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Sturm–Liouville operators and their spectral functions

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This paper is dedicated to our friend Eduard Tsekanovskiĭ on the occasion of his 65th birthday

Abstract

Assume that the differential operator $-DpD+q$ in $L^2(0,\infty)$ has 0 as a regular point and that the limit-point case prevails at ∞. If $p\equiv 1$ and q satisfies some smoothness conditions, it was proved by Gelfand and Levitan that the spectral functions $\sigma(t)$ for the Sturm–Liouville operator corresponding to the boundary conditions $(pu')(0)=\tau u(0)$, $\tau\in\mathbb{R}$, satisfy the integrability condition $\int_{\mathbb{R}} d\sigma(t)/(|t|+1)<\infty$. The boundary condition $u(0)=0$ is exceptional, since the corresponding spectral function does not satisfy such an integrability condition. In fact, this situation gives an example of a differential operator for which one can construct an analog of the Friedrichs extension, even though the underlying minimal operator is not semibounded. In the present paper it is shown with simple arguments and under mild conditions on the coefficients p and q, including the case $p\equiv 1$, that there exists an analog of the Friedrichs extension for nonsemibounded second order differential operators of the form $-DpD+q$ by establishing the above mentioned integrability conditions for the underlying spectral functions.

Keywords: Sturm–Liouville operator; Titchmarsh–Weyl coefficient; Symmetric operator; Self-adjoint extension; Spectral function; Generalized Friedrichs extension

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1. Introduction

Consider the singular Sturm–Liouville operator $-D^2 + q$ on the halfline $[0, \infty)$. Assume that the real-valued function q is locally integrable and assume that the limit-point case prevails at ∞, so that the differential operator $-D^2+q$ in $L^2(0, \infty)$ with the boundary conditions

$$u(0) = u'(0) = 0$$

is densely defined and symmetric with defect numbers (1.1). Its self-adjoint extensions $A(\tau)$, $\tau \in \mathbb{R} \cup \{\infty\}$, in $L^2(0, \infty)$ correspond to the boundary conditions

$$u'(0) = \tau u(0), \quad \tau \in \mathbb{R} \cup \{\infty\}. \tag{1.1}$$

When properly interpreted, the self-adjoint extensions $A(\tau)$ in (1.1) with $\tau \in \mathbb{R}$ behave like rank one perturbations of the extension $A(0)$ while the extension $A(\infty)$ can be obtained via a completion procedure from the corresponding minimal operator, which need not be semibounded. In particular, for all $\tau \in \mathbb{R}$ the domains $\operatorname{dom}|A(\tau)|^{1/2}$ coincide with each others and the completion of the domain of the minimal operator with respect to the form generated by the modulus $|A(\tau)|$ (or with respect to the graph topology of $|A(\tau)|^{1/2}$ which is independent of $\tau \in \mathbb{R}$) produces the domain $\operatorname{dom}|A(\infty)|^{1/2}$ as a one-dimensional restriction of $\operatorname{dom}|A(\tau)|^{1/2}$. Due to the analogy with the case where the minimal operator is semibounded, the self-adjoint extension $A(\infty)$ is called the generalized Friedrichs extension. Since the minimal operator need not be semibounded (for an example, see [17, 4.14]), the usual Friedrichs extension need not exist, but if it exists, it coincides with the generalized Friedrichs extension $A(\infty)$ (see [1], and [2] for extension). These facts follow from the results proved in [11,12] in an abstract setting. It is emphasized that there are differential operators for which one cannot construct an analog of the Friedrichs extension, cf. [12]. Therefore, it is of particular interest to find conditions or criteria which guarantee the existence of the generalized Friedrichs extension for nonsemibounded differential operators.

Denote the Titchmarsh–Weyl coefficients corresponding to the self-adjoint extensions $A(\tau)$ by $m_\tau(z)$. The facts presented above can be seen as consequences of some analytical properties of the functions $m_\tau(z)$, $\tau \in \mathbb{R} \cup \{\infty\}$. If the corresponding spectral functions are denoted by $\sigma_\tau(t)$, then

$$m_\tau(z) = -\tau + \int_{\mathbb{R}} \frac{d\sigma_\tau(t)}{t-z}, \quad \int_{\mathbb{R}} \frac{d\sigma_\tau(t)}{|t|+1} < \infty, \ \tau \in \mathbb{R}, \tag{1.2}$$

and

$$\begin{gathered} m_\tau(z) = \alpha + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\sigma_\tau(t), \\ \alpha \in \mathbb{R}, \ \int_{\mathbb{R}} \frac{d\sigma_\tau(t)}{t^2+1} < \infty, \ \int_{\mathbb{R}} \frac{d\sigma_\tau(t)}{|t|+1} = \infty, \ \tau = \infty. \end{gathered} \tag{1.3}$$

The observation that the Titchmarsh–Weyl coefficients $m_\tau(z)$ satisfy (1.2) goes back to Gelfand and Levitan [9], under certain smoothness conditions on q. More general statements with weaker smoothness conditions on q are due to Hille [14, Theorem 10.2.4] and

Levitan and Sargsjan [15, Theorem 5.2]. The proofs of Gelfand, Levitan, and Sargsjan are based on an integral representation of the resolvent operator, which involves the corresponding generalized Fourier transforms. The proof of Hille is more direct as it uses Weyl's original limit-point, limit-circle construction, see [4,7,17,18]. That is, the Titchmarsh–Weyl coefficient $m_\tau(z)$ is the limit of $m_\tau(b,z)$ as $b \to \infty$, when $m_\tau(b,z)$ is the Titchmarsh–Weyl coefficient of $-D^2+q$ on the interval $[0,b]$ with the boundary conditions (1.1) and $u(b)=0$. In this sense the generalized Friedrichs extension on $[0,\infty)$ can be seen to be the limit of Friedrichs extensions on $[0,b]$ as $b \to \infty$. The details of Hille's proof are quite technical. An attempt at simplification of them is due to Wray [19].

The purpose of the present paper is to show the existence of the generalized Friedrichs extension for nonsemibounded second order differential operators of the form $-DpD+q$ by establishing the above mentioned integrability conditions for the underlying spectral functions (cf. [12]) under mild conditions on the coefficients p and q. This involves, in particular, a coherent treatment of the problem in the context of Nevanlinna functions, i.e., analytic functions with positive imaginary part in the upper half-plane, cf. [8]. The approximation of the singular problem on $[0,\infty)$ by regular problems on $[0,b]$ is successful, due to a uniform bound given by Hille [14, Theorem 10.2.1]. This uniform bound for the Titchmarsh–Weyl functions on $[0,b]$, $b>0$, gives a justification for the convergence of the corresponding spectral functions.

