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Global existence and uniform decay for wave equation with dissipative term and boundary damping

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ABSTRACT

In this paper,we prove the existence, uniqueness and uniform stability of strong and weak solutions of the nonlinear wave equation

$$u_{tt} - \Delta u + b(x)u_t + f(u) = 0$$

in bounded domains with nonlinear damped boundary conditions, given by

$$\frac{\partial u}{\partial \nu} + g(u_t) = 0,$$

with restrictions on function $f(u)$, $g(u_t)$ and $b(x)$,. We prove the existence by means of the Glerkin method and obtain the asymptotic behavior by using of the multiplier technique from the idea of Kmornik and Zuazua (see [7]).

1. Introduction

Consider the nonlinear wave equation with a dissipative term and boundary damping

$$\begin{cases} u_{tt} - \Delta u + b(x)u_t + f(u) = 0, \quad \text{in } \Omega \times [0, \infty), \\ u = 0, \quad \text{on } \Gamma_1 \times [0, \infty), \\ \dfrac{\partial u}{\partial \nu} + g(u_t) = 0, \quad \text{on } \Gamma_0 \times [0, \infty), \\ u(x, 0) = u^0(x), \qquad u_t(x, 0) = u^1(x), \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is the bounded domain of R^n, $n \geq 1$ with a smooth C^2 boundary Γ, and let Γ_0, Γ_1 be a partition of Γ such that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \Phi$, $\Gamma_0 \neq \Phi$, $\Gamma_1 \neq 0$, $\nu = (\nu_1, \nu_2 \ldots \nu_n)$ denotes the unit outward normal to Γ and f, g and $b(x)$ are functions enjoying some properties (see(A1)–(A4) below).

First, the nonlinear wave equation reads as follows

$$u_{tt} - u_{xx} + \alpha u - \beta u^3 = 0. \tag{1.2}$$

The Eq. (1.1) arises in many fields such as nonlinear optics, the Josephson array, ferromagnetic materials, charge density waves and liquid helium. The Eq. (1.2) describes the propagation of dislocations within crystals, the Bloch wall motion of magnetic crystals, the propagation of a splay wave along a lied membrane, the unitary theory for elementary particles and

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the propagation of magnetic flux on a Josephson line, etc. (see [1]). From the point-view of physics, the authors have studied the new exact traveling wave solutions for Eq. (1.2) (see [2])

Second, the stability of a strong and weak solution for Eq. (1.1) in the case $b(x) = 0, f(u) = 0$ and $g(u_t) = \beta(x)u_t$ was studied by many authors using semigroup arguments (see [3–7]). In [8], the authors have studied the Eq. (1.1) in the case of $b(x) = 0, f(u) = a(x, t)u, g(u_t) = \beta(x)u_t$. In this paper, we use Galerkin approximation and techniques from work by [9]. According to our best knowledge, boundary stabilization for (1.1) with a dissipative term was not considered in the literature. The proof of the existence is based on the Galerkin approximation. For strong solutions to (1.1), this approximation requires a change of variables to transform (1.1) into an equivalent problem with the initial value equaling zero. In particular we overcome some difficulties, such as the presence of nonlinear terms $f(u)$ and $g(u_t)$ that bring up serious difficulties when passing the limit, which overcome combining arguments of compactly and monotonicity.

The controllability and bounded stabilization of distributed systems have attracted some attention in literature, and there are new techniques which allow us to stabilize a system through its boundary or control from an initial to a final state. As a matter of fact, when $g(s) = s$, we can see the following works (see [4–6] [10–15]). When the boundary conditions are nonlinear, we can see [16–25].

In [26], the authors have considered the wave equation

$$u_{tt} - \Delta u + f_0(\nabla u) = 0 \tag{1.3}$$

subject to the following nonlinear boundary conditions

$$\frac{\partial u}{\partial \nu} + g(u_t) = \int_0^t h(t-\tau) f_1(u(\tau))\mathrm{d}\tau \tag{1.4}$$

and they have shown the existence of solutions and uniform decay.

When $f_0 = f_1 = 0$, the problem (1.3) (1.4) has been studied by Zuazua [15], Lasiecka, and Tataru [20], Aassila [27]. When $g(s) = s$ and $f_1 = 0$, the existence and uniform decay was established by Cavalcanti et al. [14]

Recently, in [28], the authors have investigated the initial- boundary value problem for the wave equation with a localized dissipation

$$\begin{cases} u_{tt} - \Delta u + b(x)u_t = 0, & \text{in } \Omega \times [0, \infty), \\ u = 0, & \text{on } \partial\Omega \times [0, \infty), \\ u(x, 0) = u^0(x), \qquad u_t(x, 0) = u^1(x), & \text{in } \Omega, \end{cases}$$

and studied energy decay estimates for the above equation by multipliers technique and some modified interpolation and difference inequalities.

We notice that stability problems with nonlinear term $h(\nabla u)$ require careful treatment because we do not have any information about the influence of the integral $\int_\Omega f(u)u_t \mathrm{d}x$ on the energy

$$E(t) = \frac{1}{2}\int_\Omega (\|u_t(x, t)\|^2 + \|\nabla u(x, t)\|^2)\mathrm{d}x \tag{1.5}$$

or about the sign of its derivative $\frac{\mathrm{d}E}{\mathrm{d}t}$. Moreover, to have uniform stabilization, the decay constant, that is, the constant $C > 0$ which satisfies $E(t) \le C\mathrm{e}^{-\delta t}E(0)$, can not depend on the solutions u of (1.1). We take into account the dissipative term $b(x)u_t$, and make use of the perturbed energy method developed by Kmornik and Zuazua (see [7]) in order to overcome the above difficulties.

We organize the paper as follows. In Section 2, we show the governing equations and applications. In Section 3, we give the notations and state our main results. In Section 4, we prove the existence and uniqueness of strong and weak solutions to the Eq. (1.1) by Galerkin method. In Section 5, we prove the exponential decay of solutions. In Section 6, we give further remarks on this context.

2. Governing equations and applications

Eq. (1.1), is a mathematical model of the stretched string problem and has a variety of engineering applications including the study of dynamics of the transmission of stretched string lines and strings in the manufacture of fibres and textiles. In order to examine the problem of a stretched string, it is necessary to develop the equation governing its motion. This can be achieved by considering a differential element of a stretched string under a constant tension, T_0 (see Fig. 1 as follows).

We assume that during its motion, the stretched string displaces to a new position such that the segment of length dx acquires the displaced position as shown in Fig. 1. The string is assumed to be stretched between its end points under a tension T_0, which is sufficiently large that the effect of gravity can be ignored. The string is assumed to be elastic with no resistance to bending. The tension in the string is assumed to remain constant throughout the wave motion. The mass per unit length of the string is denoted by $\rho^\star$(i.e $\rho^\star = \rho\widetilde{a}$, where ρ is the mass density and $\widetilde{a}$ is the cross sectional area of the string.) In its initial position, the sting is located along the $x - axis$ and during wave motion it displaces to the position indicated in Fig. 1. We consider the equation of motion of the element $aa^\star$ shown in Fig. 1.

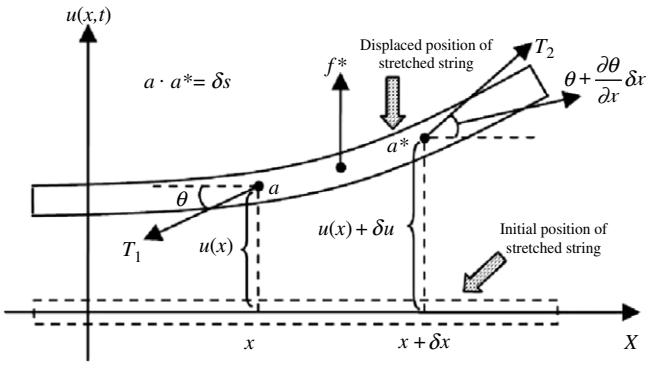

Fig. 1. Deformed configuration of a stretched string.

