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Fractional description of mechanical property evolution of soft soils during creep

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Abstract: The motion of pore water directly influences mechanical properties of soils, which are variable during creep. Accurate description of the evolution of mechanical properties of soils can help to reveal the internal behavior of pore water. Based on the idea of using the fractional order to reflect mechanical properties of soils, a fractional creep model is proposed by introducing a variable-order fractional operator, and realized on a series of creep responses in soft soils. A comparative analysis illustrates that the evolution of mechanical properties, shown through the simulated results, exactly corresponds to the motion of pore water and the solid skeleton. This demonstrates that the proposed variable-order fractional model can be employed to characterize the evolution of mechanical properties of and the pore water motion in soft soils during creep. It is observed that the fractional order from the proposed model is related to the dissipation rate of pore water pressure.

Key words: *variable-order fractional model; fractional order; soil creep; evolution of mechanical properties; soft soil*

1 Introduction

The motion of pore water directly influences the mechanical properties of soft soils, which are composed of pore water and a solid skeleton. It has long been known that the mechanical properties of soft soils change during deformation or loading (Ferry 1980). However, until now, the relationship between the evolution of mechanical properties of soils and the motion of pore water is still unclear. The main reason is the lack of a suitable method to describe the change of soil mechanical properties. In hydraulic engineering and civil engineering, creep, which is the tendency of a solid material to move slowly or deform permanently under the influence of stresses, is the main mechanical process of soft soils. In this paper, we mainly focus on the description of the evolution of mechanical properties of soft soils during creep.

Fractional calculus has been considered one of the best mathematical tools for modeling physical responses and has been applied in a number of fields. The use of fractional calculus is

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motivated in large part by the fact that fewer parameters are required to achieve accurate approximation of experimental data. Previously, the creep response has been characterized primarily with the Maxwell, Kelvin-Voigt, and standard linear solid models (Ferry 1980) for the constitutive relationship. Bagley and Torvik (1983, 1985, 1986) and Koeller (1984) have developed models using fractional calculus. Other researchers (Padovan 1987; Shah and Qi 2010; Libertiaux and Pascon 2010; Lazopoulos 2006; Enelund et al. 1999; Enelund and Olsson 1999; Eldred et al. 1996; Gaul et al. 1991) have examined various issues involved in the numerical implementation of these sorts of models. Most of the fractional models mentioned above are called component models, and are based on a linear combination of elements, Hooke springs, and the fractional derivative Abel dashpot. Here, the fractional derivative Abel dashpot obeys the following expression:

$$\sigma(t) = E\theta^{\alpha} \frac{\mathrm{d}^{\alpha}\varepsilon(t)}{\mathrm{d}t^{\alpha}} \tag{1}$$

where σ and ε are the stress and strain, respectively; E and θ are material constants; t is time; and α is the fractional order, with $0 \le \alpha \le 1$. However, the component model is a mathematical model, and it is merely used to describe the mechanical response and does not consider the mechanical properties of materials. It is well known that the ideal solid obeys Hooke's law, $\sigma(t)$-$\varepsilon(t)$, and that Newtonian fluid satisfies Newton's law of viscosity, $\sigma(t)$-$\mathrm{d}\varepsilon(t)/\mathrm{d}t$. Thus, if we regard the mechanical properties as a spectrum, one end of which is pure elasticity, with $\alpha = 0$, then the other end is pure viscosity, with $\alpha = 1$. The fractional order of Eq. (1) can denote the location of a specific mechanical property on the spectrum, which can help us distinguish the mechanical property of materials quantitatively. However, we have found that some creep behaviors still cannot be simulated by Eq. (1). The primary reason is that the constant fractional order in Eq. (1) implicates the invariability of mechanical properties, while in the real world they change during the mechanical process. Therefore, representing the evolution of mechanical properties is a challenging issue in physical modeling and phenomenological description.

The concept of fractional order calculus needs to be further generalized by a calculus of varying order so that it is applicable to more complex mechanical properties of materials. Up to now, a number of variable-order fractional calculus definitions have been proposed (Coimbra 2003; Ingman and Suzdalnitsky 2004; Soon et al. 2005), and some of them have been applied to many fields such as anomalous diffusion (Sun et al. 2009; Umarov and Steinberg 2009), viscoelasticity (Ingman and Suzdalnitsky 2005; Ramirez and Coimbra 2007), multifractional Gaussian noises (Sheng et al. 2011), processing of geographical data (Cooper and Cowan 2004), and finite impulse response filters (Tseng 2006). However, variable-order calculus has not been used to describe the evolution of mechanical properties of soft soils during creep.