In this paper the general Sturm–Liouville operator $-DpD+q$ is considered in $L^2(0,\infty)$, with $1/p$ and q being locally integrable on $[0,\infty)$. Then 0 is a regular point and it is assumed that the limit-point case prevails at ∞. Under mild conditions on p it is proved, in an elementary manner, that the spectral functions corresponding to the boundary conditions $(pu')(0)=\tau u(0)$, $\tau \in \mathbb{R}$, behave as in (1.2) and that the boundary condition $u(0)=0$ corresponds to the generalized Friedrichs extension in the sense of [12]. The results in the present paper can be extended to the case of Sturm–Liouville operators whose coefficients depend rationally on the eigenvalue parameter, cf. [3].

2. Convergence of Nevanlinna functions

A function $Q(z):\mathbb{C}\setminus\mathbb{R}\to\mathbb{C}$ is said to belong to the class **N** of Nevanlinna functions if $Q(z)$ is holomorphic, $\overline{Q(z)}=Q(\bar{z})$, and $(\operatorname{Im} z)(\operatorname{Im} Q(z))\geqslant 0$ for all $z\in\mathbb{C}\setminus\mathbb{R}$. A function $Q(z)$ belongs to **N** if and only if

$$Q(z)=\alpha+\beta z+\int_{\mathbb{R}}\left(\frac{1}{t-z}-\frac{t}{t^2+1}\right)d\sigma(t), \tag{2.1}$$

where

$$\alpha\in\mathbb{R},\qquad \beta\geqslant 0,\qquad \int_{\mathbb{R}}\frac{d\sigma(t)}{t^2+1}<\infty, \tag{2.2}$$

and $\sigma(t)$ is a nondecreasing function, see [8]. It is always assumed that the function $\sigma(t)$ is normalized such that $\sigma(t)=(\sigma(t+0)+\sigma(t-0))/2$ and $\sigma(0)=0$; in this case the above correspondence is one-to-one. Let $Q(z) \in \mathbf{N}$ and define the function $Q_\tau(z)$, $\tau \in \mathbb{R} \cup \{\infty\}$, as a fractional linear transform of $Q(z)$ by

$$Q_\tau(z) = \frac{Q(z) - \tau}{1 + \tau Q(z)}, \quad \tau \in \mathbb{R} \cup \{\infty\}. \tag{2.3}$$

Clearly, $Q_\tau(z)$ belongs to $\mathbf{N}$ for all $\tau \in \mathbb{R} \cup \{\infty\}$. The subclass $\mathbf{N}_\eta$, $0 < \eta \leqslant 1$, consists of those functions $Q(z) \in \mathbf{N}$ which satisfy the additional asymptotic condition

$$\int_1^\infty \frac{\operatorname{Im} Q(iy)}{y^\eta}\, dy < \infty.$$

It follows from this condition that $\beta = 0$ in (2.1). In fact, a function $Q(z)$ belongs to $\mathbf{N}_\eta$ if and only if

$$Q(z) = \gamma + \int_{\mathbb{R}} \frac{d\sigma(t)}{t - z}, \tag{2.4}$$

where

$$\gamma \in \mathbb{R}, \qquad \int_{\mathbb{R}} \frac{d\sigma(t)}{|t|^\eta + 1} < \infty.$$

Here $\gamma = \lim_{y\to\infty} Q(iy) \in \mathbb{R}$. The subclass $\mathbf{N}_0$ consists of those functions $Q(z) \in \mathbf{N}$ which satisfy

$$\sup_{y>0} y \operatorname{Im} Q(iy) < \infty.$$

Clearly $\mathbf{N}_0 \subset \mathbf{N}_\eta$, $0 < \eta \leqslant 1$. A function $Q(z)$ belongs to $\mathbf{N}_0$ if and only if $Q(z)$ has the integral representation (2.4) with

$$\gamma \in \mathbb{R}, \qquad \int_{\mathbb{R}} d\sigma(t) < \infty.$$

If $Q(z) \in \mathbf{N}_\eta$, $0 \leqslant \eta \leqslant 1$, and $\lim_{y\to\infty} Q(iy) = 0$, then $Q_\tau(z)$ in (2.3) belongs to $\mathbf{N}_\eta$ for all $\tau \in \mathbb{R}$, while $Q_\infty(z) \in \mathbf{N} \setminus \mathbf{N}_1$, cf. [12]. The subclass $\mathbf{N}_1$ goes back to Kac.

There is a natural topology for $\mathbf{N}$, namely the topology of uniform convergence on compact subsets of $\mathbb{C} \setminus \mathbb{R}$. Then $\mathbf{N}$ is a complete metric space, see [8, p. 32]. The following result is standard; for a proof see [6].

Proposition 2.1. *Let $Q_n(z) \in \mathbf{N}$, $n \in \mathbb{N}$, have the integral representation*

$$Q_n(z) = \alpha_n + \beta_n z + \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{t^2 + 1} \right) d\sigma_n(t)$$

with

$$\alpha_n \in \mathbb{R}, \qquad \beta_n \geqslant 0, \qquad \int_{\mathbb{R}} \frac{d\sigma_n(t)}{t^2 + 1} < \infty,$$

and let $Q(z) \in \mathbf{N}$ have the integral representation (2.1) *with* (2.2). *Then $Q_n(z)$ converges to $Q(z)$ in the sense of* $\mathbf{N}$ *if and only if for $n \to \infty$,*

$$\alpha_n \to \alpha, \qquad \beta_n \to \beta, \qquad \sigma_n(t) \to \sigma(t),$$

where the last limit takes place at the continuity points of $\sigma(t)$.

An analog of Proposition 2.1 for the subclass $\mathbf{N}_\eta$ reads as follows.

Corollary 2.1. *Assume that $0 \leqslant \eta \leqslant 1$. Let $Q_n(z) \in \mathbf{N}_\eta$, $n \in \mathbb{N}$, have the integral representation*

$$Q_n(z) = \gamma_n + \int_{\mathbb{R}} \frac{d\sigma_n(t)}{t-z},$$

where $\gamma_n \in \mathbb{R}$ and $\int_{\mathbb{R}} d\sigma_n(t)/(|t|^\eta + 1) < \infty$, and let $Q(z) \in \mathbf{N}_\eta$ have the integral representation (2.4) *with $\gamma \in \mathbb{R}$ and $\int_{\mathbb{R}} d\sigma(t)/(|t|^\eta + 1) < \infty$. Then $Q_n(z)$ converges to $Q(z)$ in the sense of* $\mathbf{N}$ *if and only if*

$$\gamma_n \to \gamma, \qquad \sigma_n(t) \to \sigma(t),$$

where the last limit takes place at the continuity points of $\sigma(t)$.