By Newton's law, we have

$$-T_1 \sin\theta + T_2 \sin(\theta + \delta\theta) + f^\star(\delta s) = \frac{\partial^2 u}{\partial t^2}, \tag{2.1}$$

$$-T_1 \cos\theta + T_2 \cos(\theta + \delta\theta) = 0. \tag{2.2}$$

By geometry and approximation, we deduce

$$\delta s = \left[1 + \left(\frac{\partial u}{\partial x}\right)^2\right]^{\frac{1}{2}} \delta x,$$

$$\sin\theta = \frac{\partial u}{\partial x}\left[1 + \left(\frac{\partial u}{\partial x}\right)^2\right]^{-\frac{1}{2}} = \frac{\partial u}{\partial x}, \tag{2.3}$$

$$\cos\theta = \left[1 + \left(\frac{\partial u}{\partial x}\right)^2\right]^{-\frac{1}{2}} = 1, \tag{2.4}$$

$$\cos(\theta + \delta\theta) = 1, \qquad \sin(\theta + \delta\theta) = \frac{\partial u(x + \delta x, t)}{\partial x}. \tag{2.5}$$

By (2.2)–(2.5), Eq. (2.2) becomes

$$-T_1 + T_2 = 0. \tag{2.6}$$

Therefore, we see $T_1 = T_2 = T_0$. Eq. (2.1) can be written as

$$-T_0\left\{\frac{\partial u(x + \delta x, t)}{\partial x} - \frac{\partial u(x, t)}{\partial x}\right\} + f^\star(\delta x) = \rho^\star \delta x \frac{\partial^2 u}{\partial t^2}. \tag{2.7}$$

If we divide (2.7) by δx and take the limit as $\delta x \to 0$, we obtain

$$T_0 \frac{\partial^2 u}{\partial x^2} + f^\star = \rho^\star \frac{\partial^2 u}{\partial t^2}. \tag{2.8}$$

However, the dynamics of the stretched string will be influenced by a variety of dissipative mechanisms, including internal friction or material hysteresis of the string, air resistance, hydrodynamic resistance (in strings immersed in fluids) and other hysteretic phenomena. The processes contributing to dissipation can be broadly classified as viscous damping and such damping effects can significantly alter the response of the stretched string. The viscous damping approach is a first approximation for the study of dissipative phenomena. Newtonian viscous effects are characterized by a linear relationship between the shear stress and the spatial gradient of velocity. We assume that the viscous damping processes follow Newtonian viscous characteristics and provide a resistive force to the motion of the string which takes the form

$$q(x, t) = \eta_D \frac{\partial u}{\partial t},$$

where η_D is the dynamic viscosity of the of the resistance with dimensions of $\frac{(Force)(Time)}{Length}^2$. This resistance can be incorporated into the basic equation of forced motion of the stretched string by assuming that $q(x, t)$ acts in a direction to $f^\star(x, t)$ and is distributed in exactly the same way. The resulting equation of forced motion of a stretched string is given by

$$T_0 \frac{\partial^2 u}{\partial x^2} + \eta_D \frac{\partial u}{\partial t} + f^\star = \rho^\star \frac{\partial^2 u}{\partial t^2}. \tag{2.9}$$

Letting $\frac{\eta_D}{T_0} = b(x)$, $\frac{f^\star}{T_0} = f(u)$, $\frac{\rho^\star}{T_0} = 1$, we obtain the first equation of (1.1) in a higher dimension R^n.

The boundary condition in the third equation of (1.1) is general and covers a family large variety of different physical configurations. The physical meaning of this boundary condition as well as the following three particular cases

$$\frac{\partial u}{\partial \nu} + \xi(x)u_t = 0, \tag{2.10}$$

$$\frac{\partial u}{\partial \nu} + \beta(x)u_t + \alpha(x)u = 0, \tag{2.11}$$

for M, there is a large constant to be determined, and we obtain

$$\begin{aligned} &m(x)\eta_{tt} + d(x)\eta_t + K(x)\eta = -u \\ &\frac{\partial u}{\partial \nu} + \eta_{tt} = 0 \end{aligned} \tag{2.12}$$

which is discussed in [29]. In [30], the authors have considered the energy decay of problem (1.1) with the boundary condition (2.9) in the case $\xi(x) \equiv C$ (a positive constant) and source term $f(u) \equiv 0$. More delicate is the same problem with the boundary condition (2.9), $f(u) \equiv 0$ without internal damping (i.e $\alpha(x) = 0$.) This is discussed in [7,15].

Remark 2.1. In our paper, we consider nonlinear damped boundary conditions in (1.1).

3. Notations and main results

We consider the Hilbert space

$$V = \{v \in H^1(\Omega);\ v = 0,\ \text{on } \Gamma_1\}$$

and we define

$$(u, v) = \int_\Omega u(x)v(x)\mathrm{d}x, \qquad |u|^2 = \int_\Omega u^2\mathrm{d}x,$$

$$(u, v)_{\Gamma_0} = \int_{\Gamma_0} u(x)v(x)\mathrm{d}x, \qquad |u|^2_{\Gamma_0} = \int_{\Gamma_0} u^2\mathrm{d}x.$$

Now, we state the general assumptions:

(A1) *Assumptions on f:*

Set $f : R \to R$ to be a $W^{1,\infty}_{loc}(R)$, piecewise $C^1(R)$ function,

$$f(s)s \geq 0, \quad \text{for } s \in R. \tag{3.1}$$

Assume that there exists $C_1 > 0$ such that

$$|f'(s)| \leq C_1(1 + |s|^{p-1}), \quad 1 < p \leq \frac{n}{n-2}, \text{ for } s \in R. \tag{3.2}$$

Setting $F(s) = \int_0^s f(\lambda)\mathrm{d}\lambda$, there exist $C_2, C_3 > 0$ satisfying

$$C_2|s|^{p+1} \leq F(s) \leq C_3 s f(s), \quad \text{for } s \in R. \tag{3.3}$$

We notice that from (3.2) and we derive that there exists $C_4 > 0$ such that

$$|f(s)| \leq C_4(1 + |s|^p), \quad \text{for } s \in R. \tag{3.4}$$

Also, we assume that there exists $C_5 > 0$ such that

$$|f(x) - f(y)| \leq C_5(|x|^{p-1} + |y|^{p-1})|x - y|, \quad \text{for } x, y \in R. \tag{3.5}$$

(A2) *Assumptions on g:*

Set $g : R \to R$ to be a non-decreasing $C^1(R)$ function,

$$g(s)s \geq 0, \quad \text{for } s \neq 0.$$

There exist C_i $(i = 6, 7, 8, 9)$, such that

$$C_6|s| \leq |g(s)| \leq C_7|s|, \quad \text{if } |s| \leq 1, \tag{3.6}$$

$$C_8|s|^q \leq |g(s)| \leq C_9|s|^q, \quad 1 < q \leq \frac{n-1}{n-2}, \text{ if } |s| > 1. \tag{3.7}$$

We have the following assumptions in order to obtain the global existence for strong solutions.

(A3) Assume that $0 < b_0 \le b(x) \le b_1$.
(A4) Assume that

$$\{u^0, u^1\} \in D(A) \times D(A), \tag{3.8}$$

satisfying the compatibility condition

$$\frac{\partial u}{\partial \nu} + g(u_t) = 0, \quad \text{on } \Gamma_0, \tag{3.9}$$

where $D(A) = \{u \in V, \Delta u \in L^2(\Omega)\}$.

Notations: In this paper C, and C_i will denote various positive constants which may be different at different occurrences and we denote $L^2(\Omega) = H$.

Now, we state our main results.

Theorem 3.1. *Under assumptions* A1 − A4, *the Eq.* (1.1) *admits a unique strong solution, that is, a function* $u(x, t) : [0, \infty) \times \Omega \to R$, *such that*

$$u \in L^\infty(0, \infty; V), \qquad u_t \in L^\infty(0, \infty; V), \qquad u_{tt} \in L^\infty(0, \infty; L^2(\Omega)).$$

Moreover, the energy determined by the strong solution $u(x, t)$ *decays exponentially. That is,*

$$E(t) = \frac{1}{2}\int_\Omega (\|u_t(x, t)\|^2 + \|\nabla u(x, t)\|^2)\mathrm{d}x + \int_\Omega F(u(x, t))\mathrm{d}x \le C\mathrm{e}^{-\gamma t}, \tag{3.10}$$

for some positive constants C *and* γ.

