) + \delta < 0$.

LEMMA 3.11. *For Γ a complete Boolean V-monoid, the restriction of σ to $P^+(\Gamma)$ defines an order-reversing, sum-preserving bijection from $P^+(\Gamma)$ onto $N^-(\Gamma)$.*

Proof. By (3.10), the restriction is an order-reversing bijection from $P^+(\Gamma)$ onto $N^-(\Gamma)$.

Say $\gamma \leqslant \delta$ in $P^+(\Gamma)$. Then $\sigma(\delta) \leqslant \sigma(\gamma)$ and hence (3.7) implies

$$\gamma + \delta = \delta \qquad \text{and} \qquad \sigma(\gamma + \delta) = \sigma(\delta) = \sigma(\gamma) + \sigma(\delta).$$

COROLLARY 3.12. *For a finite V-monoid Γ, there exists a unique non-negative integer n such that $\Gamma \cong \Gamma_n$.*

Proof. Using (2.5), one verifies that Γ is Boolean. Hence Γ is a complete Boolean V-monoid. By (3.9),

$$\Gamma \cong \Gamma(P^+(\Gamma)) \cong \Gamma_n,$$

where $|P^+(\Gamma)| = n = |P^+(\Gamma_n)|$.

EXAMPLE 3.13. For Γ a nontrivial V-monoid and $\Sigma = \Gamma \backslash \{0, \infty\}$, we write

$$R(\Gamma) = \mathbf{Z}[X_\gamma]_{\gamma \in \Sigma},$$

where $\{X_\gamma\}_{\gamma \in \Sigma}$ are indeterminates. For $\gamma \in \Sigma$ and a monomial

$M = X_{\gamma_1} \cdots X_{\gamma_n}$ such that $\gamma_1 + \cdots + \gamma_n = \gamma$, we say M has degree γ. Each $a \in \mathbf{Z}$ has degree 0. For $\gamma \in \Gamma' = \Gamma \backslash \{\infty\}$ we write R_γ for the $\mathbf{Z}$-span of all monomials of degree γ (we are regarding 1 as a monomial of degree 0). Writing R in lieu of $R(\Gamma)$, one checks that $R = \bigoplus_{\gamma \in \Gamma'} R_\gamma$ and that $R_\gamma R_\alpha \subseteq R_{\gamma+\alpha} \forall \gamma, \alpha \in \Gamma'$. Finally, we write

$$A(\Gamma) = \sum_{\gamma \in P(\Gamma')} R_\gamma.$$

THEOREM 3.14. *For a nontrivial V-monoid Γ such that either Γ is Boolean or Γ' is cancellative, $A(\Gamma)$ is a CMC subring of $R(\Gamma)$ with standard V-monoid naturally isomorphic to Γ.*

Proof. For a nonzero element r of R, write

$$r = r_{\gamma_1} + \cdots + r_{\gamma_n}$$

where $\gamma_1 < \cdots < \gamma_n$ and $0 \neq r_{\gamma_i} \in R_{\gamma_i}$ $(i = 1, \cdots, n)$. We refer to r_{γ_1} as the *initial component* of r. Define $v(r) = \gamma_1$. Define $v(0) = \infty$.

One checks that for $r, s \in R$,

$$\min\{v(r), v(s)\} \leqslant v(r+s).$$

Now suppose

$$0 \neq r = r_{\gamma_1} + \cdots + r_{\gamma_n}$$

$$0 \neq s = s_{\alpha_1} + \cdots + s_{\alpha_m}.$$

In the cancellative case, one observes that the initial component of rs is $r_{\gamma_1} s_{\alpha_1}$ and hence, $v(rs) = v(r) + v(s)$.

Hence we assume Γ is Boolean. Without loss, by (3.7), we may assume that $\gamma_1 + \alpha_1 = \gamma_1$.

Case 1. $\gamma_1 < \alpha_1$. Using (3.7), we see that for $\gamma_i > \gamma_1, \gamma_i + \alpha_j \neq \gamma_1$ $(1 \leqslant j \leqslant n)$. Thus the γ_1-component of rs is

$$r_{\gamma_1}\left(\sum s_{\alpha_j}\right),$$

where we sum over all α_j such that $\gamma_1 + \alpha_j = \gamma_1$. As the rightmost factor is a sum of nonzero components of distinct degrees, $\sum s_{\alpha_j} \neq 0$. Hence $r_{\gamma_1}(\sum s_{\alpha_j})$ is nonzero and is the initial component of rs. Hence

$$v(rs) = v(r) + v(s).$$

Case 2. $\alpha_1 < \gamma_1$. This case is handled similarly to case 1.

Case 3. $\alpha_1=\gamma_1$. When $\gamma_1=0$ the γ_1 component of rs is $r_{\gamma_1}s_{\alpha_1}$. So assume $\gamma_1\neq 0$. By (2.5) the γ_1-component of rs is

$$r_{\gamma_1}s_{\alpha_1}+r_{\gamma_1}\left(\sum s_{\alpha_i}\right)+s_{\alpha_1}\left(\sum r_{\gamma_j}\right),$$

where the first sum is over all α_i such that $\alpha_1<\alpha_i$ and $\alpha_i+\gamma_1=\gamma_1$ and the second sum is over all γ_j such that $\gamma_1<\gamma_j$ and $\gamma_j+\alpha_1=\alpha_1$.

We write $B=\mathbf{Z}[X_\beta]_{\beta\in S}$, where

$$S=\{\delta\in\Sigma\mid\delta\neq\gamma_1,\delta+\gamma_1=\gamma_1\}.$$

Write Y in lieu of X_{γ_1}. By Lemma 3.16 to follow, $R_{\gamma_1}=YB[Y]$. One checks, using (3.16), that $-r_{\gamma_1}s_{\alpha_1}$ and $r_{\gamma_1}(\sum s_{\alpha_i})+s_{\alpha_1}(\sum r_{\gamma_j})$ are in $B[Y]$, and that $(\sum s_{\alpha_i})$ and $(\sum r_{\gamma_j})$ are in B. Hence, examining the degree in Y, we see that

$$-r_{\gamma_1}s_{\alpha_1}\neq r_{\gamma_1}\left(\sum s_{\alpha_i}\right)+s_{\alpha_1}\left(\sum r_{\gamma_j}\right).$$

Thus, the initial component of rs has degree γ_1, that is,

$$v(rs)=v(r)+v(s).$$

Summarizing, $v\colon R\to\Gamma$ is a formally finite V-valuation with $A_v=A(\Gamma)$.

LEMMA 3.15. *For a Boolean V-monoid Γ, any positive integer n, and $\gamma_1,\cdots,\gamma_n$ in Γ,*

$$\gamma_1+\cdots+\gamma_n=\gamma_1\Leftrightarrow\gamma_1+\gamma_j=\gamma_1\qquad(j=1,\cdots,n).$$

Proof. One uses induction on n.