Proof. The weak convergence of the spectral measures $d\sigma_n$ implies

$$\alpha_n - \gamma_n = \int_{\mathbb{R}} \frac{t}{t^2+1}\, d\sigma_n(t) \to \int_{\mathbb{R}} \frac{t}{t^2+1}\, d\sigma(t) = \alpha - \gamma.$$

Now, clearly $\alpha_n \to \alpha$ is equivalent to $\gamma_n \to \gamma$. □

The convergence properties in Proposition 2.1 and its corollary may be augmented by means of the following compactness criterion, see [8].

Proposition 2.2. *Let* $\mathbf{P}$ *be an infinite family of functions in* $\mathbf{N}$ *such that for some $z_0 \in \mathbb{C} \setminus \mathbb{R}$ there is a constant M such that $|Q(z_0)| \leqslant M$ for all $Q(z) \in \mathbf{P}$. Then there is a sequence in* $\mathbf{P}$ *which is convergent.*

In particular, if a sequence of Nevanlinna functions $Q_n(z)$ converges at some point $z_0 \in \mathbb{C} \setminus \mathbb{R}$, then there is a subsequence of $Q_n(z)$ which converges in the sense of $\mathbf{N}$. Moreover, if the Nevanlinna functions $Q_n(z)$ converge pointwise for each $z \in \mathbb{C} \setminus \mathbb{R}$ to a function $Q(z)$, then $Q(z)$ is a Nevanlinna function and the convergence is in the sense of $\mathbf{N}$. The next result deals with the convergence of functions in $\mathbf{N}_\eta$, $0 \leqslant \eta \leqslant 1$.

Proposition 2.3. *Let $Q_n(z)$, $n \in \mathbb{N}$, belong to* $\mathbf{N}$ *and assume that $Q_n(z) \to Q(z)$ as $n \to \infty$ for all $z \in \mathbb{C} \setminus \mathbb{R}$. If, for $0 < \eta \leqslant 1$, there is a constant N such that for all $n \in \mathbb{N}$,*

$$\int_1^\infty \frac{\operatorname{Im} Q_n(iy)}{y^\eta}\, dy \leqslant N, \tag{2.5}$$

then

$$\int_1^\infty \frac{\operatorname{Im} Q(iy)}{y^\eta}\,dy \leqslant N \tag{2.6}$$

and $Q(z) \in \mathbf{N}_\eta$. *Moreover, if there is a constant* N *such that for all* $n \in \mathbb{N}$,

$$\sup_{y>0} y \operatorname{Im} Q_n(iy) \leqslant N, \tag{2.7}$$

then

$$\sup_{y>0} y \operatorname{Im} Q(iy) \leqslant N \tag{2.8}$$

and $Q(z) \in \mathbf{N}_0$. *The same assertions are also true when* $\operatorname{Im} Q_n$ *in* (2.5), (2.7) *and* $\operatorname{Im} Q$ *in* (2.6), (2.8) *are replaced by* $|Q_n|$ *and* $|Q|$, *respectively.*

Proof. By assumptions, the functions $Q_n(z)$ tend to $Q(z)$ in the sense of $\mathbf{N}$, so that $Q(z) \in \mathbf{N}$ and the convergence is uniform on compact subsets. This together with (2.5) implies that for each $R > 1$,

$$\int_1^R \frac{\operatorname{Im} Q(iy)}{y^\eta}\,dy = \lim_{n\to\infty} \int_1^R \frac{\operatorname{Im} Q_n(iy)}{y^\eta}\,dy \leqslant N.$$

Now letting $R \to \infty$ one obtains (2.6) by the monotone convergence theorem. In particular, $Q(z) \in \mathbf{N}_\eta$. The assertion that conditions (2.7) imply (2.8) and $Q(z) \in \mathbf{N}_0$ is now also obvious. The proofs with $\operatorname{Im} Q_n$ and $\operatorname{Im} Q$ replaced by their absolute values $|Q_n|$ and $|Q|$ are similar. $\square$

Assume that $0 \leqslant \eta \leqslant 1$. With the notations from Corollary 2.1 the conditions in Proposition 2.3 are equivalent to

$$\gamma_n \to \gamma, \qquad \sigma_n(t) \to \sigma(t), \qquad \int_{\mathbb{R}} \frac{d\sigma_n(t)}{|t|^\eta + 1} \leqslant A.$$

The conclusion is then equivalent to

$$\int_{\mathbb{R}} \frac{d\sigma(t)}{|t|^\eta + 1} \leqslant A.$$

This resembles the approach of [19]. In the present approach the conditions concerning the growth of the spectral functions are more general.

3. Some estimates for Sturm–Liouville operators

Let p and q be complex-valued functions on $[0, \infty)$ such that $1/p$ and q are locally integrable. Let $\alpha(z)$ and $\beta(z)$ be (locally) holomorphic functions on $\mathbb{C} \setminus \mathbb{R}$. Consider the solution $u(\cdot, z)$ of the initial value problem

$$-(pu')' + (q - z)u = 0, \qquad u(0, z) = \alpha(z), \qquad (pu')(0, z) = \beta(z). \tag{3.1}$$

Clearly, the unique solution of the following “unperturbed equation:”

$$-(pu')' = 0, \qquad u(0,z) = \alpha(z), \qquad (pu')(0,z) = \beta(z), \tag{3.2}$$

is given by

$$\alpha(z) + \beta(z)P(x), \quad P(x) = \int_0^x \frac{1}{p(t)}\,dt. \tag{3.3}$$

As to (3.1), integration gives

$$(pu')(x,z) = \beta(z) + \int_0^x \big(q(t) - z\big)u(t,z)\,dt,$$

and a further integration leads to the following integral equation for the solution of the “perturbed equation” (3.1):

$$u(x,z) = \alpha(z) + \beta(z)P(x) + \int_0^x \frac{1}{p(t)}\left(\int_0^t \big(q(s) - z\big)u(s,z)\,ds\right)dt. \tag{3.4}$$

Conversely, the solution of (3.4) satisfies (3.1). Define the functions Q_0, P_0, and P_1 by

$$Q_0(x) = \int_0^x \big|q(t)\big|\,dt, \qquad P_0(x) = \int_0^x \frac{dt}{|p(t)|}, \qquad P_1(x) = \int_0^x \frac{t}{|p(t)|}\,dt. \tag{3.5}$$

Clearly, these functions are real-valued, continuous, and nondecreasing on $[0,\infty)$ and $Q_0(0) = P_0(0) = P_1(0) = 0$. In the following, the function Ψ stands for $\Psi(t) = te^t$, so that also Ψ is real-valued, continuous, and monotonically increasing on $[0,\infty)$ with $\Psi(0) = 0$. The next lemma gives an estimate for the difference of the solution $u(x,z)$ in (3.1) and solution (3.3) of the unperturbed equation (3.2).