Theorem 3.2. *Assume that* $\{u^0, u^1\} \in V \times L^2(\Omega)$ *and assumptions* A1 − A3 *hold, then* (1.1) *has a unique weak solution,* $u(x, t) : [0, \infty) \times \Omega \to R$, *in the class*

$$u \in C(0, \infty; V) \cap C^1(0, \infty; L^2(\Omega)).$$

Furthermore, if $q = 1$, *then* (3.10) *holds for the weak solution* u.

4. Existence of strong and weak solutions

In this section, we prove the existence and uniqueness of strong solutions of the Eq. (1.1), when u^0, u^1 are smooth. First, we consider strong solutions by the Galerkin approximation and we extend the same result to weak solutions using a density argument.

Now, we consider the variational function of (1.1) as follows

$$(u_{tt}, w) + (\nabla u, \nabla w) + (b(x)u_t, w) + (g(u_t), w)_{\Gamma_0} + (f(u), w) = 0, \quad \forall w \in V. \tag{4.1}$$

Strong solutions to (1.1) with a boundary condition $(g(u_t), w)_{\Gamma_0}$ cannot be obtained by the method of "special basis", hence, a basis formed by eigenfunctions of the operator $-\Delta$ cannot be used for it. This leads us to differentiate the variational formulation related with (1.1) with respect to time t. However, this brings up serious difficulties when we estimate $u_{tt}(0)$.

In order to overcome these difficulties, we can transform the boundary value problem (1.1) into an equivalent one with zero initial data. In fact, we introduce the new variables

$$v(x, t) = u(x, t) - \phi(x, t), \tag{4.2}$$

where

$$\phi(x, t) = u^0(x) + tu^1(x), \quad t \in [0, T]. \tag{4.3}$$

Due to (3.8) (3.9) (4.1)–(4.3), we get the equivalent problem for variables v:

$$\begin{cases} v_{tt} - \Delta v + b(x)v_t + f(v + \phi) = \mathcal{F}, & \text{in } \Omega \times [0, \infty), \\ v = 0, \quad \text{on } \Gamma_1 \times [0, \infty), \\ \dfrac{\partial v}{\partial \nu} + g(v_t + \phi_t) = \mathcal{G}, & \text{on } \Gamma_0 \times [0, \infty), \\ v(0) = v_t(0) = 0, \quad \text{in } \Omega, \end{cases} \tag{4.4}$$

where

$$\mathcal{F} = \Delta\phi + b(x)\phi_t, \qquad \mathcal{G} = -\frac{\partial \phi}{\partial \nu}. \tag{4.5}$$

We note that if v is a solution of (4.4) in $[0, T]$, then $u = v + \phi$ is a solution of (1.1) in the same interval. From the estimates obtained below, we can prove that

$$\|\Delta v\|^2 + \|\nabla v'\|^2 \le C, \quad \forall t \in [0, T].$$

Hence, using standard methods, we can extend the solution u to the interval $(0, \infty)$. It is sufficient to prove that (4.4) has a local solution by using Galerkin method.

Let $(\omega_\upsilon)_{\upsilon\in N}$ be a basis in $D(A) = V \cap H^2(\Omega)$ which is orthonormal in $L^2(\Omega)$ and setting $V_m = span\{\omega_1, \omega_2, \ldots \omega_m\}$.

Now, we define $v_m(t) = \sum_{j=1}^{n} g_{jm}(t)\omega_j$, where $v_m(t)$ is the solution the Cauchy problem as follows:

$$\begin{aligned}&(v_m'', w) + (\nabla v_m, \nabla w) + (b(x)v_m', w) + (g(v_m' + \phi'), w)_{\Gamma_0} + (f(v_m + \phi), w) = (\mathcal{F}, w) + (\mathcal{G}, w)_{\Gamma_0}, \quad \forall w \in V_m,\\ &v_m(0) = v_m'(0) = 0.\end{aligned} \tag{4.6}$$

By standard methods of differential equations, we prove the existence of a solution (4.6) on some interval $[0, t_m]$, then, this solution can be extended to the whole interval $[0, T]$ by using the first estimate as follows.

First estimate

Taking $w = v_m'$ in (4.6), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|v_m'\|^2 + \|\nabla v_m\|^2 + 2\int_\Omega F(v_m + \phi)\mathrm{d}x\right\} + \frac{1}{2}b(x)\|v_m'\|^2 + (g(v_m' + \phi'), v_m' + \phi')_{\Gamma_0}\\ &\quad = (\mathcal{F}, v_m') + \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{G}, v_m)_{\Gamma_0} - (\mathcal{G}', v_m)_{\Gamma_0} + (f(v_m + \phi), v_m') + (g(v_m' + \phi'), \phi')_{\Gamma_0}.\end{aligned}$$

Due to (A3), we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|v_m'\|^2 + \|\nabla v_m\|^2 + 2\int_\Omega F(v_m + \phi)\mathrm{d}x\right\} + (g(v_m' + \phi'), v_m' + \phi')_{\Gamma_0}\\ &\quad \le (\mathcal{F}, v_m') + \frac{\mathrm{d}}{\mathrm{d}t}I_1(t) + I_2(t) + I_3(t) + I_4(t).\end{aligned} \tag{4.7}$$

Next, we shall estimate for I_i $(i = 1, 2, 3, 4)$ respectively. From (3.4) and applying Young's inequality, we get

$$\begin{aligned}|I_3| &= |(f(v_m + \phi), v_m')|\\ &\le C\int_\Omega (1 + |v_m + \phi|^p)|\phi'|\mathrm{d}x\\ &\le C\left\{\int_\Omega |\phi'|\mathrm{d}x + \int_\Omega |v_m + \phi|^{p+1}\mathrm{d}x + \int_\Omega |\phi'|^{p+1}\mathrm{d}x\right\}\\ &\le C + C\int_\Omega |v_m + \phi|^{p+1}\mathrm{d}x.\end{aligned} \tag{4.8}$$

At the same time, due to $\frac{q}{q+1} + \frac{1}{q+1} = 1$, we obtain

$$\begin{aligned}|I_4| &= |(g(v_m' + \phi'), \phi')|\\ &\le \varepsilon\int_{\Gamma_0} g(v_m' + \phi')^{\frac{q+1}{q}}\mathrm{d}\Gamma + C(\varepsilon)\int_{\Gamma_0} |\phi'|^{q+1}\mathrm{d}\Gamma.\end{aligned} \tag{4.9}$$

where ε is an arbitrary positive constant.

Also, from (3.7), we deduce

$$|g(s)|^{\frac{q+1}{q}} = |g(s)||g(s)|^{\frac{1}{q}} \le C_{11}|g(s)||s|, \quad |s| > 1. \tag{4.10}$$

Owing to the Poincare inequality, we conclude

$$\|v\|_{\Gamma_0} \le C\|\nabla v\|, \quad \forall v \in V. \tag{4.11}$$

From Cauchy–Schwarz's inequality, we have

$$\begin{aligned}|I_2| &= |(\mathcal{G}', v_m)_{\Gamma_0}|\\ &\le C\{\|\mathcal{G}'\|_{\Gamma_0}^2 + \|\nabla v_m\|^2\}.\end{aligned} \tag{4.12}$$

Combining (4.7)–(4.12), it follows that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|v_m'\|^2 + \|\nabla v_m\|^2 + 2\int_\Omega F(v_m + \phi)\mathrm{d}x\right\} + (1 - \varepsilon)\int_{|v_m' + \phi'| > 1} |g(v_m' + \phi')|^{\frac{q+1}{q}}\mathrm{d}\Gamma\\ &\quad \le C(\varepsilon) + \|\mathcal{F}\|^2 + \|\mathcal{G}'\|_{\Gamma_0}^2 + \frac{\mathrm{d}}{\mathrm{d}t}I_1(t) + C\left\{\int_\Omega |v_m + \phi|^{p+1}\mathrm{d}x + \|v_m'\|^2 + \|\nabla v_m\|^2\right\}.\end{aligned} \tag{4.13}$$