LEMMA 3.16. *For a nontrivial Boolean V-monoid Γ and $\gamma\in\Gamma\backslash\{0,\infty\}$,*

$$R_\gamma=X_\gamma\mathbf{Z}[X_\alpha]_{\alpha\in A_\gamma},$$

where $A_\gamma=\{\beta\in\Gamma\backslash\{0,\infty\}\mid\beta+\gamma=\gamma\}$.

Proof. This follows easily from (3.15).

EXAMPLE 3.17. For G a totally ordered abelian group and Δ a V-monoid we define

$$G\oplus\Delta=G\times\Delta'\cup\{\infty\},$$

where $\Delta'=\Delta\backslash\{\infty\}$ and + and $\leqslant$ are defined as follows: addition on

$G \times \Delta'$ is component-wise and ∞ is absorbent; ∞ is maximal and $G \times \Delta'$ is ordered lexicographically. One checks that $G \otimes \Delta$ is a V-monoid.

Remark 3.18. For G and H totally ordered abelian groups and Δ a V-monoid, one checks that

$$G \otimes (H \otimes \Delta) \cong (G \times H) \otimes \Delta,$$
$$1 \otimes \Delta \cong \Delta \qquad \text{and} \qquad G \otimes 1 \cong 1,$$

where $G \times H$ is the lexicographic product of G and H and 1 denotes the trivial group.

Notation and Remarks 3.19. Recall that for a V-valuation $v: R \to \Gamma$ we write

$$A_v = \{r \in R \mid 0 \leqslant v(r)\}.$$

A_v is a CMC subset of R. For Γ nontrivial, we further define

$$P_v = \{r \in R \mid 0 < v(r)\}$$

and

$$I_v = \{r \in R \mid v(r) = \infty\}.$$

Notice that $P_v = \{r \in R \mid \exists s \in R \backslash A_v \text{ s.t. } rs \in A_v\}$ and that $I_v = (A_v : R) = (P_v : R)$ is a prime ideal of R. If $A_v + I_v \subseteq A_v$, then v naturally induces a V-valuation

$$\bar{v}: R/I_v \to \Gamma$$

such that $A_{\bar{v}} = A_v/I_v$ and $I_{\bar{v}} = \{0\}$.

In the formally finite case, P_v is precisely the set of zero divisors of the A_v-module R/A_v and we always have $A_v + I_v \subseteq A_v$. Thus if Γ is realizable as the target of a formally finite V-valuaton v, we may assume that the source of v is an integral domain and that $I_v = \{0\}$.

THEOREM 3.20. *Suppose $\bar{w}: R \to \Delta$ is a formally finite V-valuation of an integral domain R such that $I_{\bar{w}} = \{0\}$. Suppose G is a totally ordered abelian group. Then, there exists a formally finite V-valuation*

$$v: R(G) \to G \otimes \Delta,$$

where $R(G)$ denotes the group ring of G over R.

Proof. We may and shall assume G is nontrivial. Write $R(G)=\bigoplus_{g\in G} Rt_g$. For $0\neq x=\sum r_g t_g$ in $R(G)$ let

$$\text{Supp}(x)=\{g\in G \mid r_g\neq 0\}.$$

Define u: $R(G)\to G_\infty$ by

$$u(x)=\begin{cases}\min \text{Supp}(x) & \text{if } x\neq 0\\ \infty & \text{if } x=0.\end{cases}$$

One checks that u is a formally finite V-valuation and that $A_u/P_u\cong R$.

Let w: $A_u\to\Delta$ be the canonical homomorphism $A_u\to A_u/P_u$ followed by $\bar{w}$: $R\to\Delta$, where we identify A_u/P_u with R.

Define v: $R(G)\to G\otimes\Delta$ by

$$v(x)=\begin{cases}(u(x), w(t_{-u(x)}x)) & \text{if } x\neq 0\\ \infty & \text{if } x=0.\end{cases}$$

One checks that v is a formally finite V-valuation with image $G\otimes\Delta$.

DEFINITION 3.21. We say a V-monoid Γ is *divisible* if for all $\gamma\in\Gamma$ and all $n\in\mathbf{N}^*=\mathbf{N}\backslash\{0\}$,

$$nx=\gamma$$

has a solution in Γ.

Remark 3.22. In a V-monoid Γ, $\alpha<\beta$ implies $n\alpha<n\beta$ for all $n\in\mathbf{N}^*$; this follows from (2.5) and induction on n. Hence in a divisible V-monoid, for each $\gamma\in\Gamma$ and each $n\in\mathbf{N}$, $nx=\gamma$ has a unique solution in Γ.

DEFINITION 3.23. By a *divisible hull* of a V-monoid Γ we mean a pair (θ, Δ) where:

(i) Δ is a divisible V-monoid.

(ii) θ: $\Gamma\to\Delta$ is an injective order-homomorphism such that $\theta(0)=0$ and $\theta(\infty)=\infty$.

(iii) For all $\delta\in\Delta$, $\exists n\in\mathbf{N}^*$ such that $n\delta\in im\,\theta$.

THEOREM 3.24. *Let Γ be a V-monoid. There exists an essentially unique divisible hull $(\theta, D(\Gamma))$ of Γ, where essentially unique means that if (ψ, Δ) is another divisible hull of Γ, then there exists a unique isomorphism σ: $D(\Gamma)\to\Delta$ such that $\sigma\circ\theta=\psi$.*

Proof. Define an equivalence relation on $\Gamma \times \mathbf{N}^*$ by

$$(\alpha, n) \sim (\beta, m) \Leftrightarrow m\alpha = n\beta.$$

Write α/n for the class of (α, n) and let $D(\Gamma)$ denote the set of all such equivalence classes.

Define

$$\alpha/n + \beta/m = (m\alpha + n\beta)/nm.$$

One checks this is well-defined, associative, and commutative with $0/1$ as the zero element.

Define $\leqslant$ on $D(\Gamma)$ by

$$\alpha/n \leqslant \beta/m \Leftrightarrow m\alpha \leqslant n\beta.$$

One checks that $\leqslant$ is well-defined and that $(D(\Gamma), \leqslant)$ is a totally ordered set with absorbent maximum $\infty/1$. Finally, one checks that $(D(\Gamma), +, \leqslant)$ is a V-monoid.

Define $\theta: \Gamma \to D(\Gamma)$ by $\theta(\gamma) = \gamma/1$ for all $\gamma \in \Gamma$. It is clearly an order-homomorphism and it is injective by (3.22).

Now suppose (ψ, Δ) is another divisible hull of Γ. For $\alpha/n \in D(\Gamma)$, let $\sigma(\alpha/n) = \delta$, where $\delta \in \Delta$ is the unique solution to $nx = \psi(\alpha)$. One checks this is well-defined and is an isomorphism of V-monoids. The uniqueness of σ is easily checked.

DEFINITION 3.25. Let Γ be a V-monoid.