Lemma 3.1. *Let $u(\cdot,z)$ be the solution of* (3.1). *Then for all $x \in [0,\infty)$ and $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$\begin{aligned}&\big|u(x,z) - \alpha(z) - \beta(z)P(x)\big| \\ &\quad \leqslant \big(\big|\alpha(z)\big| + \big|\beta(z)\big|P_0(x)\big)\Psi\left(\int_0^x \frac{Q_0(t) + |z|t}{|p(t)|}\,dt\right).\end{aligned} \tag{3.6}$$

Proof. Let $M(t,z) = \max_{0 \leqslant s \leqslant t} |u(s,z)|$. Then it follows from (3.4) and (3.5) that

$$\big|u(x,z)\big| \leqslant \big|\alpha(z)\big| + \big|\beta(z)\big|P_0(x) + \int_0^x \frac{Q_0(t) + |z|t}{|p(t)|} M(t,z)\,dt.$$

Therefore, also

$$M(x,z) \leqslant \big|\alpha(z)\big| + \big|\beta(z)\big|P_0(x) + \int_0^x \frac{Q_0(t) + |z|t}{|p(t)|}\,M(t,z)\,dt. \tag{3.7}$$

Inequality (3.7) can be solved in the usual manner by applying Gronwall's lemma, see [14]. In fact, for each $z \in \mathbb{C} \setminus \mathbb{R}$ the function $|\alpha(z)| + |\beta(z)|P_0(x)$ is nondecreasing, and therefore (3.7) leads to

$$M(x,z) \leqslant \big(\big|\alpha(z)\big| + \big|\beta(z)\big|P_0(x)\big) \exp\left(\int_0^x \frac{Q_0(t) + |z|t}{|p(t)|}\,dt\right). \tag{3.8}$$

Now, from (3.4) and (3.8) one obtains

$$\big|u(x,z) - \alpha(z) - \beta(z)P(x)\big| \leqslant \big(\big|\alpha(z)\big| + \big|\beta(z)\big|P_0(x)\big)\Psi\left(\int_0^x \frac{Q_0(t) + |z|t}{|p(t)|}\,dt\right),$$

which proves the statement. □

For each $z \in \mathbb{C} \setminus \mathbb{R}$ the values of the function Ψ in Lemma 3.1 can be made arbitrarily small when x is restricted to a sufficiently small interval $[0, \delta(z)]$. It will be enough to define such intervals for $z = iy$, $y > 0$. One selection is presented in the next lemma.

Lemma 3.2. *For each* $0 < c_0 < 1$ *there exist* $y_0 > 0$ *and a monotonically decreasing function* $\delta : [y_0, \infty) \to (0, 1)$ *such that for* $y \to \infty$,

(i) $\delta(y) \to 0$,
(ii) $y\delta(y) \to \infty$,

and for all $y \geqslant y_0$ *and* $x \in [0, \delta(y)]$,

$$\Psi\left(\int_0^x \frac{Q_0(t) + yt}{|p(t)|}\,dt\right) \leqslant c_0. \tag{3.9}$$

Consequently,

$$\big|u(x,iy) - \alpha(iy) - \beta(iy)P(x)\big| \leqslant c_0\big(\big|\alpha(iy)\big| + \big|\beta(iy)\big|P_0(x)\big). \tag{3.10}$$

Proof. Let $0 < c_0 < 1$ and let d_0 be the unique positive number such that $\Psi(d_0) = c_0$. Then there exists $y_1 > 0$ such that

$$\int_0^{\delta(y)} \frac{Q_0(t) + yt}{|p(t)|}\,dt = d_0 \tag{3.11}$$

uniquely determines a function $\delta(y)$ on $[y_1, \infty)$, which is monotonically decreasing. Using $P_1(x)$ defined in (3.5), identity (3.11) gives

$$yP_1\big(\delta(y)\big) = d_0 - \int_0^{\delta(y)} \frac{Q_0(t)}{|p(t)|}\,dt \leqslant d_0. \tag{3.12}$$

This implies that $\delta(y) \to 0$ as $y \to \infty$, and thus proves (i). Now, the integral term in (3.12) can be made smaller than $d_0/2$ when, say, $y \geqslant y_2 \geqslant y_1$. Therefore, for $y \geqslant y_2$ one has $yP_1(\delta(y)) \geqslant d_0/2$ and

$$\frac{1}{y\delta(y)} \leqslant \frac{2}{d_0}\frac{P_1(\delta(y))}{\delta(y)} \leqslant \frac{2}{d_0}P_0\bigl(\delta(y)\bigr) \to 0, \quad y \to \infty,$$

which shows (ii). Finally, choose $y_0 \geqslant y_2$ such that $\delta(y_0) < 1$. Clearly, for $y \geqslant y_0$ estimate (3.9) is satisfied, and thus, by Lemma 3.1 inequality (3.10) holds for every $y \geqslant y_0$ and $0 \leqslant x \leqslant \delta(y)$. □

Lemma 3.3. *Let $u(\cdot, z)$ be the solution of* (3.1). *Choose $0 < c_0 < 1$ and let $\delta(y) : [y_0, \infty) \to (0, 1)$ be a function as in Lemma* 3.2.

(i) *Then for $y \geqslant y_0$ and $b > 1$,*

$$\int_0^b \bigl|u(t, iy)\bigr|^2\,dt > (1-c_0)^2\delta(y)\bigl|\alpha(iy)\bigr|^2 - 2(1+c_0)^2\bigl|\alpha(iy)\beta(iy)\bigr|\int_0^{\delta(y)} P_0(t)\,dt.$$

(ii) *If, in addition, $p \geqslant 0$ in a neighborhood of 0, then for $y \geqslant y_0$ and $b > 1$,*

$$\int_0^b \bigl|u(t, iy)\bigr|^2\,dt > (1-c_0)^2\left(\delta(y)\bigl|\alpha(iy)\bigr|^2 + \bigl|\beta(iy)\bigr|^2\int_0^{\delta(y)} P_0(t)^2\,dt\right) - 2(1+c_0)^2\bigl|\alpha(iy)\beta(iy)\bigr|\int_0^{\delta(y)} P_0(t)\,dt.$$