Integrating (4.13) over the interval $(0, t)$, observing that $v_m(0) = v_m'(0) = 0$ and taking (3.3) into account, it follows that

$$\frac{1}{2}\|v_m'\|^2 + \frac{1}{2}\|\nabla v_m\|^2 + \int_\Omega F(v_m+\phi)\mathrm{d}x + (1-\varepsilon)\int_0^t \int_{|v_m'+\phi'|>1} |g(v_m'+\phi')|^{\frac{q+1}{q}}\mathrm{d}\Gamma\mathrm{d}s$$
$$\le C + C\int_0^t \left\{\int_\Omega F(v_m+\phi)\mathrm{d}x + \int_{|v_m'+\phi'|>1} |g(v_m'+\phi')|^{\frac{q+1}{q}}\mathrm{d}\Gamma + \|v_m'\|^2 + \|\nabla v_m\|^2\right\}\mathrm{d}s + (\mathcal{G}, v_m)_{\Gamma_0}. \tag{4.14}$$

Due to (4.12), we have

$$|I_1| = |(\mathcal{G}, v_m)_{\Gamma_0}|$$
$$\le \frac{C^2}{4\varepsilon}\|\mathcal{G}\|_{\Gamma_0}^2 + \varepsilon\|\nabla v_m\|^2. \tag{4.15}$$

From (4.14) (4.15), choosing a small enough $\varepsilon > 0$ and using Gronwall's inequality, we get

$$\|v_m'\|^2 + \|\nabla v_m\|^2 + \int_\Omega F(v_m+\phi)\mathrm{d}x + \int_0^t\int_{\Gamma_0} |g(v_m'+\phi')|^{\frac{q+1}{q}}\mathrm{d}\Gamma\mathrm{d}s \le C. \tag{4.16}$$

Second estimate

Taking $w = v_m''(0)$ in (4.6) and noticing that $v_m(0) = v_m'(0) = 0$, we obtain

$$\|v_m''(0)\|^2 + (g(u^1), v_m''(0))_{\Gamma_0} + (f(u^0), v_m''(0)) = (\Delta u^0 + b(x)\phi_t, v_m''(0)) + \left(-\frac{\partial u^1}{\partial \nu}, v_m''(0)\right)_{\Gamma_0}. \tag{4.17}$$

From (A3), (3.9) and (4.17), we obtain

$$\|v_m''(0)\|^2 \le (\|f(u^0)\| + \|\Delta u^0(0)\| + b_1 u^1)\|v_m''(0)\|.$$

Owing to (3.2) (3.6), we deduce that

$$\|v_m''(0)\| \le C. \tag{4.18}$$

Also, taking the derivative of (4.6) with respect to time t and taking $w = v_m''(t)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}\|v_m''\|^2 + \frac{1}{2}\|\nabla v_m'\|^2\right\} + b(x)\|v_m''\|^2 + \int_{\Gamma_0} g(v_m'+\phi')(v_m'')^2\mathrm{d}\Gamma + \int_\Omega f'(v_m+\phi)(v_m'+\phi')v_m''\mathrm{d}\Gamma$$
$$= (\mathcal{F}', v_m'') + \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{G}', v_m')_{\Gamma_0}. \tag{4.19}$$

Next, we shall estimate some terms of (4.19).

Firstly, we estimate $I_5 = \int_\Omega f'(v_m+\phi)(v_m'+\phi')v_m''\mathrm{d}\Gamma$.

Owing to (3.2), we can get that

$$|I_5| \le C\int_\Omega (1 + |v_m+\phi|^{p-1})|v_m'+\phi'||v_m''|\mathrm{d}x. \tag{4.20}$$

Noticing that $\frac{p-1}{2p} + \frac{1}{2p} + \frac{1}{2} = 1$, from (4.20) and applying the generalized Holder's inequality, we obtain that

$$|I_5| \le C(\|v_m'+\phi'\|^2 + |v_m''|^2) + C(\|v_m+\phi\|_{\frac{2p}{p-1}}^{p-1}\|v_m'+\phi'\|_{2p}\|v_m''\|). \tag{4.21}$$

Since $p \le \frac{n}{n-2}$, and using the Sobolev embedding theory (see [31]), we get

$$H^1(\Omega) \hookrightarrow L^{2p}(\Omega). \tag{4.22}$$

From (4.21) (4.22), we obtain that

$$|I_6| \le C(1 + \|v_m''\|^2 + \|\nabla v_m'\|^2). \tag{4.23}$$

Combining (4.19) (4.21) (4.23) and observing that $g'(s) \ge 0$, $b(x) \ge b_0 > 0$, we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}\|v_m''\|^2 + \frac{1}{2}\|\nabla v_m'\|^2\right\} \le \|\mathcal{F}'\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{G}', v_m')_{\Gamma_0} + C(1 + \|v_m''\|^2 + \|\nabla v_m'\|^2). \tag{4.24}$$

Integrating (4.24) over the interval $(0, t)$, and noticing that (4.18) and $v_m'(0) = 0$, it follows that

$$\|v_m''\|^2 + \|\nabla v_m'\|^2 \le C + (\mathcal{G}', v_m')_{\Gamma_0} + C\int_0^t (\|v_m''\|^2 + \|\nabla v_m'\|^2)\mathrm{d}s. \tag{4.25}$$

Also, by Young's inequality, we have

$$(\mathcal{G}', v_m')_{\Gamma_0} \le \frac{C^2}{4\varepsilon}\|\mathcal{G}'\|_{\Gamma_0}^2 + \varepsilon\|\nabla v_m'\|^2. \tag{4.26}$$

From (4.25) (4.26), choosing a $\varepsilon > 0$ that is sufficiently small enough, and applying Gronwall inequality, we get

$$\|v_m''\|^2 + \|\nabla v_m'\|^2 \le C. \tag{4.27}$$

Next, we shall prove the existence of solutions and analyze the nonlinear terms f, g.

Analysis of f:

From (3.4) and the first estimate, we have

$$\int_0^t \int_\Omega f(v_m+\phi)^{\frac{p+1}{p}}\,\mathrm{d}x\mathrm{d}s \le C\int_0^t \int_\Omega 1 + |v_m+\phi|^{\frac{p+1}{p}}\,\mathrm{d}x\mathrm{d}s \le C. \tag{4.28}$$

The inequality (4.28) implies that

$$\{f(v_m+\phi)\} \quad \text{is bounded in } L^{\frac{p+1}{p}}(Q_T), \tag{4.29}$$

where $Q_T = \Omega \times (0, T)$.

From (4.27), we can find a subsequence v_m(still denote v_m) such that

$$v_m \to v \quad \text{strong in } L^2(Q_T), \tag{4.30}$$

then, $v_m \to v$ a.e in Q_T. Hence, we get

$$f(v_m+\phi) \to f(v+\phi) \quad \text{a.e in } Q_T. \tag{4.31}$$

Due to (4.29) (4.31) and applying Lion's lemma (see [32]), we deduce

$$f(v_m+\phi) \to f(v+\phi) \quad \text{weakly in } L^{\frac{p+1}{p}}(Q_T). \tag{4.32}$$

Analysis of g:

From the first estimate, we obtain

$$\{g(v_m'+\phi')\} \quad \text{is bounded in } L^{\frac{q+1}{q}}(\Sigma),$$

where $\Sigma = \Gamma_0 \times (0, T)$. Therefore, there exists $\varrho \in L^{\frac{q+1}{q}}(\Sigma)$, such that

$$g(v_m'+\phi') \to \varrho \quad \text{weakly in } L^{\frac{q+1}{q}}(\Sigma). \tag{4.33}$$

From the first and second estimate, and the Sobolev embedding theory (see [31]), it is easy to see that

$$v_m' \to v \quad \text{in } L^2(0,\infty, L^2(\Omega)).$$

We also notice that

$$\begin{aligned}\int_{\Gamma_0} |g(v_m'+\phi')|^2\mathrm{d}\Gamma &= \int_{|v_m'+\phi'|\le 1} |g(v_m'+\phi')|^2\mathrm{d}\Gamma + \int_{|v_m'+\phi'|>1} |g(v_m'+\phi')|^2\mathrm{d}\Gamma \\ &\le C + C\|v_m'+\phi'\|_{2q,\Gamma_0}^{2q} \\ &\le C + C\|v_m'+\phi'\|_2^{2q} \\ &\le C.\end{aligned}$$

Then, we deduce that

$$g(v_m'+\phi') \to \varrho \quad \text{weakly in } L^2(\Sigma). \tag{4.34}$$

From (4.6) and above our arguments, we have

$$v_{tt} - \Delta v + b(x)v_t + f(v+\phi) = \mathcal{F}, \quad \text{in } L^2(0,\infty, L^2(\Omega)). \tag{4.35}$$

Also, we have

$$\frac{\partial v}{\partial \nu} + \varrho = \mathcal{G}, \quad \text{in } L^2(0,\infty, L^2(\Gamma_0)). \tag{4.36}$$

Next, we shall show $\varrho = g(v+\phi')$. For this goal, we shall use monotonicity arguments.