If for all $\alpha, \beta \in P^+(\Gamma')$, $\exists$ a positive integer n such that $\beta \leqslant n\alpha$ we say *$P(\Gamma)$ is Archimedean.*

If for all $\alpha, \beta \in N^-(\Gamma)$, $\exists$ a positive integer n such that $n\alpha \leqslant \beta$ we say *$N(\Gamma)$ is Archimedean.*

If $P(\Gamma)$ and $N(\Gamma)$ are Archimedean we say *Γ is Archimedean.*

THEOREM 3.26. *Suppose Γ is an Archimedean V-monoid with a nontrivial unit; let $\alpha \in \Gamma$ be a positive unit. Then, there exists an injective order-homomorphism $\theta: \Gamma' \to \mathbf{R}$ into the additive reals such that $\theta(\alpha) = 1$. In particular, Γ' is cancellative.*

Proof. For $\beta < \infty$, let

$$S_\beta = \{m/n \in \mathbf{Q} \mid n > 0, m\alpha \leqslant n\beta\}.$$

One checks that S_β is a nonempty proper subset of $\mathbf{Q}$ and that if $m/n \in S_\beta$ and $p/q \leqslant m/n$, then $p/q \in S_\beta$. Hence S_β is bounded from above. Let

$$\theta(\beta) = \sup S_\beta.$$

One checks that $S_\beta + S_\gamma \subseteq S_{\beta+\gamma}$. Hence $\sup(S_\beta + S_\gamma) \leqslant \sup S_{\beta+\gamma}$. Thus, $\theta(\beta) + \theta(\gamma) \leqslant \theta(\beta+\gamma)$ for all $\beta, \gamma \in \Gamma'$.

Let

$$T_\beta = \{m/n \in \mathbf{Q} \mid n > 0, m\alpha > n\beta\}.$$

Then $T_\beta = \mathbf{Q} \backslash S_\beta$. One checks that $\inf T_\beta = \sup S_\beta$.

One checks that $T_\beta + T_\gamma \subseteq T_{\beta+\gamma}$. Thus $\inf T_{\beta+\gamma} \leqslant \inf(T_\beta + T_\gamma)$. Hence $\theta(\beta+\gamma) \leqslant \theta(\beta) + \theta(\gamma)$. Combining, we have $\theta(\beta+\gamma) = \theta(\beta) + \theta(\gamma)$ for all $\beta, \gamma \in \Gamma'$.

Notice that $S_0 = \{m/n \in \mathbf{Q} \mid n > 0, m\alpha \leqslant 0\} = \{m/n \in \mathbf{Q} \mid n > 0, m \leqslant 0\}$. Hence $\theta(0) = 0$.

One checks that if $\beta \leqslant \gamma$, then $S_\beta \subseteq S_\gamma$ and, consequently, $\theta(\beta) \leqslant \theta(\gamma)$. If $\beta < \gamma$, then $\exists\, \delta \in \Gamma'$ such that $\beta + \delta < 0 \leqslant \gamma + \delta$. Since $\beta + \delta < 0$, $S_{\beta+\delta} \subseteq N^-(\mathbf{Q})$. Since $0 \leqslant \gamma + \delta$, $0/1 \in S_{\gamma+\delta}$. Hence $\theta(\beta) < \theta(\gamma)$. Therefore, θ is an injective order-homomorphism.

We will find the following two results useful in the last section.

LEMMA 3.27. *Let Γ be a V-monoid with a negative unit β and suppose $N(\Gamma)$ is Archimedean. Then, Γ is Archimedean.*

Proof. Let $\alpha = -\beta$. Suppose $0 < \gamma < \infty$. Choose $\delta \in \Gamma$ such that $\gamma + \delta < 0$. Note that $\delta < 0$. By hypothesis, there exists $m \geqslant 1$ such that $m\beta \leqslant \delta$. Now $\gamma + m\beta \leqslant \gamma + \delta < 0$ implies $\gamma + m\beta < 0$ and hence $\gamma < m\alpha$. Thus for every $\gamma \in P^+(\Gamma')$, there exists $m \geqslant 1$ such that $\gamma < m\alpha$.

Again, suppose $0 < \gamma < \infty$. Choose ε in Γ such that $\varepsilon < 0 \leqslant \gamma + \varepsilon$. By hypothesis, there exists $n \geqslant 1$ such that $n\varepsilon < \beta$. Now $0 \leqslant n(\gamma + \varepsilon) = n\gamma + n\varepsilon \leqslant n\gamma + \beta$ implies $\alpha \leqslant n\gamma$. Thus for every $\gamma \in P^+(\Gamma')$ there exists $n \geqslant 1$ such that $\alpha \leqslant n\gamma$.

Now suppose $0 < \gamma, \delta < \infty$. Without loss, assume that $\gamma < \delta$. Choose $m, n \geqslant 1$ such that $\delta < m\alpha$ and $\alpha \leqslant n\gamma$. Then $\delta < m\alpha \leqslant mn\gamma$. Hence $P(\Gamma)$ is Archimedean and, therefore, Γ is Archimedean.

LEMMA 3.28. *Let Γ be a V-monoid with a negative unit β and suppose that for each $\gamma \in N^-(\Gamma)$ there exist positive integers m and n such that $n\beta \leqslant \gamma$ and $m\gamma < \beta$. Then, Γ is Archimedean.*

Proof. By (3.27) it suffices to show that $N(\Gamma)$ is Archimedean. Suppose $\gamma, \delta \in N^-(\Gamma)$. Without loss, assume that $\gamma < \delta$. Choose positive integers m and n such that $m\delta < \beta$ and $n\beta \leqslant \gamma$. Then

$$nm\delta < n\beta \leqslant \gamma.$$

Hence $N(\Gamma)$ is Archimedean.

DEFINITION AND NOTATION 3.29. For a V-monoid Γ, let $\mathscr{F}(\Gamma)$ denote the set of all convex submonoids of $N(\Gamma)$. One checks that $\mathscr{F}(\Gamma)$ is totally ordered by inclusion. We define the *rank of* Γ as the order type of $\mathscr{F}(\Gamma)$.

THEOREM 3.30. *A V-monoid Γ has rank* $1 \Leftrightarrow N(\Gamma)$ *is Archimedean.*

Proof. Assume that Γ has rank one. Then $\{0\} \subset N(\Gamma)$ are the sole convex submonoids of $N(\Gamma)$. Suppose $\alpha < \beta < 0$ and suppose, to the contrary, that $\alpha < n\beta\ \forall n \geqslant 1$. Let $K = \{\gamma \in N(\Gamma) \mid \exists n \geqslant 1 \text{ such that } n\beta \leqslant \gamma\}$. One checks that K is a convex submonoid of $N(\Gamma)$. Since $\{0\} \subset K \subset N(\Gamma)$, this is a contradiction.