Proof. Let $\delta(y)$ be a function with the properties in Lemma 3.2. It follows from (3.10) that there are complex numbers ζ_1 $(= \zeta_1(x, y))$ and ζ_2 $(= \zeta_2(x, y))$ with $|\zeta_1|$, $|\zeta_2| \leqslant c_0$, such that for all $y \geqslant y_0$ and $0 \leqslant x \leqslant \delta(y)$,

$$u(x, iy) = \alpha(iy)(1+\zeta_1) + \beta(iy)\bigl(P(x) + \zeta_2 P_0(x)\bigr).$$

The triangle inequality implies

$$\bigl|u(x, iy)\bigr|^2 \geqslant (1-c_0)^2\bigl|\alpha(iy)\bigr|^2 - 2(1+c_0)^2\bigl|\alpha(iy)\beta(iy)\bigr|P_0(x) + \bigl|\beta(iy)\bigl(P(x) + \zeta_2 P_0(x)\bigr)\bigr|^2. \tag{3.13}$$

To obtain estimate (i) delete the last term in the right-hand side of (3.13), integrate both sides of the resulting inequality from 0 to $\delta(y)$, and then use $\delta(y) \leqslant \delta(y_0) < 1 < b$.

If $p \geqslant 0$, then $P(x) = P_0(x)$ and the following inequality holds:

$$\bigl|P(x) + \zeta_2 P_0(x)\bigr| = \bigl|(1+\zeta_2)P_0(x)\bigr| \geqslant (1-c_0)P_0(x).$$

Estimate (ii) is now obtained by substituting the previous estimate into (3.13), integrating both sides of the resulting inequality from 0 to $\delta(y)$, and finally using $\delta(y) \leqslant \delta(y_0) < 1 < b$. $\square$

4. Titchmarsh–Weyl coefficients

Let p and q be real-valued functions on the interval $[0, \infty)$ such that $1/p$ and q are locally integrable. The eigenvalue problem

$$(-DpD + q)u = zu \tag{4.1}$$

has two linearly independent solutions $u_1(\cdot, z)$ and $u_2(\cdot, z)$, entire in z, satisfying the initial conditions

$$u_1(0, z) = 1, \quad (pu_1')(0, z) = 0, \quad u_2(0, z) = 0, \quad (pu_2')(0, z) = -1.$$

Define the meromorphic function $m(b, z)$ by

$$m(b, z) = -\frac{u_2(b, z)}{u_1(b, z)}.$$

Then the function $\chi_b(\cdot, z)$ defined by

$$\chi_b(\cdot, z) = m(b, z)u_1(\cdot, z) + u_2(\cdot, z) \tag{4.2}$$

satisfies the boundary conditions

$$\chi_b(b, z) = 0, \qquad \chi_b(0, z) = m(b, z), \qquad (p\chi_b')(0, z) = -1. \tag{4.3}$$

By Green's formula

$$\frac{m(b, z) - \overline{m(b, w)}}{z - \bar{w}} = \int_0^b \chi_b(t, z)\overline{\chi_b(t, w)}\, dt, \tag{4.4}$$

and, in fact, $m(b, z)$ is the Titchmarsh–Weyl coefficient for the self-adjoint realization in $L^2(0, b)$ corresponding to the self-adjoint boundary conditions

$$(pu')(0) = 0, \qquad u(b) = 0.$$

Now assume that $-DpD + q$ is in the limit-point case at ∞. Then the restriction by the boundary conditions $u(0) = 0$ and $(pu')(0) = 0$ defines a symmetric, completely nonself-adjoint operator with defect numbers $(1, 1)$ (see, e.g., [10] for the complete nonself-adjointness). The assumption of the limit-point case at ∞ is equivalent to the functions $m(b, z)$ having a unique limit $m(z)$ for each $z \in \mathbb{C}$ as $b \to \infty$. In this case

$$\frac{m(z) - \overline{m(w)}}{z - \bar{w}} = \int_0^\infty \chi(t, z)\overline{\chi(t, w)}\, dt,$$

where

$$\chi(\cdot, z) = m(z)u_1(\cdot, z) + u_2(\cdot, z)$$

is the unique solution of (4.1) which belongs to $L^2(0,\infty)$. In fact, $m(z)$ is the Titchmarsh–Weyl coefficient for the self-adjoint realization in $L^2(0,\infty)$ corresponding to the self-adjoint boundary condition $(pu')(0) = 0$. The fractional linear transformation

$$m_\tau(z) = \frac{m(z) - \tau}{1 + \tau m(z)}$$

defines the Titchmarsh–Weyl coefficients for the self-adjoint realization in $L^2(0,\infty)$ corresponding to the self-adjoint boundary condition

$$(pu')(0) = \tau u(0), \quad \tau \in \mathbb{R} \cup \{\infty\}.$$

For the underlying theory see, for instance, [4,7,14,15,17].

Theorem 4.1. *Let p and q be real-valued functions on the interval $[0,\infty)$ such that $1/p$ and q are locally integrable. Assume that the differential operator $-DpD + q$ is in the limit-point case at ∞. Let $\delta(y): [y_0,\infty) \to (0,1)$ be a function as in Lemma* 3.2 *and assume that*

$$\int_{y_0}^{\infty} \frac{1 + y \int_0^{\delta(y)} P_0(t)\,dt}{y^2\delta(y)}\,dy < \infty. \tag{4.5}$$

Then $m(z) \in \mathbf{N}_1$ and

$$\lim_{y\to\infty} m(iy) = 0.$$

Moreover, $m_\tau(z) \in \mathbf{N}_1$ for all $\tau \in \mathbb{R}$, and the boundary condition $u(0) = 0$, corresponding to $\tau = \infty$, determines the generalized Friedrichs extension.