In our study, we attempted to describe the evolution of mechanical properties of soft soils through variable order calculus, and then discussed the impact of pore water on the evolution of soil mechanical properties.

2 Variable-order integral operator

Variable-order fractional integral operator definitions come from either direct extension from fractional calculus or generalization from the Laplace or Fourier transform. In the direct extension, the constant exponent in the fractional operator is replaced by a function. Lorenzo and Hartley (1998) proposed a generalized linear Riemann-Liouville integration operator with the form of

$$ {}_0D_t^{-\alpha(t)} f(t) = \int_0^t \frac{(t-\tau)^{\alpha(t,\tau)-1}}{\Gamma[\alpha(t,\tau)]} f(\tau)\,\mathrm{d}\tau \qquad \alpha(t) > 0 \tag{2} $$

where $\alpha(t,\tau)$ is the fractional order operator, and it may have three different arguments, i.e., $\alpha(t)$, $\alpha(\tau)$, and $\alpha(t-\tau)$. Based on the behavior of the operator for different $f(t)$ values, Eq. (2) can be stated as three expressions:

$$ {}_0^1D_t^{-\alpha(t)} f(t) = \int_0^t \frac{(t-\tau)^{\alpha(t)-1}}{\Gamma[\alpha(t)]} f(\tau)\,\mathrm{d}\tau \qquad \alpha(t) > 0 \tag{3} $$

$$ {}_0^2D_t^{-\alpha(t)} f(t) = \int_0^t \frac{(t-\tau)^{\alpha(\tau)-1}}{\Gamma[\alpha(\tau)]} f(\tau)\,\mathrm{d}\tau \qquad \alpha(t) > 0 \tag{4} $$

$$ {}_0^3D_t^{-\alpha(t)} f(t) = \int_0^t \frac{(t-\tau)^{\alpha(t-\tau)-1}}{\Gamma[\alpha(t-\tau)]} f(\tau)\,\mathrm{d}\tau \qquad \alpha(t) > 0 \tag{5} $$

where the superscript before the derivative symbol indicates the arguments of $\alpha(t,\tau)$. Since Eqs. (4) and (5) involve the variable of integration within the exponent, they imply memory of the fractional order, meaning that past states have a strong effect on the fractional order (Lorenzo and Hartley 2002).

It is known that the current mechanical response of viscoelastic materials depends not only on the loading or deformation histories but also the change process of mechanical properties. Thus, the variable-order operator definition applied to describing the evolution of a physical feature should have the memory of not only the prior history but also the fractional order. We therefore used Eq. (5) to characterize mechanical properties of soft soils during creep in this study.

In order to use Eq. (5) conveniently, we discuss this definition for $f(t) = c$, where c is a constant:

$$ {}_0^3D_t^{-\alpha(t)} c = c\int_0^t \frac{(t-\tau)^{\alpha(t-\tau)-1}}{\Gamma[\alpha(t-\tau)]}\mathrm{d}\tau \overset{t-\tau=\lambda}{=} c\int_0^t \frac{\lambda^{\alpha(\lambda)-1}}{\Gamma[\alpha(\lambda)]}\mathrm{d}\lambda \tag{6} $$

If $\alpha(t) = \alpha$, where α denotes a constant, we can rewrite Eq. (6) as

$$ {}_0^3D_t^{-\alpha}c = c\int_0^t \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}\lambda = \frac{ct^\alpha}{\Gamma(1+\alpha)} = g(t,\alpha) \tag{7} $$

We assume that the fractional order function $\alpha(t)$ in Eq. (6) is of a piece-wise constant manner, taking the values of $\alpha_1, \alpha_2, \cdots, \alpha_n$ for $0 < t \le t_1,\ t_1 < t \le t_2,\ \cdots,\ t_{n-1} \le t \le t_n$, respectively. Based on Eq. (7), Eq. (6) can also be written as

$$ {}_0^3D_t^{-\alpha(t)}c = \sum_{k=1}^{n}\left[g(t_k,\alpha_k) - g(t_{k-1},\alpha_k)\right] \quad 0 = t_0 < t_1 < t_2 < \cdots < t_n \tag{8} $$

Because Eq. (8) is a discrete form, which is more suitable for numerical approximation and fitting of experimental data, we will use it to establish a variable-order fractional creep model in the next section.

3 Establishment of variable-order fractional creep model

3.1 Variable-order fractional viscoelastic model

Coimbra (2003), Ingman and Suzdalnitsky (2005), and Soon (2005) presented a variable-order fractional viscoelastic model:

$$ \sigma(t) = X\frac{\mathrm{d}^{\alpha(t)}\varepsilon(t)}{\mathrm{d}t^{\alpha(t)}} \tag{9} $$

where X is considered a material parameter. Because the fractional order function is involved in the dimension of X in Eq. (9), X has an unclear physical meaning and can be influenced by time resolution. Thus, there are some problems in Eq. (9).