Now assume that $N(\Gamma)$ is Archimedean. Suppose $\{0\} \subset K$ and K is a convex submonoid of $N(\Gamma)$. Suppose, to the contrary, that there exists $\alpha \in N(\Gamma) \backslash K$. Choose $\beta < 0$ in K. Since K is convex we must have $\alpha < \beta$. Let $n \geqslant 1$ be such that $n\beta \leqslant \alpha$. Then $n\beta \in K$ implies $\alpha \in K$, a contradiction.

4. CMC SUBRINGS

DEFINITIONS 4.1. We define a *maxoid* as a totally ordered commutative monoid with an absorbent maximum, which we denote by ∞.

We say a submonoid Λ of a maxoid Γ is *convex* if $\alpha < \beta < \gamma$ and $\alpha, \gamma \in \Lambda$ implies $\beta \in \Lambda$.

Given any submonoid Λ of a maxoid Γ, let

$$U(\Lambda) = \{\gamma \in \Gamma \mid \exists \alpha \in \Lambda \text{ such that } \alpha \leqslant \gamma\}.$$

Notice that both $U(\Lambda)$ and $\Gamma \backslash U(\Lambda)$ are closed under addition.

EXAMPLE 4.2. For a maxoid Γ, let K be a convex submonoid of $N(\Gamma)$. Notice that $U(K) = K + P(\Gamma)$. Define an equivalence relation on Γ by

$$\alpha \sim \beta \Leftrightarrow U(K){:}\ \alpha = U(K){:}\ \beta.$$

Write $[\alpha]$ for the class of α and let Γ/K denote the set of all such equivalence classes.

Define a commutative addition on Γ/K by

$$[\alpha] + [\beta] = [\alpha + \beta];$$

one checks this is well-defined and associative with $[0]$ as its zero.

Define $\leqslant$ on Γ/K by

$$[\alpha] \leqslant [\beta] \Leftrightarrow U(K){:}\ \alpha \subseteq U(K){:}\ \beta.$$

One checks that $(\Gamma/K, \leqslant)$ is a totally ordered set with absorbent maximum $[\infty]$. Finally, one verifies that $(\Gamma/K, +, \leqslant)$ is a V-monoid.

Let $\tau: \Gamma \to \Gamma/K$ denote the canonical map. Then τ is a surjective order-homomorphism. Letting $K(\tau) = \{\gamma \in \Gamma \mid \gamma \leqslant 0, \tau(\gamma) = [0]\}$, one checks that $K(\tau) = K$.

Notation 4.3. For a surjective order-homomorphism of maxoids $\sigma: \Gamma \to \Delta$, we let

$$K(\sigma) = \{\gamma \in \Gamma \mid \gamma \leqslant 0, \sigma(\gamma) = 0\}.$$

THEOREM 4.4. *Let Γ be a maxoid, Δ a V-monoid, and let $\sigma: \Gamma \to \Delta$ be a surjective order-homomorphism. Then, $K(\sigma)$ is a convex submonoid of $N(\Gamma)$ and there exists a unique map $\varphi: \Gamma/K(\sigma) \to \Delta$ such that $\varphi \circ \tau = \sigma$ and φ is an isomorphism of V-monoids.*

Proof. Clearly, $K(\sigma)$ is a submonoid of $N(\Gamma)$. Suppose $\alpha \in K(\sigma)$ and $\alpha < \beta < 0$. Then, $0 = \sigma(\alpha) \leqslant \sigma(\beta) \leqslant \sigma(0) = 0$ and, hence, $\sigma(\beta) = 0$. Thus $K(\sigma)$ is convex.

Define $\varphi: \Gamma/K(\sigma) \to \Delta$ by $\varphi([\gamma]) = \sigma(\gamma)$. Let $K = K(\sigma)$ and notice that $\gamma \in U(K) \Leftrightarrow 0 \leqslant \sigma(\gamma)$. To see φ is well-defined, suppose $\sigma(\alpha) < \sigma(\beta)$. Since φ is surjective, there exists $\gamma \in \Gamma$ such that

$$\sigma(\alpha + \gamma) = \sigma(\alpha) + \sigma(\gamma) < 0 \leqslant \sigma(\beta) + \sigma(\gamma) = \sigma(\beta + \gamma).$$

Thus $0 \notin U(K): \alpha + \gamma$, and $0 \in U(K): \beta + \gamma$. Hence $[\alpha] + [\gamma] < [\beta] + [\gamma]$ and, consequently, $[\alpha] < [\beta]$. Thus φ is well-defined. Clearly φ is the unique map such that $\varphi \circ \tau = \sigma$.

φ is a surjective order-homomorphism. One checks that $K(\varphi) = \{[0]\}$. By the following lemma φ is an isomorphism.

LEMMA 4.5. *Let $\sigma: \Gamma \to \Delta$ be a surjective order-homomorphism of V-monoids such that $K(\sigma) = \{0\}$. Then, σ is an isomorphism.*

Proof. Suppose $\sigma(\alpha) = \sigma(\beta)$ in Δ. We may and shall assume that $\alpha \leqslant \beta$ in Γ. Suppose, to the contrary, that $\alpha < \beta$. Then, there exists $\gamma \in \Gamma$ such that $\alpha + \gamma < 0 \leqslant \beta + \gamma$. Thus $\sigma(\alpha) + \sigma(\gamma) = \sigma(\alpha + \gamma) < 0 \leqslant \sigma(\beta + \gamma) = \sigma(\beta) + \sigma(\gamma)$, contradicting the assumption that $\sigma(\alpha) = \sigma(\beta)$. Thus σ is injective and, hence, is an isomorphism.

DEFINITIONS 4.6. Let $A \subseteq R$ be a CMC subring. We say a subring B of R is an *intermediate ring* if $A \subseteq B$, B is CMC in R, and

$$B: x \subset B: y \Rightarrow A: x \subset A: y \qquad \forall x, y \in R.$$

Let $v\colon R \to \Gamma$ be the standard V-valuation associated with A. We say a formally finite V-valuation $w\colon R \to \Delta$ is *composite with* v if there exists an order-homomorphism $\sigma\colon \Gamma \to \Delta$ such that $\sigma \circ v = w$.

THEOREM 4.7. *Let $v\colon R \to \Gamma$ be a formally finite V-valuation and let $A = A_v$. There exists a bijective correspondence between the set of all intermediate rings to A in R and the class of all isomorphism classes of formally finite V-valuations composite with v.*

Proof. Let B be intermediate to A in R. Let $w\colon R \to \Delta$ be the standard V-valuation associated with B. Define $\sigma\colon \Gamma \to \Delta$ by $\sigma(v(r)) = w(r)\ \forall r \in R$; one checks that σ is a well-defined order-homomorphism using the assumption that B is intermediate. Hence w is composite with v.

Suppose $w\colon R \to \Delta$ is a formally finite V-valuation and $\sigma\colon \Gamma \to \Delta$ is an order-homomorphism such that $\sigma \circ v = w$. Letting $B = A_w$, one checks that B is intermediate to A in R. One checks that this depends only on the isomorphism class of w and, finally, that these associations are inverse to each other.