Proof. It follows from the boundary conditions (4.3) that the function $u(\cdot,z) = \chi_b(\cdot,z)$, defined in (4.2), satisfies (3.1) with

$$\alpha(z) = m(b,z), \qquad \beta(z) = -1. \tag{4.6}$$

Equality (4.4) implies that for $y > 0$,

$$\bigl|m(b,iy)\bigr| \geqslant \operatorname{Im} m(b,iy) = y \int_0^b \bigl|u(t,iy)\bigr|^2\,dt. \tag{4.7}$$

Choose $b > 1$. Then it follows from (4.6), (4.7), and (i) of Lemma 3.3 that

$$\bigl|m(b,iy)\bigr| > (1-c_0)^2 y\delta(y)\bigl|m(b,iy)\bigr|^2 - 2(1+c_0)^2 y\bigl|m(b,iy)\bigr| \int_0^{\delta(y)} P_0(t)\,dt,$$

where c_0, y_0, and the function $\delta(y)$ are as in Lemma 3.2. This implies

$$\bigl|m(b,iy)\bigr| < \frac{1 + 2(1+c_0)^2 y \int_0^{\delta(y)} P_0(t)\,dt}{(1-c_0)^2 y\delta(y)} \leqslant C\frac{1 + y\int_0^{\delta(y)} P_0(t)\,dt}{y\delta(y)}, \tag{4.8}$$

and the estimate in the right-hand side is independent of b. Integration of (4.8) leads to

$$\int_{y_0}^{\infty} \frac{|m(b,iy)|}{y}\,dy < C \int_{y_0}^{\infty} \frac{1 + y\int_0^{\delta(y)} P_0(t)\,dt}{y^2\delta(y)}\,dy < \infty.$$

By Proposition 2.3 also $m(z)$ satisfies $\int_{y_0}^{\infty} |m(iy)|/y\,dy < \infty$. Hence, clearly $m(z) \in \mathbf{N}_1$ and $m(iy) \to 0$ as $y \to \infty$. The second statement is now a consequence of the facts presented in Section 2. □

In Section 5 sufficient conditions will be given to guarantee assumption (4.5). However, it is not difficult to see that assumption (4.5) is satisfied when the function p is continuous and $\lim_{x\to 0} p(x) > 0$. In fact, in this case the results in Theorem 4.1 can be sharpened; see [14, Theorem 10.2.4] for the special case $p \equiv 1$.

Theorem 4.2. *Let the assumptions be as in Theorem* 4.1 *and assume, in addition, that p is continuous and that $\lim_{x\to 0} p(x) > 0$. Then there exist $A > 0$ and $B > 0$ so that*

$$A \leqslant \sqrt{y}\,\bigl|m(iy)\bigr| \leqslant B \tag{4.9}$$

for sufficiently large y. In particular, $m(z) \in \mathbf{N}_\eta$, $1/2 < \eta \leqslant 1$, and

$$\lim_{y\to\infty} m(iy) = 0.$$

Moreover, $m_\tau(z) \in \mathbf{N}_\eta$, $1/2 < \eta \leqslant 1$, for all $\tau \in \mathbb{R}$, and the boundary condition $u(0) = 0$, corresponding to $\tau = \infty$, determines the generalized Friedrichs extension.

Proof. Let the function $\delta(y)$ be as in the proof of Lemma 3.2. One may assume that $p(x) \geqslant 0$ for $x \in [0, \delta(y_0)]$. Then (4.6), (4.7), and (ii) of Lemma 3.2 imply that

$$\bigl|m(b,iy)\bigr|^2 - \frac{1 + 2(1+c_0)^2 y\int_0^{\delta(y)} P_0(t)\,dt}{(1-c_0)^2 y\delta(y)}\bigl|m(b,iy)\bigr| + \frac{\int_0^{\delta(y)} P_0(t)^2\,dt}{\delta(y)} < 0. \tag{4.10}$$

Take the limit as $b \to \infty$ in (4.10), and rewrite the result as follows:

$$\begin{aligned}\bigl(\sqrt{y}\bigl|m(iy)\bigr|\bigr)^2 &- \frac{1 + 2(1+c_0)^2 y\int_0^{\delta(y)} P_0(t)\,dt}{(1-c_0)^2\sqrt{y}\,\delta(y)}\sqrt{y}\bigl|m(iy)\bigr| \\ &+ \frac{y^{3/2}\int_0^{\delta(y)} P_0(t)^2\,dt}{\sqrt{y}\,\delta(y)} \leqslant 0.\end{aligned} \tag{4.11}$$

Then the definition of $\delta(y)$ in (3.11) and the equality in (3.12) show that

$$yP_1(\delta) = d_0 - \int_0^{\delta(y)} \frac{Q_0(t)}{p(t)}\,dt = d_0 + o(1), \quad y \to \infty. \tag{4.12}$$

Moreover, if $p_0 = \lim_{x\to 0} p(x) > 0$, then

$$P_1(\delta) = \int_0^{\delta(y)} \frac{t}{p(t)}\,dt = \frac{1}{p_0}\int_0^{\delta(y)} \frac{t}{1+o(1)}\,dt$$
$$= \frac{1}{p_0}\int_0^{\delta(y)} t\bigl(1+o(1)\bigr)\,dt = \frac{1}{2p_0}\delta(y)^2\bigl(1+o(1)\bigr), \quad y\to\infty.$$

Substituting this into (4.12) leads to

$$\sqrt{y}\,\delta(y) = \sqrt{2p_0d_0} + o(1), \quad y\to\infty. \tag{4.13}$$

Similarly it is seen that for $y\to\infty$,

$$y\int_0^{\delta(y)} P_0(t)\,dt \to d_0, \qquad y^{3/2}\int_0^{\delta(y)} P_0(t)^2\,dt \to \frac{(2p_0d_0)^{3/2}}{3p_0^2}. \tag{4.14}$$

It follows from (4.13) and (4.14) that the limit as $y\to\infty$ of the quadratic polynomial in the left-hand side of (4.11) is given by

$$X^2 - \frac{1+2(1+c_0)^2d_0}{(1-c_0)^2\sqrt{2p_0d_0}}\,X + \frac{2d_0}{3p_0} \tag{4.15}$$

with $X = \sqrt{y}|m(iy)|$. Clearly, both roots of (4.15) are positive (and distinct), which implies (4.9). The remaining assertions follow from the facts presented in Section 2. $\square$

Remark 4.1. It is of independent interest to notice that in the case where $p\equiv 1$ Atkinson [5] has proved the existence of the limit $\lim_{y\to\infty}\sqrt{y}|m(iy)|$ with a different method, which is stronger than estimate (4.9). However, for the purposes in the present paper the key result needed to guarantee the existence of a generalized Friedrichs extension is to prove the integrability conditions in (1.3), or equivalently, that at least for some (and then for all) $\tau\in\mathbb{R}$ the condition $m_\tau(z)\in\mathbf{N}_1$ is satisfied.

5. Further sufficient conditions

In this section it is shown that (4.5) is a mild condition which can be verified in simple terms. Define the real-valued function $P_{01}(x)$ by

$$P_{01}(x) = \int_0^x P_0(t)\,dt.$$

Clearly, $P_{01}(x)$ is continuous and nondecreasing on $[0,\infty)$ with $P_{01}(0)=0$.