First of all, we notice that from the first and second estimates and applying Aubin–Lion's Theorem (see [32]), then we have

$$v_m' \to v' \quad \text{weakly in } L^2(0,T;H). \tag{4.37}$$

Considering $w = v_m'$ in (4.6), integrating over $[0,T]$, we deduce

$$v'' - \nabla v + b(x)v' + f(v+\phi) = \mathcal{F} \in L^2(0,\infty;H),$$
$$\frac{\partial v}{\partial \nu} + \varrho = \mathcal{G} \in L^2(0,\infty;L^2(\Gamma_0)).$$

Also, we have

$$\lim_{m\to\infty}\int_0^T (g(v_m'+\phi'), v_m'+\phi')\mathrm{d}t = \int_0^T (\varrho, v'+\phi')_{\Gamma_0}\mathrm{d}t. \tag{4.38}$$

Owing to function g is a non-decreasing monotone function, we have

$$\int_0^T \langle g(v_m'+\phi') - g(\psi), v_m'+\phi'-\psi\rangle \mathrm{d}t \ge 0, \quad \forall \psi \in L^{q+1}(\Gamma_0),$$

where $\langle\cdot,\cdot\rangle$ means the duality between Sobolev space $L^{\frac{q+1}{q}}(\Gamma_0)$ and $L^{q+1}(\Gamma_0)$. So, we deduce

$$\int_0^T \langle g(v_m'+\phi'), \psi\rangle \mathrm{d}t + \int_0^T \langle g(\psi), v_m'+\phi'-\psi\rangle \mathrm{d}t \le \int_0^T \langle g(v_m'+\phi'), v_m'+\phi'\rangle \mathrm{d}t. \tag{4.39}$$

From (4.39), we get

$$\begin{aligned}&\lim_{m\to\infty}\inf\int_0^T \langle g(v_m'+\phi'), \psi\rangle \mathrm{d}t + \lim_{m\to\infty}\inf\int_0^T \langle g(\psi), v_m'+\phi'-\psi\rangle \mathrm{d}t\\ &\quad\le \lim_{m\to\infty}\inf\int_0^T \langle g(v_m'+\phi'), v_m'+\phi'\rangle \mathrm{d}t.\end{aligned} \tag{4.40}$$

Since

$$\|v_m\|_{q+1,\Gamma_0} \le C\|\nabla v_m'\| \le C,$$

then, it follows that

$$v_m' \to v' \quad \text{weakly star in } L^\infty(0,T;L^{q+1}(\Gamma_0)). \tag{4.41}$$

From (4.33) (4.38) (4.40) (4.41), we obtain

$$\int_0^T \langle \varrho - g(\psi), v_m'+\phi'-\psi\rangle \mathrm{d}t \ge 0. \tag{4.42}$$

Finally, we apply the monotone method to obtain $\varrho = g(v'+\phi')$.

Setting $\psi = (v'+\phi') + \lambda\xi$ in (4.42), where ξ is an arbitrary element of $L^{q+1}(\Gamma_0)$ and $\lambda > 0$, we obtain

$$\int_0^T \langle \varrho - g(v'+\phi'+\lambda\xi), -\lambda\xi\rangle \mathrm{d}t \ge 0.$$

So,

$$\int_0^T \langle \varrho - g(v'+\phi'+\lambda\xi), -\xi\rangle \mathrm{d}t \le 0, \quad \forall \xi \in L^{q+1}(\Gamma_0).$$

The operator

$$g : L^{q+1}(\Gamma_0) \to L^{\frac{q+1}{q}}(\Gamma_0) : v| \to g(v)$$

is hemi-continuous and we get

$$\int_0^T \langle \varrho - g(v'+\phi'), \xi\rangle \mathrm{d}t \le 0, \quad \forall \xi \in L^{q+1}(\Gamma_0).$$

Also, we have

$$\int_0^T \langle \varrho - g(v'+\phi'), \xi\rangle \mathrm{d}t \ge 0, \quad \forall \xi \in L^{q+1}(\Gamma_0).$$

Hence,

$$\int_0^T \langle \varrho - g(v' + \phi'), \xi \rangle \mathrm{d}t = 0, \quad \forall \xi \in L^{q+1}(\Gamma_0),$$

which implies

$$\varrho = g(v' + \phi'). \tag{4.43}$$

Uniqueness:

Let u_1 and u_2 be two smooth solutions to problem (1.1), then, $z = u_1 - u_2$ satisfies

$$\begin{aligned} &(z'', w) + (\nabla z, \nabla w) + (b(x)z', w) + (g(u_1') - g(u_2'), w)_{\Gamma_0} + (f(u_2) - f(u_1), w) = 0, \quad \forall w \in V, \\ &z(0) = z'(0) = 0. \end{aligned} \tag{4.44}$$

Putting $w = z'(t)$ in (4.44), we deduce

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z'\|^2 + \|\nabla z\|^2 + b(x)\|z'\|^2 + (g(u_1') - g(u_2'), z')_{\Gamma_0} = (f(u_2) - f(u_1), z'). \tag{4.45}$$

Next, we estimate the terms on the right hand side of (4.45).

Estimate for $J = (f(u_2) - f(u_1), z')$

From (3.5), we get

$$\begin{aligned} |J| &\le C \int_\Omega (|u_2|^{p-1} + |u_1|^{p-1})|z||z'|\mathrm{d}x \\ &\le C(\|u_2\|_{2p} + \|u_1\|_{2p})\|z\|_{2p}\|z'\| \\ &\le C(\|\nabla z\|^2 + \|z'\|^2). \end{aligned} \tag{4.46}$$

From (A3) (4.45)–(4.46),observing that g is monotone function and making using of Gronwall's Lemma (see [33]), we deduce $\|\nabla z\| = \|z'\| = 0$ and so, $u_2 = u_1$.

Then, we complete the proof of the existence and uniqueness of smooth solutions.

Existence of weak solutions

Consider

$$\{u^0, u^1\} \in V \times H \tag{4.47}$$

and set

$$D(-\Delta) = \left\{ v \in V \cap H^2(\Omega);\ \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma_0 \right\}.$$

Due to the Sobolev space theory (see [31]), we get

$$D(-\Delta) \text{ is dense in } V \quad \text{and} \quad H_0^1(\Omega) \cap H^2(\Omega) \text{ is dense in } H,$$

then, there exist $\{u_\mu^0\} \subset D(-\Delta)$ and $\{u_\mu^1\} \subset H_0^1(\Omega) \cap H^2(\Omega)$, such that

$$u_\mu^0 \to u^0 \quad \text{strongly in } V, \tag{4.48}$$

$$u_\mu^1 \to u^1 \quad \text{strongly in } H. \tag{4.49}$$

Furthermore, $\frac{\partial u^0}{\partial \nu} + g(u_\mu^1) = 0$ on Γ_0. Therefore, there exists $u_\mu : Q \to R$, a smooth solution of problem (1.1), satisfying

$$\begin{cases} u_\mu'' - \Delta u_\mu + b(x)u_\mu' + f(u_\mu) = 0, \quad \text{in } L^2[0, \infty; H), \\ u_\mu = 0, \quad \text{on } \Gamma_1, \\ \dfrac{\partial u_\mu}{\partial \nu} + g(u_\mu') = 0, \quad \text{on } L^2[0, \infty; L^2(\Gamma_0)), \\ u_\mu(0) = u_\mu^0, \qquad u_\mu'(0) = u_\mu^1. \end{cases} \tag{4.50}$$

Repeating the same discussions, we get

$$\|u_\mu'\|^2 + \|\nabla u_\mu\|^2 + \int_0^t \int_\Omega |u_\mu|^{p+1}\mathrm{d}x\mathrm{d}t + \int_0^t \int_{\Gamma_0} |g(u_\mu')|^{\frac{q+1}{q}} \mathrm{d}\Gamma \mathrm{d}s \le C, \tag{4.51}$$

and

$$\int_0^t \int_\Omega |f(u_\mu)|^{\frac{p+1}{p}} \mathrm{d}x\mathrm{d}s \le C, \tag{4.52}$$

$$\int_0^t \int_{\Gamma_0} |u'_\mu|^{q+1} \mathrm{d}\Gamma \mathrm{d}s \le C, \tag{4.53}$$

for all $t \in [0, T]$ and $\mu \in N$.