COROLLARY 4.8. *Let A be a CMC subring of a ring R and let $v\colon R \to \Gamma$ be the corresponding V-valuation. Then, there exists a bijective correspondence between the set of convex submonoids of $N(\Gamma)$ and the set of all rings intermediate to A in R given by $K \mapsto A_{\tau \circ v}$, where $\tau\colon \Gamma \to \Gamma/K$ is the canonical map.*

Proof. This follows from (4.7), (4.4), and (2.15).

DEFINITION 4.9. Let A be a subring of a ring R. We say an ideal P of A is *strongly prime in* R if $P \subset A$ and $R \setminus P$ is closed under multiplication.

Notation and Remark 4.10. For $A \subset R$ a CMC subring, let

$$P = Z(R/A) \qquad \text{and} \qquad I = (A\colon R),$$

where $Z(R/A)$ denotes the set of zero divisors of the A-module R/A. Note that P is strongly prime in R. In addition, if $v\colon R \to \Gamma$ is the standard V-valuation associated with A, then $P = \{r \in R \mid 0 < v(r)\}$ and $I = \{r \in R \mid v(r) = \infty\}$.

For $B \subset R$ an intermediate ring, let

$$Q = Z(R/B).$$

Notice that Q is strongly prime in R and that $I \subseteq Q \subseteq P$.

Proof. The first containment is apparent. Suppose $b \in Q$; choose $s \in R \backslash B$ such that $bs \in B$. Since

$$B: s \subset B \subseteq B: bs$$

and B is intermediate, there exists $a \in R$ such that as $\notin A$ and $abs \in A$. Thus $b \in Z(R/A) = P$.

LEMMA 4.11. *Suppose A is a proper CMC subring of a ring R and that the A-ideal Q is strongly prime in R. Then, for each $x \in R$ either $Q \subset A: x$ or $A: x \subseteq Q$.*

Proof. Suppose there exists $b \in Q \backslash (A : x)$ for some $x \in R$. Let $a \in (A : x)$. Now $b \in Q$ and $ax \in A$ implies bax $\in Q$. Since $bx \in R \backslash A \subseteq R \backslash Q$ and Q is strongly prime, $a \in Q$. Thus $A : x \subseteq Q$ or $Q \subset A : x$ for all $x \in R$.

LEMMA 4.12. *Let A and B be CMC subrings of a ring R such that $A \subseteq B$. Then, B is intermediate to A in R if and only if for all $s \in R \backslash B$ and $b \in B$, there exists $a \in R$ such that $as \notin A$ and $ab \in A$.*

Proof. Suppose B is intermediate to A in R. Let $s \in R \backslash B$ and $b \in B$. Then $B : s \subset B \subseteq B : b$ and hence $A : s \subset A : b$. Thus there exists $a \in R$ such that $as \notin A$ and $ab \in A$.

Now assume that for all $s \notin B$ and $b \in B$ there exists $a \in R$ such that $as \notin A$ and $ab \in A$. Suppose $B : x \subset B : y$. Then there exists $u \in R$ such that $ux \notin B$ and $uy \in B$. By hypothesis, there exists $a \in R$ such that $aux \notin A$ and $auy \in A$. Hence $A : x \subset A : y$.

Notation 4.13. Suppose A is a proper CMC subring of a ring R. Let Q be a strongly prime A-ideal such that $I \subseteq Q \subseteq P$. Define $B(Q)$ and $B\langle Q \rangle$ by

$$B(Q) = \{r \in R \mid Q \subset A : r\}$$

and

$$B\langle Q \rangle = \{r \in R \mid Q \subseteq A : r\}.$$

We point out that $B\langle Q \rangle$ need not be a subring of R.

LEMMA 4.14. *Suppose A is a proper CMC subring of a ring R. Let Q be a strongly prime A-ideal such that $I \subseteq Q \subseteq P$. Then, $B(Q)$ is an intermediate ring, Q is an ideal of $B(Q)$, and $Z(R/B(Q)) \subseteq Q$. For C an intermediate ring such that $Q = Z(R/C)$, either $C = B(Q)$ or $C = B\langle Q \rangle$.*

Proof. One checks, using the fact that Q is strongly prime in R, that $B=B(Q)$ is a subring of R.

Suppose $r, s \in R\backslash B$. Then, by (4.11), $A:r \subseteq Q$ and $A:s \subseteq Q$. Suppose $brs \in A$. Then $br \in A:s \subseteq Q$ and $r \notin Q$ implies $b \in Q$. Hence $A:rs \subseteq Q$, i.e., $rs \in R\backslash B$. Thus B is CMC in R.

Suppose $s \in R\backslash B$ and $b \in B$. Then, by (4.11), $A:s \subseteq Q \subset A:b$. Thus B is intermediate to A in R by (4.12).

Suppose $b \in B(Q)$ and $a \in Q$. Then $Q \subset (A:b^2)$ implies $ab^2 \in A$ and $a^2b^2 \in Q$. Hence $ab \in Q$. Thus Q is an ideal of $B(Q)$.

Suppose $s \in R\backslash B$ and $bs \in B$. Then, $A:s \subseteq Q \subset A:bs$. Choose $a \in (A:bs)\backslash Q$. Then $abs \in A$ implies $ab \in Q$. Since $a \notin Q$ and Q is strongly prime, $b \in Q$. Thus $Z(R/B) \subseteq Q$.

Let C be an intermediate ring such that $Q = Z(R/C)$. Suppose $Q \subset A:r$. Let $a \in (A:r)\backslash Q$. Then $ar \in C$ and $a \notin Z(R/C)$ implies $r \in C$. Thus $B(Q) \subseteq C$. Now Q is a C-ideal implies $Q \subseteq A:c \ \forall c \in C$. Thus $C \subseteq B\langle Q\rangle$. Suppose $B(Q) \subset C$. Then there exists $c \in C$ such that $Q=(A:c)$. Since C is intermediate, C must contain each element y of R such that $A:y=Q$, i.e., $C=B\langle Q\rangle$.

THEOREM 4.15. *Let A be a proper CMC subring of a ring R.*

(i) For $B \subset R$ intermediate to A in R and $Q=Z(R/B)$, Q is strongly prime in R and $I \subseteq Q \subseteq P$. If $Q \neq A:r \ \forall r \in R$ and there exists $b \in Q$ such that $A:b \subseteq A:c \ \forall c \in Q$, then there exists $x \in R$ such that $A:x \subset Q \subset A:bx$.

(ii) Suppose the A-ideal Q is strongly prime in R and $I \subseteq Q \subseteq P$.

Case (a). Suppose that $Q \neq (A:r) \ \forall r \in R, \exists b \in Q$ such that $A:b \subseteq A:c \ \forall c \in Q$, and $\forall x \in R$ either $Q \subset A:x$ or $A:bx \subset Q$. Then, there does not exist an intermediate ring B such that $Z(R/B)=Q$.