Considering the evolution of mechanical properties during the loading process, the modified viscoelastic model (Yin et al. 2013) should be

$$ \sigma(t) = E\theta^{\alpha(t)}\frac{\mathrm{d}^{\alpha(t)}\varepsilon(t)}{\mathrm{d}t^{\alpha(t)}} \quad 0 \le \alpha(t) \le 1 \tag{10} $$

Obviously, Eq. (10) is a direct extension of the constant-order fractional model (Eq. (1)).

3.2 Variable-order fractional creep model and its parameters

During creep, $\sigma(t) = c_0$, where c_0 is a constant and represents the constant stress, Eq. (10) may be rewritten as

$$ \varepsilon(t) = = \frac{{}_0^3D_t^{-\alpha(t)}c_0}{E\theta^{\alpha(t)}} \quad 0 \le \alpha(t) \le 1 \tag{11} $$

Based on Eq. (8), Eq. (11) may be rewritten as

$$ \varepsilon(t) = \sum_{k=1}^{n}\left[\varepsilon(t_k,\alpha_k) - \varepsilon(t_{k-1},\alpha_k)\right] \quad 0 = t_0 < t_1 < t_2 < \cdots < t_n \tag{12} $$

According to Eq. (7), $\varepsilon(t,\alpha_k)$ is expressed as

$$ \varepsilon(t,\alpha_k) = \frac{c_0}{E\theta^{\alpha_k}}\frac{t^{\alpha_k}}{\Gamma(1+\alpha_k)} \quad 0 \le \alpha_k \le 1 \tag{13} $$

After substituting Eq. (13) into Eq. (12), we can obtain

$$\varepsilon(t)=\sum_{k=1}^{n}\left[\frac{c_0}{E\theta^{\alpha_k}}\frac{t_k^{\alpha_k}-t_{k-1}^{\alpha_k}}{\Gamma(1+\alpha_k)}\right] \qquad 0=t_0<t_1<t_2<...<t_n \tag{14}$$

Using Eq. (14) for $t=t_k$ and $t=t_{k-1}$, and subtracting the second expression from the first one, the following equation is obtained

$$\varepsilon(t_k)-\varepsilon(t_{k-1})=\frac{c_0}{E\theta^{\alpha_k}}\frac{t_k^{\alpha_k}-t_{k-1}^{\alpha_k}}{\Gamma(1+\alpha_k)} \qquad 0=t_0<t_1<t_2<...<t_n \tag{15}$$

When $0<t\le t_1$, Eq. (14) may be written as

$$\varepsilon(t)=\frac{c_0}{E\theta^{\alpha_1}}\frac{t^{\alpha_1}}{\Gamma(1+\alpha_1)}=\eta t^{\alpha_1} \tag{16}$$

where $\eta=\dfrac{c_0}{\Gamma(1+\alpha_1)E\theta^{\alpha_1}}$. Certainly, Eq. (16) can also be written as

$$\ln\varepsilon(t)=\ln\eta+\alpha_1\ln t \tag{17}$$

The coefficients η and α_1 can be obtained by the first several strain-time (ε-t) experimental data, whose $\ln\varepsilon$-$\ln t$ curve can be fitted as a straight line.

Because strain increases consistently and gradually approaches a fixed value ε_0 in plenty of creep processes, we define it as

$$E=\frac{c_0}{\varepsilon_0} \tag{18}$$

The reason for the strain reaching the fixed value is that materials exhibit elasticity during creep. As mentioned above, the order corresponding to ε_0 is zero, i.e., $\alpha_k=0$. Based on Eqs. (18) and the definition of η, the coefficient θ is stated as

$$\theta=\left[\frac{\varepsilon_0}{\eta\Gamma(1+\alpha_1)}\right]^{\frac{1}{\alpha_1}} \tag{19}$$