Setting $z_{\mu\sigma} = u_\mu - u_\sigma$, $\mu, \sigma \in N$, where u_μ, u_σ are smooth solutions of (4.50).

Repeating the same discussions used in finding the existence and uniqueness of strong solutions, we deduce that there exists $u : Q \to R$ such that

$$u_\mu \to u \quad \text{in } L^\infty([0, T]; V), \tag{4.54}$$

$$u'_\mu \to u' \quad \text{in } L^\infty([0, T]; H). \tag{4.55}$$

Furthermore, from (4.51) (4.52), we get

$$u'_\mu \to u' \quad \text{weakly in } L^{q+1}(\Sigma), \tag{4.56}$$

$$f(u_\mu) \to \eta \quad \text{weakly in } L^{\frac{p+1}{p}}(Q_T), \tag{4.57}$$

$$g(u'_\mu) \to \chi \quad \text{weakly in } L^{\frac{q+1}{q}}(\Sigma). \tag{4.58}$$

From (4.54) and Lion's Lemma (see [32]), it is easy to obtain that $\eta = f(u)$.

Furthermore, we have

$$\begin{cases} u'' - \Delta u + b(x)u' + f(u) = 0 \\ u(0) = u^0, \qquad u'(0) = u^1 \end{cases} \tag{4.59}$$

Next, we shall prove $\chi = g(u')$.

Indeed, by multiplying the first Eq. (4.49) by u_μ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{\|u'_\mu\|^2 + \|\nabla u_\mu\|^2\} + b(x)\|u'_\mu\|^2 + (f(u_\mu), u'_\mu) + (g(u'_\mu), u'_\mu)_{\Gamma_0} = 0.$$

Also, we have

$$\frac{1}{2}\|u'_\mu\|^2 + \frac{1}{2}\|\nabla u_\mu\|^2 + b(x)\|u'_\mu\|^2 + \int_0^t (f(u_\mu(s)), u'_\mu(s))\mathrm{d}s$$
$$+ \int_0^t (g(u'_\mu(s)), u'_\mu(s))_{\Gamma_0}\mathrm{d}s = \frac{1}{2}\|u^1_\mu\|^2 + \frac{1}{2}\|\nabla u^0_\mu\|^2. \tag{4.60}$$

From (4.47) (4.48) (4.53) (4.54) (4.56) (4.57) (4.59), we deduce

$$\lim_{\mu\to\infty} \int_0^t (g(u'_\mu(s)), u'_\mu(s))_{\Gamma_0}\mathrm{d}s = \frac{1}{2}\|u^1\|^2 + \frac{1}{2}\|\nabla u^0\|^2 - \frac{1}{2}\|u'\|^2 - \frac{1}{2}\|\nabla u\|^2 - b(x)\|u'\|^2 - \int_0^t (f(u(s)), u'(s))\mathrm{d}s. \tag{4.61}$$

At the same time, assuming that ω is a weak solution to problem

$$\begin{cases} \omega'' - \Delta\omega + b(x)\omega' + f(\omega) = 0, \quad \text{in } \Omega \times (0, \infty) \\ \omega = 0, \quad \text{on } \Gamma_1 \times (0, \infty) \\ \dfrac{\partial\omega}{\partial\nu} + \chi = 0, \quad \text{on } \Gamma_0 \times (0, \infty) \\ \omega(0) = u^0, \qquad \omega'(0) = u^1 \end{cases} \tag{4.62}$$

then, we have

$$\int_0^t (\chi(s)), \omega'(s)_{\Gamma_0}\mathrm{d}s = \frac{1}{2}\|u^1\|^2 + \frac{1}{2}\|\nabla u^0\|^2 - \frac{1}{2}\|\omega'\|^2 - \frac{1}{2}\|\nabla\omega\|^2 - b(x)\|\omega'\|^2 - \int_0^t (f(\omega(s)), \omega'(s))\mathrm{d}s. \tag{4.63}$$

Since u is a weak solution to problem (4.62), hence, from (4.61) and (4.63), we get

$$\lim_{\mu\to\infty} \int_0^t (g(u'_\mu(s)), u'_\mu(s))_{\Gamma_0}\mathrm{d}s = \int_0^t (\chi(s)), u'(s)_{\Gamma_0}\mathrm{d}s.$$

According to the above arguments, it is easy to show that $\chi = g(u')$.

Remark 4.1. For the uniqueness of weak solutions, we require a regularization procedure using standard arguments (see [32]).

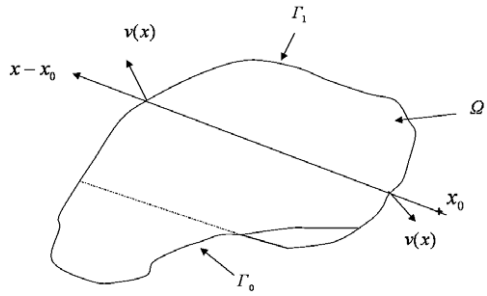

Fig. 2. It shows a compact Γ such that there exist Γ_0 and Γ_1, where $\Gamma = \partial\Omega$.

5. Asymptotic behavior

In this section, we shall prove the exponential decay for a strong solution of (1.1), and by density arguments we obtain the same results for the weak solutions.

Let x_0 be a fixed point in space R^n, set

$$m = m(x) = x - x_0 = (x_1 - x_0, x_2 - x_0, \ldots, x_n - x_0),$$
$$R(x_0) = \max\{\|m(x)\|, x \in \overline{\Omega}\},$$

and partition the boundary Γ into two sets as follows:

$$\Gamma_0 = \{x \in \Gamma, m(x) \cdot \nu(x) \geq 0\}, \qquad \Gamma_1 = \{x \in \Gamma, m(x) \cdot \nu(x) < 0\},$$

where

$$m(x) \cdot \nu(x) = \sum_{i=1}^{n} (x_i - x_0)\nu_i, \quad (i = 1, \ldots, n)$$

and $\nu = (\nu_1, \ldots, \nu_n)$ denotes the unit normal on Γ directed towards the outward of the bounded domain Ω (see Fig. 2 as follows)

The derivative of the energy is given as follows:

$$E'(t) + b(x) \int_{\Omega} |u'|^2 \mathrm{d}x = -\int_{\Gamma_0} g(u')u' \mathrm{d}\Gamma. \tag{5.1}$$

Let us introduce perturbed energy functional as follows:

$$E_\varepsilon(t) = E(t) + \varepsilon\mathscr{E}(t), \tag{5.2}$$

where

$$\mathscr{E}(t) = \int_{\Omega} u'(2m \cdot \nabla u + (n-1)u)\mathrm{d}x. \tag{5.3}$$

Also, let C_{10} be positive constant, such that

$$\int_{\Gamma_0} (m \cdot \nu)v^2 \mathrm{d}\Gamma \leq C_{10} \int_{\Omega} |v|^2 \mathrm{d}x, \quad \forall v \in V, \tag{5.4}$$

and $C_{11} > 0$ such that

$$|v|^2 \leq C_{11}|\nabla v|^2, \quad \forall v \in V. \tag{5.5}$$

Next, we need the following lemma and shall prove it.