Case (b). Suppose that $Q \neq (A:r) \ \forall r \in R$ and if there exists $b \in Q$ such that $A:b \subseteq A:c \ \forall c \in Q$, then $\exists x \in R$ such that $A:x \subset Q \subset A:bx$. Then $B(Q)$ is the unique intermediate ring B such that $Z(R/B)=Q$.

Case (c). Suppose that $Q=(A:r)$ for some $r \in R$, and $Qy \nsubseteq Q$ some y such that $Q=A:y$ Then, $B(Q)$ is the unique intermediate ring B such that $Z(R/B)=Q$.

Case (d). Suppose that $Q=(A:r)$ for some $r \in R$, $Qy \subseteq Q \ \forall y$ such that $Q=A:y, \exists b \in Q$ such that $A:b \subseteq A:c \ \forall c \in Q$, and $\forall x \in R$ either $Q \subseteq A:x$ or $A:bx \subset Q$. Then, $B(Q)$ is the unique intermediate ring B such that $Z(R/B)=Q$.

Case (e). Suppose that $Q=(A:r)$ for some $r \in R$, $Qy \subseteq Q \ \forall y$ such that $Q=A:y$, and if $\exists b \in Q$ such that $A:b \subseteq A:c \ \forall c \in Q$, then $\exists x \in R$ such that $A:x \subset Q \subseteq A:bx$. Then there are precisely two intermediate rings B with $Z(R/B)=Q$, namely, $B(Q)$ and $B\langle Q\rangle$.

Moreover, for any strongly prime A-ideal Q such that $I \subseteq Q \subseteq P$, exactly one of the above cases holds.

Proof.

(i) We already observed that Q is strongly prime and $I \subseteq Q \subseteq P$. Suppose $Q \neq A : r \,\forall r \in R$ and there exists $b \in Q$ such that $A : b \subseteq A : c \,\forall c \in Q$. Since B is intermediate, $B : b \subseteq B : c \,\forall c \in Q$ as in the proof of (4.7). As $b \in Q = Z(R/B)$, there exists $x \in R \backslash B$ such that $bx \in B$. Since $x \notin B$, $A : x \subseteq B : x \subseteq Q$. Since $bx \in B$ and Q is a B-ideal that is contained in A, $Q \subseteq A : bx$. Thus $A : x \subset Q \subset A : bx$ by our hypotheses.

(ii) Now suppose Q is a strongly prime A-ideal and $I \subseteq Q \subseteq P$.

Case (a). This follows directly from (i).

Case (b). Let $b \in Q$. Suppose there exists $c \in C$ such that $A : c \subset A : b$. Choose $z \in (A : b) \backslash (A : c)$. Since $cz \notin A$ and $QB(Q) \subseteq Q$, $z \notin B(Q)$. Thus $b \in Z(R/B(Q))$. Now suppose $A : b \subseteq A : c \,\forall c \in Q$. Let x be such that $A : x \subset Q \subset A : bx$. Then $x \notin B(Q)$ and $bx \in B(Q)$ implies $b \in Z(R/B(Q))$. Thus $Q \subseteq Z(R/B(Q))$. By (4.14), $B(Q)$ is an intermediate ring such that $Q = Z(R/B(Q))$. By the case assumption, $B(Q) = B\langle Q \rangle$, so it is the unique intermediate ring B with $Q = Z(R/B)$ by (4.14).

Case (c). $Q = (A : r)$ implies $Q \subseteq Z(R/B(Q))$ and hence $B(Q)$ is an intermediate ring with $Q = Z(R/B(Q))$. Since $Qy \nsubseteq Q$ for some y such that $A : y = Q$, Q is not an ideal of $B\langle Q \rangle$ and consequently $B\langle Q \rangle$ is not an intermediate ring B such that $Q = Z(R/B)$. The conclusion follows from (4.14).

Case (d). As in (c), $B(Q)$ is an intermediate ring with $Q = Z(R/B(Q))$. Let $b \in Q$ be such that $A : b \subseteq A : c \,\forall c \in Q$. Since either $Q \subseteq A : x$ or $A : bx \subset Q$ for all $x \in R$, b is not a zero divisor for $B\langle Q \rangle$. Thus $B(Q)$ is the unique intermediate ring B such that $Z(R/B) = Q$.

Case (e). Since $Qy \subseteq Q$ for all y such that $Q = (A : y)$, $B\langle Q \rangle$ is a subring of R and Q is an ideal of $B\langle Q \rangle$. One further checks that $B\langle Q \rangle$ is an intermediate ring. Suppose $s \in R \backslash B\langle Q \rangle$ and $bs \in B\langle Q \rangle$. Then $A : s \subset Q$. Choose $a \in Q \backslash (A : s)$. Since $QB\langle Q \rangle = Q$, $abs \in Q$. Since $as \in R \backslash A \subseteq R \backslash Q$, $b \in Q$. Thus $Z(R/B\langle Q \rangle) \subseteq Q$. Let $b \in Q$. Suppose there exists $c \in Q$ such that $A : c \subset A : b$. Choose $x \in (A : b) \backslash (A : c)$. Then $bx \in A$ and $Q \nsubseteq (A : x)$ implies $b \in Z(R/B\langle Q \rangle)$. Now suppose $A : b \subseteq A : c$ for all $c \in Q$. Let x be such that $A : x \subset Q \subseteq A : bx$. Then $x \notin B\langle Q \rangle$ and $bx \in B\langle Q \rangle$. Thus $b \in Z(R/B\langle Q \rangle)$. So $B\langle Q \rangle$ is an intermediate ring with $Z(R/B\langle Q \rangle) = Q$. As in (4.14), these are the only possibilities.

DEFINITIONS 4.16. We say a proper CMC subring A of a ring R is *minimal* if there does not exist a CMC subring B of R such that $B \subset A$.

Let $f: \mathbf{Z} \to R$ be the canonical homomorphism. R is said to be *absolutely integral* if $f(\mathbf{Z}) \subseteq R$ is an integral extension.

By an *algebraic number field* we mean an algebraic extension of $\mathbf{Q}$.

Remark 4.17. For any proper CMC subring A of a ring R, there exists a minimal CMC subring B of R such that $B \subseteq A$. This follows directly from Zorn's lemma.

LEMMA 4.18. *A ring R has no proper CMC subrings $\Leftrightarrow$ R is absolutely integral.*

Proof. Suppose R has no proper CMC subrings. As the integral closure of $f(\mathbf{Z})$ in R is the intersection of all CMC subrings of R ([7, Théorème 8]), R is absolutely integral.

Assume that R is absolutely integral. As every CMC subring of R is integrally closed in R [7, Théorème 1], R has no proper CMC subrings.

LEMMA 4.19. *Let A be an integral domain. Then, A is absolutely integral $\Leftrightarrow$ A is either a locally finite field or is isomorphic to a subring of the ring of integers in an algebraic number field.*

Proof. Assume that A is absolutely integral.