Proposition 5.1. *Let p and q be as in Theorem* 4.1, *and assume that one of the following functions*:

$$\frac{P_{01}(\delta)}{P_1(\delta)|p(\delta)|} \tag{5.1}$$

or

$$\frac{P_0(\delta)}{\delta} \tag{5.2}$$

is locally integrable in a neighborhood of 0*. Then there exists a monotonically decreasing, absolutely continuous function* $\delta(y):[y_0,\infty)\to(0,1)$ *with the properties in Lemma* 3.2 *and such that*

$$\int_{y_0}^{\infty} \frac{1+yP_{01}(\delta(y))}{y^2\delta(y)}\,dy<\infty. \tag{5.3}$$

Proof. Assume that (5.1) is integrable near 0. Let $0<c_0<1$ and let $d_0>0$ be defined by $\Psi(d_0)=c_0$. The identity

$$P_1\big(\delta(y)\big)=\frac{d_0}{2y} \tag{5.4}$$

uniquely determines a function $\delta(y)$ on $[y_1,\infty)$, for some $y_1>0$, which is monotonically decreasing. Clearly, $\delta(y)\to 0$ as $y\to\infty$. Definition (5.4) implies that

$$\frac{1}{y\delta(y)}=\frac{2}{d_0}\frac{P_1(\delta(y))}{\delta(y)}\leqslant\frac{2}{d_0}P_0\big(\delta(y)\big)\to 0,\quad y\to\infty.$$

Thus, $\delta(y)$ admits properties (i) and (ii) in Lemma 3.2. Using the notations in (3.5), choose $y_0\geqslant y_1$ such that $P_0(\delta(y_0))Q_0(\delta(y_0))\leqslant d_0/2$, and such that $\delta(y_0)<1$. Then clearly for all $y\geqslant y_0$ and $0\leqslant x\leqslant\delta(y)$,

$$\int_0^x \frac{Q_0(t)+yt}{|p(t)|}\,dt\leqslant P_0(x)Q_0(x)+yP_1(x)\leqslant d_0.$$

Hence, estimate (3.9) in Lemma 3.2 is satisfied, and thus, inequality (3.10) holds for every $y\geqslant y_0$ and $0\leqslant x\leqslant\delta(y)$. Clearly, $P_1(x)$ and hence also $\delta(y)$ is absolutely continuous by Lemma A.1. Now (5.4) gives

$$\delta'(y)=-\frac{d_0|p(\delta(y))|}{2y^2\delta(y)}. \tag{5.5}$$

Since (5.1) is integrable near 0, also

$$\left(\frac{P_{01}(\delta)}{P_1(\delta)}+1\right)\frac{1}{|p(\delta)|}$$

is integrable near 0. The change of variables $\delta=\delta(y)$, (5.5), and the local integrability of (5.1) lead to (5.3).

Next assume that (5.2) is integrable near 0. The function $xP_0(x)$ is absolutely continuous and monotonically increasing. Hence, for $d_0>0$,

$$\delta(y)P_0\big(\delta(y)\big)=\frac{d_0}{2y} \tag{5.6}$$

uniquely determines a function $\delta(y)$, which is monotonically decreasing on $[y_1, \infty)$, $y_1 > 0$. Clearly, $\delta(y) \to 0$ as $y \to \infty$ and

$$\frac{1}{y\delta(y)} = \frac{2}{d_0} P_0\big(\delta(y)\big) \to 0, \quad y \to \infty,$$

so that properties (i) and (ii) in Lemma 3.2 are satisfied. Observe that

$$y P_1\big(\delta(y)\big) = \frac{d_0 P_1(\delta(y))}{2\delta(y) P_0(\delta(y))} \leqslant \frac{d_0}{2}.$$

Choose $y_0 \geqslant y_1$ such that $P_0(\delta(y_0)) Q_0(\delta(y_0)) \leqslant d_0/2$ and such that $\delta(y_0) < 1$. Then estimate (3.9) in Lemma 3.2 is satisfied. The absolute continuity of $\delta(y)$ again follows from Lemma A.1. Hence, (5.6) gives

$$\left(\frac{P_0(\delta(y))}{\delta(y)} + \frac{1}{|p(\delta(y))|}\right)\delta'(y) = -\frac{d_0}{2y^2\delta(y)}. \tag{5.7}$$

It follows from

$$y P_{01}\big(\delta(y)\big) \leqslant y\delta(y) P_0\big(\delta(y)\big) = \frac{d_0}{2}$$

that assertion (5.3) holds if

$$\int_{y_0}^{\infty} \frac{1}{y^2\delta(y)}\, dy < \infty. \tag{5.8}$$

Since (5.2) is locally integrable, also

$$\frac{P_0(\delta)}{\delta} + \frac{1}{|p(\delta)|}$$

is locally integrable. The change of variables $\delta = \delta(y)$, (5.7), and local integrability of (5.2) now lead to (5.8). $\square$

The function $\delta(y)$ that was constructed in the proof of Lemma 3.2 satisfies $d_0/2 \leqslant y P_1(\delta(y)) \leqslant d_0$ for all $y \geqslant y_0$, so that $P_1(\delta(y)) \sim 1/y$, see (3.12) in the proof of Lemma 3.2. In this sense it is asymptotically equivalent to the function $\delta(y)$ defined in (5.4).

The assumptions imposed on p and q to prove the main theorems are rather mild. Moreover, the inequalities that were used in proving these results have been obtained for $|m(iy)|$ rather than for $\operatorname{Im} m(iy)$. The conclusion is that, even under the more general assumptions used in the present paper, the above results actually lead to somewhat stronger properties for the Titchmarsh–Weyl coefficients associated with Sturm–Liouville problems of the form (4.1) than what is needed for the Kac subclasses $\mathbf{N}_1$ of Nevanlinna functions and for the existence of generalized Friedrichs extensions.

6. A lower bound

The method by which Theorems 4.1 and 4.2 were proved also provides a lower bound for $|m(iy)|$ when $p(x)$ is nonnegative. Denote

$$P_{02}(x) = \int_0^x P_0(t)^2\,dt.$$

Theorem 6.1. *Let the function p be nonnegative, let $\delta(y):[y_0,\infty) \to (0,1)$ be defined by* (5.4)*, and let $\delta_0 = \delta(y_0)$. Then for some $C > 0$,*

$$\int_{y_0}^{\infty} \frac{|m(iy)|}{y}\,dy \geqslant C \int_0^{\delta_0} \frac{P_{02}(\delta)}{P_0(\delta)P_1(\delta)|p(\delta)|}\,d\delta. \tag{6.1}$$