Lemma 5.1. *There exists $\theta_1 > 0$, such that*

$$|E_\varepsilon(t) - E(t)| \leq \varepsilon\theta_1 E(t), \quad \forall t \geq 0, \ \forall \varepsilon > 0.$$

Proof. From (5.3) (5.5) and Young's inequality, we have

$$|\mathscr{E}(t)| \leq 2R(x^0)|u'\|\nabla u| + (n-1)\sqrt{C_{11}}|u'\|\nabla u| \leq (2R(x^0) + (n-1)\sqrt{C_{11}})E(t). \tag{5.6}$$

Putting $\theta_1 = (2R(x^0) + (n-1)\sqrt{C_{11}})$, then from (5.2) and (5.6), we deduce

$$|E_\varepsilon(t) - E(t)| = \varepsilon|\mathscr{E}(t)| \leq \varepsilon\theta_1 E(t).$$

Finally, we shall show uniform decay for solutions we consider, then we shall prove the result. □

Theorem 5.1. *There exist $\theta_2 > 0$, such that*

$$E'_\varepsilon(t) \le -\theta_2 E(t), \quad \forall t \ge 0.$$

Proof. From (5.3), we have

$$\mathscr{E}'(t) = \int_\Omega 2u''(m\cdot\nabla u) + 2u'(m\cdot\nabla u') + (n-1)u''u + (n-1)(u')^2 \mathrm{d}x. \tag{5.7}$$

Firstly, we are going to analyze the terms in the right hand side of (5.7). □

Estimate for $J_1 = 2\int_\Omega u''(m\cdot\nabla u)\mathrm{d}x$.
Applying Green and Gauss's Theorem (see [32]), we get

$$\begin{aligned} J_1 &= 2\int_\Omega (\Delta u - b(x)u' - f(u))(m\cdot\nabla u)\mathrm{d}x \\ &= (n-2)\int_\Omega |\nabla u|^2\mathrm{d}x - \int_{\Gamma_0}(2u'(m\cdot\nabla u) + |\nabla u|^2)\mathrm{d}\Gamma + 2n\int_\Omega F(u)\mathrm{d}x - 2\int_\Omega b(x)u'(m\cdot\nabla u)\mathrm{d}x. \end{aligned} \tag{5.8}$$

Estimate for $J_2 = 2\int_\Omega u'(m\cdot\nabla u')\mathrm{d}x$.

$$\begin{aligned} J_2 &= \int_\Omega m\cdot(\nabla u')^2\mathrm{d}x \\ &= -n\int_\Omega |u'|^2\mathrm{d}x + \int_{\Gamma_0}|u'|^2\mathrm{d}\Gamma. \end{aligned} \tag{5.9}$$

Estimate for $J_3 = \int_\Omega u''u\mathrm{d}x$.
Taking into account the generalized Green formula, we infer

$$\begin{aligned} J_3 &= \int_\Omega (\Delta u - b(x)u' - f(u))u\mathrm{d}x \\ &= -\int_\Omega |\nabla u|^2\mathrm{d}x - \int_\Omega b(x)u'u\mathrm{d}x - \int_\Omega f(u)u\mathrm{d}x - \int_{\Gamma_0} g(u')u\mathrm{d}\Gamma. \end{aligned} \tag{5.10}$$

Next, we estimate for $E'_\varepsilon(t)$. Combining (5.1) (5.2) (5.8)–(5.10), it follows that

$$\begin{aligned} E'_\varepsilon(t) \le& -\int_{\Gamma_0} g(u')u'\mathrm{d}\Gamma + \varepsilon\Bigg\{(n-2)\int_\Omega |\nabla u|^2\mathrm{d}x - \int_{\Gamma_0}(2u'(m\cdot\nabla u) + |\nabla u|^2)\mathrm{d}\Gamma + 2n\int_\Omega F(u)\mathrm{d}x \\ &- 2\int_\Omega b(x)u'(m\cdot\nabla u)\mathrm{d}x - n\int_\Omega |u'|^2\mathrm{d}x + \int_{\Gamma_0}|u'|^2\mathrm{d}\Gamma - (n-1)\int_{\Gamma_0} g(u')u\mathrm{d}\Gamma \\ &- (n-1)\int_\Omega |u|^2\mathrm{d}x - (n-1)\int_\Omega b(x)u'u\mathrm{d}x - (n-1)\int_\Omega f(u)u\mathrm{d}x + (n-1)\int_\Omega |u'|^2\mathrm{d}x\Bigg\}. \end{aligned} \tag{5.11}$$

Applying Young's inequality and the Poincare inequality, we have

$$\left|\int_\Omega b(x)u'u\mathrm{d}x\right| \le b_1\int_\Omega |u'|^2\mathrm{d}x + b_1\int_\Omega |u|^2\mathrm{d}x \tag{5.12}$$

where $\int_\Omega |u|^2\mathrm{d}x \le C_{12}\int_\Omega |\nabla u|^2\mathrm{d}x$.
From (5.11) and (5.12), we deduce

$$\begin{aligned} E'_\varepsilon(t) \le& -\left(\varepsilon - \frac{1}{2}\right)\int_\Omega |u'|^2\mathrm{d}x - \varepsilon[(n-2) - (n-1)b_1C_{12}]\int_\Omega |\nabla u|^2\mathrm{d}x + 2\varepsilon n\int_\Omega F(u)\mathrm{d}x \\ &- \varepsilon(n-1)\int_\Omega f(u)u\mathrm{d}x - 2\varepsilon\int_\Omega b(x)u'(m\cdot\nabla u)\mathrm{d}x - \varepsilon(n-1)\int_{\Gamma_0} g(u')u\mathrm{d}\Gamma - \int_{\Gamma_0} g(u')u'\mathrm{d}\Gamma \\ &- \varepsilon\int_{\Gamma_0}(2u'(m\cdot\nabla u) + |\nabla u|^2)\mathrm{d}\Gamma + \varepsilon\int_{\Gamma_0}|u'|^2\mathrm{d}\Gamma. \end{aligned} \tag{5.13}$$

On the other hand, applying Young's inequality, we have

$$\int_{\Gamma_0} |2u'(m\cdot\nabla u)|\mathrm{d}\Gamma \le R^2\int_{\Omega_0}|u'|^2\mathrm{d}\Gamma + \int_{\Omega_0}|\nabla u|^2\mathrm{d}\Gamma, \tag{5.14}$$

$$\int_{\Gamma_0} |(n-1)g(u')u|\mathrm{d}\Gamma \le \frac{(n-1)^2}{2}\int_{\Omega_0}|g(u')|^2\mathrm{d}\Gamma + C_{13}E(t). \tag{5.15}$$

Due to (A3) and Young's inequality, we get

$$2\int_{\Omega}|b(x)u'(m\cdot\nabla u)|\mathrm{d}x \le b_1R^2\int_{\Omega_0}|u'|^2\mathrm{d}x + b_1\int_{\Omega_0}|\nabla u|^2\mathrm{d}x. \tag{5.16}$$

From (5.13)–(5.16), we obtain

$$\begin{aligned} E_\varepsilon'(t) \le &-\left(\varepsilon+\varepsilon b_1R^2-\frac{1}{2}\right)\int_{\Omega}|u'|^2\mathrm{d}x - \{b_1+\varepsilon[(n-2)-(n-1)b_1C_{12}]\}\int_{\Omega}|\nabla u|^2\mathrm{d}x + 2\varepsilon n\int_{\Omega}F(u)\mathrm{d}x \\ &-\varepsilon(n-1)\int_{\Omega}f(u)u\mathrm{d}x - \int_{\Gamma_0}g(u')u'\mathrm{d}\Gamma - \varepsilon R^2\int_{\Gamma}|u'|^2\mathrm{d}\Gamma - \varepsilon\int_{\Gamma_0}|\nabla u|^2\mathrm{d}\Gamma \\ &-\frac{(n-1)^2\varepsilon}{2}\int_{\Gamma_0}|g(u')|^2\mathrm{d}\Gamma - C_{13}\varepsilon E(t). \end{aligned} \tag{5.17}$$