Case (a). Suppose that char $A = p > 0$. Let $F = \mathbf{Z}/p\mathbf{Z}$ and identify F with its image in A. Then $F \subseteq A$ is an integral extension of domains implies A is a locally finite field.

Case (b). Suppose char $A = 0$. Identifying $\mathbf{Z}$ with its image in A, $\mathbf{Z} \subseteq A$ is an integral extension of domains. Letting K denote the quotient field of A, $\mathbf{Q} \subseteq K$ is an algebraic extension of fields, i.e., K is a number field. Let $\mathcal{O}$ denote the ring of integers in K. Then $A \subseteq \mathcal{O}$ since $\mathcal{O}$ is integrally closed in K.

Now assume that either A is a locally finite field or is isomorphic to the ring of integers in an algebraic number field. One checks that A is absolutely integral.

THEOREM 4.20. *Let A be a proper CMC subring of a ring R and let $\bar{A}$ denote A/P, where $P = Z(R/A)$. Then, A is minimal $\Rightarrow \bar{A}$ is either a locally finite field or is isomorphic to a subring of the ring of integers in an algebraic number field.*

Proof. Suppose A is minimal. Then $\bar{A}$ has no proper CMC subrings. For if $\bar{B}$ is a proper CMC subring of $\bar{A}$ and $g: A \to \bar{A}$ is the canonical homomorphism, then $g^{-1}(\bar{B})$ is a CMC subring of R by Lemma 4.21 to follow. Thus $\bar{A}$ is either a locally finite field or is isomorphic to a subring of the ring of integers in an algebraic number field by (4.17) and (4.18).

LEMMA 4.21. *Let A be a proper CMC subring of a ring R, let $P = Z(R/A)$, and let B be a CMC subring of A. If $P \subseteq B$, then B is CMC in R.*

Proof. Suppose $P \subseteq B$. Let $r, s \in R \backslash B$. Without loss, we may assume that either $r, s \in R \backslash A$ or $r, s \in A \backslash B$ or $r \in R \backslash A$, $s \in A \backslash B$. If $r, s \in R \backslash A$, then $rs \in R \backslash A \subseteq R \backslash B$. If $rs \in A \backslash B$, then $rs \in A \backslash B \subseteq R \backslash B$. Suppose $r \in R \backslash A$ and $s \in A \backslash B$. Since $s \notin Z(R/A) = P$, $rs \in R \backslash A \subseteq R \backslash B$. Thus B is CMC in R.

5. NUMBER RINGS

For background material on absolute values of number fields, the reader is referred to the lecture notes of E. Artin [1].

DEFINITION 5.1. We call a ring a *number ring* if for every proper CMC subring A of R, A/P is a locally finite field, where $P = Z(R/A)$.

Remark 5.2. It is well known that an algebraic number field (i.e., an algebraic extension of $\mathbf{Q}$) is an example of a number ring.

THEOREM 5.3. *Let v be a formally infinite V-valuation of a number ring R. Then, there exists a ring homomorphism $f: R \to \mathbf{C}$ such that v is isomorphic to w, where $w(r) = |f(r)|$ for all $r \in R$.*

Proof. Let Γ be the target of v, let t be a unit of R satisfying (2.10b), and let $\alpha = v(t)$. Since $\min\{v(1), v(0)\} = 0 \leqslant v(1+0) + v(t) = v(t)$ and v is formally infinite, α is a positive unit of Γ. Let $\beta = -\alpha$ and let

$$K = \{\gamma \in N(\Gamma) \mid \exists n \geqslant 1 \text{ such that } n\beta \leqslant \gamma\}.$$

One checks that K is a convex submonoid of $N(\Gamma)$ and that $\beta \in K$.

Let $\tau: \Gamma \to \Gamma/K$ be the canonical map. Then, $\tau(\beta) = 0$ and, consequently, $\tau(\alpha) = 0$. Write $\bar{v} = \tau \circ v$. One checks that

$$\min\{\bar{v}(r), \bar{v}(s)) \leqslant \bar{v}(r+s) \qquad \forall r, s \in R.$$

Thus $\bar{v}$ is a formally finite V-valuation on R. Let $A = A_{\bar{v}}$.

We claim that $A = R$. Suppose, to the contrary, that $A \subset R$ and let $P = P_{\bar{v}}$. Let $s = t^{-1}$. Since $v(s) = \beta$ and $\bar{v}(s) = \tau(\beta) = 0$, $s \in A \backslash P$. Since R is a number ring and $A \subset R$ is a CMC subring, A/P is a locally finite field. Hence there exists an integer $n \geqslant 2$ such that $s^n = 1 + p$ for some $p \in P$. Now

$$v(s^n) = nv(s) = n\beta < \beta,$$

since β is a negative unit in Γ. On the other hand,

$$0 = \min\{v(1), v(p)\} \leqslant v(1 + p) + \alpha,$$

which implies

$$\beta \leqslant v(1 + p) = v(s^n) = n\beta,$$

a contradiction. Thus $A = R$ and consequently, $K = N(\Gamma)$ by (2.15) and (4.2).

We recall that since t satisfies (2.10b), for all $\gamma < 0$ in Γ there exists a positive integer n such that $n\gamma < \beta$. Combining this with the fact that $K = N(\Gamma)$ we deduce that $N(\Gamma)$ is Archimedean by (3.28). Hence Γ is Archimedean by (3.27). By (3.26) there exists an injective order-homomorphism $\sigma\colon \Gamma \to \mathbf{R}_\infty$ with $\sigma(v(t)) = 1$. Let $\Delta = \sigma(\Gamma)$.

Write $u = \sigma \circ v\colon R \to \Delta$. Then $u(t) = 1$ satisfies (2.10b). Notice that v and u are isomorphic; in particular, u is formally infinite. Consider $P(\mathbf{R})$ as a V-monoid where the binary operation is multiplication and the order is the dual of the usual order on $P(\mathbf{R})$. Then,

$$\eta\colon \mathbf{R}_\infty \to P(\mathbf{R})$$

defined by $\eta(x) = 2^{-x}$ is an isomorphism of V-monoids. Let $\Lambda = \eta(\Delta)$ and let $w = \eta \circ u$. Notice that w is isomorphic to v and hence w is formally infinite.

One checks that:

(i) $w(rs) = w(r)\, w(s)\ \forall r, s \in R$.

(ii) $w(r) = 0 \Leftrightarrow v(r) = \infty\ \forall r \in R$.

(iii)′ $w(r + s) \leqslant 2 \max\{w(r), w(s)\}\ \forall r, s \in R$.

By Lemma 5.4 to follow, we see that w satisfies

(iii) $w(r + s) \leqslant w(r) + w(s)\ \forall r, s \in R$.

Also notice that $1 = w(1) = w((-1)^2) = w(-1)^2$ implies $w(-1) = 1$.