Proof. For a polynomial $z^2 - az + c$, where $a, c > 0$ and $a^2 - 4c \geqslant 0$, with the real roots r_1, r_2 (> 0) it is easy to check that $c/a < \min\{r_1, r_2\}$. For $\delta(y):[y_0,\infty) \to (0,1)$ in (5.4) the quadratic inequality (4.10) is satisfied. Therefore, one obtains the following estimate for $|m(b,iy)|$ from below:

$$\bigl|m(b,iy)\bigr| \geqslant \frac{(1-c_0)^2 y P_{02}(\delta)}{1 + 2(1+c_0)^2 y P_{01}(\delta)} \geqslant C_1 \frac{P_{02}(\delta)}{1/y + P_{01}(\delta)}, \quad \delta = \delta(y). \tag{6.2}$$

Here $C_1 > 0$ depends only on c_0, $0 < c_0 < 1$. Integration by parts yields $P_1(\delta) = \delta P_0(\delta) - P_{01}(\delta)$. Now, taking into account (5.4) one obtains from (6.2) the estimate

$$\bigl|m(b,iy)\bigr| \geqslant C \frac{P_{02}(\delta)}{P_1(\delta) + P_{01}(\delta)} = C \frac{P_{02}(\delta)}{\delta P_0(\delta)}, \quad \delta = \delta(y), \tag{6.3}$$

where $C > 0$ depends only on c_0. Hence, one may take the limit $b \to \infty$ in (6.3) to obtain the corresponding inequality for $|m(iy)|$. Therefore, the previous inequality and identities (5.4) and (5.5) lead to

$$\int_{y_0}^{\infty} \frac{|m(iy)|}{y}\,dy \geqslant C \int_{y_0}^{\infty} \frac{P_{02}(\delta(y))}{y\delta(y)P_0(\delta(y))}\,dy = C \int_0^{\delta_0} \frac{P_{02}(\delta)}{P_0(\delta)P_1(\delta)|p(\delta)|}\,d\delta,$$

which proves (6.1). □

The conditions in Proposition 5.1 are satisfied by a function p which behaves near 0 like t^{1-c} for $c > 0$ or like $t|\ln t|^{1+c}$ for $c > 1$, but they are not satisfied if p behaves like $t|\ln t|^{1+c}$ for $0 < c \leqslant 1$. This is due to the fact that the present methods give estimates for $|m(b,iy)|$ and not for $\operatorname{Im} m(b,iy)$. Using Theorem 6.1 it is shown that indeed if p behaves like $t|\ln t|^{1+c}$ for $1 < c \leqslant 2$, then $|m(b,iy)|/y$ is not integrable on (y_0,∞).

Example 6.1. Let $p(t) = t|\ln t|^{1+c}$, $0 < c \leqslant 1$, for, say, $0 < t < 1/2$. Then $1/(c|\ln t|^c)$ is a primitive for $1/p$, so that $1/p \in L^1(0,1/2)$. But

$$\delta \mapsto \frac{P_{02}(\delta)}{P_0(\delta)P_1(\delta)|p(\delta)|} \tag{6.4}$$

is not integrable over $(0,\delta_0)$ for any $\delta_0 \in (0,1/2)$. Then, according to Theorem 6.1, the function $|m(b,iy)|/y$ is not integrable on (y_0,∞).

Proof. A straightforward calculation gives

$$P_0(\delta) = \frac{1}{c|\ln\delta|^c}, \qquad P_1(\delta) = \int_0^\delta \frac{dt}{|\ln t|^{1+c}}, \qquad P_{02}(\delta) = \frac{1}{c^2}\int_0^\delta \frac{dt}{|\ln t|^{2c}}.$$

Hence, the function $S(\delta)$, defined by (6.4), satisfies

$$S(\delta) = \frac{1}{c}\frac{\int_0^\delta (1/|\ln t|^{2c})\,dt}{\delta|\ln\delta|^c(\delta/|\ln\delta|^{2c})(\int_0^\delta (1/|\ln t|^{1+c})\,dt\,(|\ln\delta|^{1+c}/\delta))}.$$

Next observe that for $\nu > 0$ by l'Hôpital's rule,

$$\lim_{\delta\to 0}\frac{\int_0^\delta (1/|\ln t|^\nu)\,dt}{\delta/|\ln\delta|^\nu} = \lim_{\delta\to 0}\frac{1}{1-\nu/|\ln\delta|} = 1.$$

Hence, as $\delta \to 0$,

$$S(\delta) \sim \frac{1}{c}\frac{1}{\delta|\ln\delta|^c}.$$

Therefore, $S(\delta)$ is not integrable on $(0,\delta_0)$. □

Appendix A

The various functions δ which were constructed above are absolutely continuous. This fact is a consequence of the following lemma.

Lemma A.1. *Let $F:[a,b]\to[\alpha,\beta]$ be a strictly increasing, absolutely continuous bijection, where $-\infty < a < b < \infty$ and $-\infty < \alpha < \beta < \infty$. Assume that $F'(x)$ exists and that $F'(x) > 0$ almost everywhere on $[a,b]$. Then the inverse function F^{-1} is absolutely continuous.*

In particular, if $f \in L^1[a,b]$ and $f > 0$ almost everywhere, then the function F, defined by

$$F(x) = \int_a^x f(t)\,dt, \quad x \in [a,b],$$

has an absolutely continuous inverse F^{-1}.

Proof. Let N be the subset of $[a,b]$ of Lebesgue measure 0 such that $F'(x)$ exists and $F'(x) > 0$ except for $x \in N$. Then $(F^{-1})'(y)$ exists with $0 \leqslant (F^{-1})'(y) < \infty$ except for $y \in F(N)$, a set of Lebesgue measure 0. Now one can proceed as follows. The set

$$E = \{x \in [a,b]:\ (D^+F^{-1})(F(x)) = \infty\} \subset N,$$

where D^+ denotes the upper-right derivative, clearly has Lebesgue measure 0. By [13, (18.35)], F^{-1} is absolutely continuous.

Another, more direct, way to proceed is to use

$$(F^{-1})'\bigl(F(x)\bigr)F'(x) = 1, \quad x \notin N.$$

Due to the change of variables formula for an absolutely continuous transformation applied to integrable functions (see, for instance, [16, Section 26]),

$$\begin{aligned}\int_\alpha^\beta (F^{-1})'(y)\,dy &= \int_a^b (F^{-1})'\bigl(F(x)\bigr)F'(x)\,dx \\ &= \int_a^b dx = b - a = F^{-1}(\beta) - F^{-1}(\alpha).\end{aligned}$$

Here, $(F^{-1})'$ is a nonnegative measurable, and hence integrable function. Likewise,

$$F^{-1}(\xi) = \int_\alpha^\xi (F^{-1})'(y)\,dy + F^{-1}(\alpha), \quad \xi \in [\alpha, \beta],$$

which shows that F^{-1} is absolutely continuous. $\square$

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