From (5.17), taking the (3.7) into account and assuming that there exist positive constants C_{14}, C_{15} such that

$$C_{14}|s| \le |g(s)| \le C_{15}|s|, \quad \forall s\in R,$$

we deduce

$$\begin{aligned} E_\varepsilon'(t) \le &-\left(\varepsilon+\varepsilon b_1R^2-\frac{1}{2}-\frac{C_{13}\varepsilon}{2}\right)\int_{\Omega}|u'|^2\mathrm{d}x - \{b_1+\varepsilon[(n-2)-(n-1)b_1C_{12}]\}\int_{\Omega}|\nabla u|^2\mathrm{d}x \\ &-\left(-2\varepsilon n+\frac{\varepsilon(n-1)}{C_3}-\frac{C_{13}\varepsilon}{2}\right)\int_{\Omega}F(u)\mathrm{d}x - \left(1-\varepsilon\left(\frac{2(1+R^2)}{C_{14}}+2(n-1)^2C_{15}\right)\right)\int_{\Gamma_0}g(u')u'\mathrm{d}\Gamma. \end{aligned} \tag{5.18}$$

Owing to g being a non-decreasing function and $g(s)s \ge 0$, if we choose the constants $b_1, \varepsilon, C_3, C_{12}, C_{13}, C_{14}, C_{15}$ such that

$$\begin{aligned} &\varepsilon+\varepsilon b_1R^2-\frac{1}{2}-\frac{C_{13}\varepsilon}{2} > 0, \\ &b_1+\varepsilon[(n-2)-(n-1)b_1C_{12}] > 0, \\ &-2\varepsilon n+\frac{\varepsilon(n-1)}{C_3}-\frac{C_{13}\varepsilon}{2} > 0, \\ &1-\varepsilon\left(\frac{2(1+R^2)}{C_{14}}+2(n-1)^2C_{15}\right) > 0, \end{aligned}$$

then, we conclude the proof of Theorem 5.1 and consequently we obtain the exponential decay of the energy.

Remark 5.1. If we consider the case $f(u)=0$ and $\Gamma_0=0$ in the equation (1.1), then we can see [28].

Remark 5.2. Martinez [34] has studied the energy decay of solutions without damping term and source term

$$\begin{cases} u_{tt}-\Delta u = 0, & \text{in } \Omega\times[0,\infty), \\ u=0, & \text{on } \Gamma_1\times[0,\infty), \\ \dfrac{\partial u}{\partial\nu}+g(u_t)=0, & \text{on } \Gamma_0\times[0,\infty), \\ u(x,0)=u^0(x), \quad u_t(x,0)=u^1(x), & \text{in } \Omega, \end{cases}$$

where for some positive constants C_1, C_2 one has

$$C_1|s| \le |g(s)| \le C_2|s|, \quad |s|\ge 1.$$

Remark 5.3. It is worth mentioning that Cavalcanti et al. [35] recently investigated the uniform decay rates of solutions to the wave equation with Cauchy–Ventael boundary conditions

$$\begin{cases} u_{tt}-\Delta u+b(x)h(u_t)=0, & \text{in } \Omega\times[0,\infty), \\ u=0, & \text{on } \Gamma_1\times[0,\infty), \\ \dfrac{\partial u}{\partial\nu}-\Delta_{\Gamma_0}u=0, & \text{on } \Gamma_0\times[0,\infty), \\ u(x,0)=u^0(x), \quad u_t(x,0)=u^1(x), & \text{in } \Omega, \end{cases}$$

under some assumptions on the nonlinear feedback function h and $b(x)$. Furthermore, as for the above system, it is known that if $b(x) = 0$ then the uniform exponential stability never holds, even if a linear frictional feedback is applied to the entire boundary of the domain (we can see Theorem 3.1 in reference [36]).

Remark 5.4 (*Counter Example*). In this remark, we give an example where one would not be sufficient to produce an exponential decay of the solutions of problem (1.1). In our work, we show that the solution decays exponentially. More precisely, the method used is based on the construction of suitable Lyapunov functionals, $E'_\varepsilon(t)$, satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\varepsilon(t) \le -c_1E_\varepsilon(t) + c_2E(t)^{-\gamma t}$$

for some positive constants c_1, c_2, γ and $p > 1$.

To construct such a functional $E_\varepsilon(t)$ we start from the energy of problem (1.1), Then we seek other functions whose derivatives introduce negative terms such as: $-\int_\Omega |\nabla u|^2\mathrm{d}x$, $-\int_\Omega |\nabla u_t|^2\mathrm{d}x$ until it is possible to construct the whole energy on the right hand side of the above inequality with a negative sign. If we consider nonlinear feedback

$$g(x) = \frac{-x}{\sqrt{1+x^2}},$$

then we deduce $g(x)x \le 0$. Furthermore, letting $b(x) \le 0$, then by (5.1) and the above discussion, we conclude that

$$E'(t) = -\int_{\Gamma_0} g(u')u'\mathrm{d}\Gamma - b(x)\int_\Omega |u'|^2\mathrm{d}x \ge 0.$$

Therefore, the uniform exponential stability never holds, even if a linear frictional feedback is applied to the entire boundary of the domain.

6. Further remarks

(1) Let $-\Delta$ be the operator defined by the triple $\{V, H, a(u, v)\}$, where

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v\mathrm{d}x, \quad \forall u, v \in V$$

and

$$D(-\Delta) = \left\{u \in V \cap H^2(\Omega),\ \frac{\partial u}{\partial \nu} + g(u_t) = 0,\ \text{on } \Gamma_0 \times [0, \infty)\right\},$$

where Ω is a bounded domain of R^n.

Now we recall that the spectral theorem for self-adjoint operators guarantees the existence of a complete orthonormal system (ω_ν) of the Hilbert space H, given by the eigenfunctions of the operator $-\Delta$. If (λ_ν) are the eigenvalues of the operator $-\Delta$, then $\lambda_\nu \to +\infty$, as $\nu \to +\infty$.

Due to $-\Delta$ being positive, for a given $\beta > 0$, we have

$$D[(-\Delta)^\beta] = \left\{u \in H;\ \sum_{\nu=1}^{\infty} \lambda_\nu^{2\beta}|(u, \omega_\nu)|^2 < \infty\right\}$$

and

$$(-\Delta)^\beta u = \sum_{\nu=1}^{\infty} \lambda_\nu^\beta (u, \omega_\nu)\omega_\nu, \quad \forall u \in D[(-\Delta)^\beta].$$

Especially, in $D[(-\Delta)^\beta]$, we consider the topology given by

$$\|u\|_{D[(-\Delta)^\beta]} = |(-\Delta)^\beta u|_H.$$

Furthermore, we notice that such operators are self-adjointed, that is,

$$(-\Delta)^\beta u, v = (u, (-\Delta)^\beta v), \quad \forall u,\ v \in D[(-\Delta)^\beta],$$

and $D[(-\Delta)^{\frac{1}{2}}] = V$.

choosing smooth initial data, discussing a basis in $D[(-\Delta)^\beta]$, and observing that the injection $D[(-\Delta)^\beta] \hookrightarrow D[(-\Delta)^{\frac{1}{2}}] = V$ is continuous, we can repeat all the discussions used in the above estimates in order to extend our results for the generalized β-dissipative wave equation as follows

$$\begin{cases} u_{tt} - \Delta u + (-\Delta)^\beta u_t + f(u) = 0, & \text{in } \Omega \times [0, \infty), \\ u(x, t) = 0, & \text{on } \Gamma_1 \times [0, \infty), \\ \dfrac{\partial u}{\partial \nu} + g(u_t) = 0, & \text{on } \Gamma_0 \times [0, \infty), \\ u(x, 0) = u^0(x), \qquad u_t(x, 0) = u^1(x), & \text{in } \Omega. \end{cases}$$

(2) We notice that we can avoid the characterization of the boundary condition if we employ semigroup theory. The linear part of the system generates an analytic semigroup, and the nonlinear terms can be treated as perturbation, and it is enough to show the global existence for the above system. In spite of all these facilities, we need the a priori estimates for global solvability. At the same time, we make use of the perturbed energy method to obtain the energy decay for the above system if we construct an appropriate energy functional.

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