Let $I = I_t = \{r \in R \mid v(r) = \infty\}$. Recall that I is a prime ideal of R. Write F for the quotient field of R/I. Notice that $w(r + x) = w(r)$ for all $r \in R$, $x \in I$. Thus w induces a map

$$\rho\colon R/I \to P(\mathbf{R}),$$

given by $\rho(\bar{r}) = w(r)\ \forall r \in R$. Extend ρ to F by

$$\rho(\bar{r}/\bar{s}) = w(r)/w(s) \qquad \forall r \in R,\ s \in R \backslash I.$$

One checks that ρ is well-defined and that ρ satisfies:

(i) $\rho(xy) = \rho(x)\,\rho(y) \qquad \forall x, y \in F.$

(ii) $\rho(x) = 0 \Leftrightarrow x = 0 \qquad \forall x \in F.$

(iii) $\rho(x+y) \leqslant \rho(x) + \rho(y) \qquad \forall x, y \in F.$

Thus ρ is a classical absolute value on the field F. One checks that since w is a formally infinite V-valuation, ρ is Archimedean in the sense of Cassels and Fröhlich [3]. By the Gelfand–Tornheim theorem [3, p. 45] there exists an injective homomorphism

$$\pi\colon F \to \mathbf{C}$$

such that

$$\rho(x) = |\pi(x)| \qquad \forall x \in F.$$

Let $f\colon R \to \mathbf{C}$ denote $\pi \circ i \circ g$, where $g\colon R \to R/I$ is the canonical map, and $i\colon R/I \to F$ is the inclusion. Then $f(r) = \pi(\bar{r}/\bar{1})$ and hence

$$w(r) = \rho(\bar{r}/\bar{1}) = |f(r)|.$$

LEMMA 5.4. *Let R be a ring and suppose $w\colon R \to P(\mathbf{R})$ satisfies*:

(a) $w(rs) = w(r)\,w(s) \qquad \forall r, s \in R.$

(b) $w(r+s) \leqslant 2 \max\{w(r), w(s)\} \qquad \forall r, s \in R.$

Then, w satisfies

(c) $w(r+s) \leqslant w(r) + w(s) \qquad \forall r, s \in R.$

Proof. By (b) and induction, for all $m \geqslant 1$, $l = 2^m$, and $r_1, \cdots, r_l$ in R,

$$w(r_1 + \cdots + r_l) \leqslant l \max\{w(r_i)\}.$$

Given $n \geqslant 1$, choose m such that $2^{m-1} < n \leqslant 2^m$. Notice that $2^m < 2n$. Now

$$\begin{aligned} w(r_1 + \cdots + r_n) &= w(r_1 + \cdots + r_n + 0 + \cdots + 0) \\ &\leqslant 2^m \max\{w(r_i)\} \\ &\leqslant 2n \max\{w(r_i)\}. \end{aligned}$$

Thus, for all $n \geqslant 1$ and $r_1, \cdots, r_n$ in R,

$$w(r_1 + \cdots + r_n) \leqslant 2n \max\{w(r_i)\}.$$

In particular, for all $n \geqslant 1$ and $r_1, \cdots, r_n$ in R,

$$w(r_1 + \cdots + r_n) \leqslant 2n[w(r_1) + \cdots + w(r_n)].$$

Let $r, s \in R$. Then

$$\begin{aligned}
[w(r+s)]^n &= w([r+s]^n) \\
&= w\left(\sum_{i=0}^{n} \binom{n}{i} r^{n-i} s^i\right) \\
&\leqslant 2(n+1)\left(\sum_{i=0}^{n} w\left(\binom{n}{i} r^{n-i} s^i\right)\right) \\
&\leqslant 2(n+1)\left(\sum_{i=0}^{n} 2\binom{2}{i} w(r)^{n-i} w(s)^i\right) \\
&= 4(n+1)\left(\sum_{i=0}^{n} \binom{n}{i} w(r)^{n-i} w(s)^i\right) \\
&= 4(n+1)[w(r) + w(s)]^n.
\end{aligned}$$

Thus

$$[w(r+s)]^n \leqslant 4(n+1)[w(r)+w(s)]^n \qquad \forall n \geqslant 1,$$

and hence

$$w(r+s) \leqslant 4^{1/n}(n+1)^{1/n}[w(r)+w(s)] \qquad \forall n \geqslant 1.$$

Letting $n \to \infty$, we deduce that

$$w(r+s) \leqslant w(r) + w(s) \qquad \forall r, s \in R.$$

Notation 5.5. For R a ring, we write $X(R)$ for the set of all proper CMC subsets of R, $X_0(R)$ for the set of all proper CMC subrings of R, and let $X_\infty(R) = X(R) \backslash X_0(R)$. For $A \in X(R)$, we write v_A for the standard V-valuation associated with A.

THEOREM 5.6. *Let R be a field. Then, R is a finite dimensional number field if and only if the following five conditions hold*:

(i) For all $A \in X(R)$, the t of Definition 2.13 may be chosen such that either $t = 1$ or $2t = 1$.

(ii) $X_\infty(R)$ is nonempty and finite and $X_0(R)$ is infinite.

(iii) For all nonzero $r \in R$, $\{A \in X(R) \mid v_A(r) \neq 0\}$ is finite.

(iv) For all nonzero $r \in R$, $v_A(r) = 0 \ \forall A \in X(R) \Leftrightarrow \exists 0 \neq n \in \mathbf{N}$ with $r^n = 1$.

(v) For distinct elements $A_1, ..., A_n$ in $X(R)$, $\exists r \in R$ with $r \notin A_1$ and $r \in A_2 \cap \cdots \cap A_n$.

Proof. First we assume R is a finite dimensional number field. The proper CMC subsets of R correspond bijectively to the usual primes of R, the nonrings corresponding to the real and complex primes, and the CMC subrings corresponding to the proper Krull valuation subrings. This follows from Theorem 5.3 and a tedious, but straightforward, check; in particular, one verifies that noncomplex conjugate embeddings of R into $\mathbf{C}$ give rise to distinct CMC subsets of R. With this observation, (i)–(v) follow from well-known results (e.g., see p. 60 of [3] for (iii), p. 164 of [1] for (iv), and p. 39 of [1] for (v)).

Conversely, assume (i)–(v) hold for the field R. By (ii) and Theorem 5.3, the characteristic of R is 0. By p. 111 of [8], R is an algebraic extension of $\mathbf{Q}$ (here one uses (v) to check that the topological space $X_0(R)$ is T_1). All that remains is to show that R is finite dimensional. Using (i), we see that for any finite dimensional subfield F of R and any A in $X_\infty(R)$, $A \cap F$ is a CMC subset of F. By (5.3), $A \cap F \subset F$ and $2 \notin A \cap F$. Hence $A \cap F$ is in $X_\infty(F)$.

Let n denote the number of elements in $X_\infty(R)$. Let F be a finite dimensional subfield of R, let s denote the number of real primes of F, and let t denotes the number of complex primes of F. Then, $\dim F = s + 2t \leqslant 2(s+t) \leqslant 2n$. Hence R is finite dimensional.

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