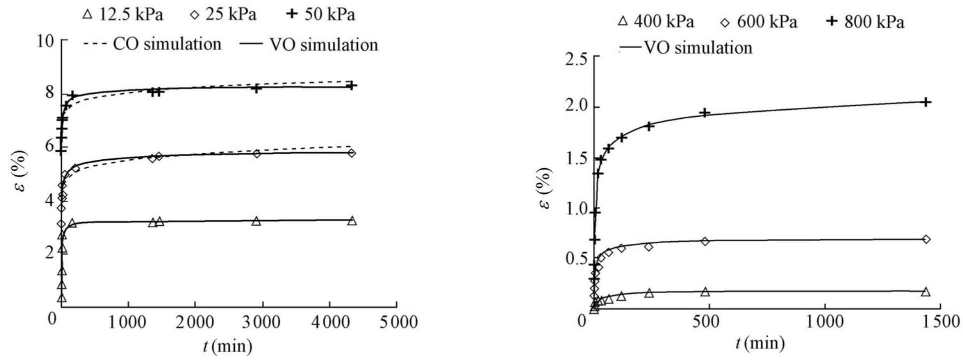
When the experimental data are substituted into Eq. (15), we can obtain a sequence of fractional orders, α_k, and the order-time curve. Then, we can determine the evolution of mechanical properties of materials on the basis of the relationship between the fractional order and the mechanical property.

4 Evolution of mechanical properties of soft soils during creep

In this section, we will examine the evolution of mechanical properties of soft soils during creep through the variable-order fractional creep model proposed in section 3.

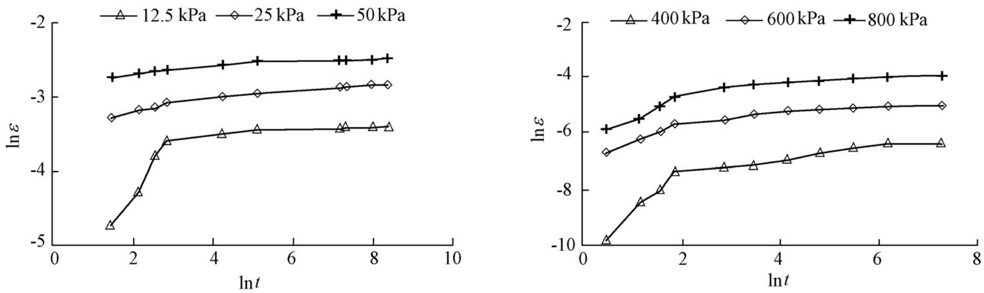
A series of triaxial creep experiments on soft soils have been performed by Zhou and Chen (2006) and Luo (2010). The corresponding experimental data are shown in Fig. 1. In order to validate the variable-order fractional creep model, Eq. (12) was used to fit these creep tests.

Fig. 2 shows that some $\ln\varepsilon$-$\ln t$ curves of the creep test cannot be fitted with a straight line, implying that the mechanical property of soils changes with time. Based on the variable-



(a) Experimental data from Zhou and Chen (2006)   (b) Experimental data from Luo (2010)

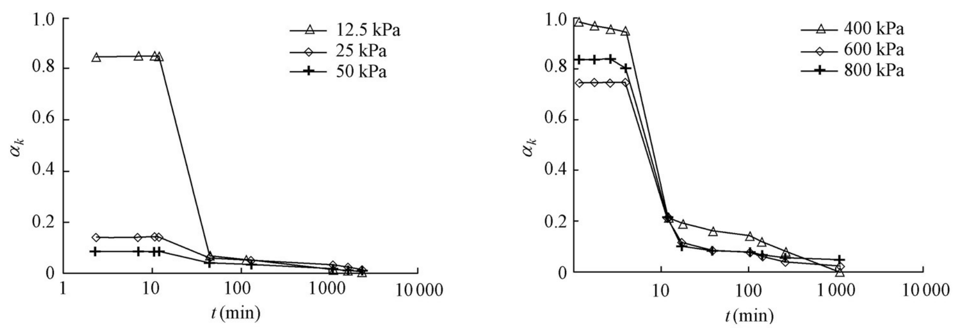
Fig. 1 Simulated ε-t curves of soils at different stresses (VO denotes the variable-order fractional model, and CO denotes the constant-order model.)



(a) $\ln\varepsilon$ - $\ln t$ curves (Zhou and Chen 2006)   (b) $\ln\varepsilon$ - $\ln t$ curves (Luo 2010)

Fig. 2 $\ln\varepsilon$-$\ln t$ curves obtained by experiments at different stresses

order fractional method mentioned above, the α_k-t curves were obtained, as shown in Fig. 3. We observe from Fig. 3 that the fractional order declines with time, and that all the α_k-t curves have an almost uniform shape.



(a) Experimental data from Zhou and Chen (2006)   (b) Experimental data from Luo (2010)

Fig. 3 Simulated α_k-t curves obtained by variable-order fractional model at different stresses

It is of interest that the order-time curves in Fig. 3 can be split into three stages: in the first stage, the fractional order is close to a constant, the second stage is a mutation stage of the fractional order, and in the third stage the fractional order decreases slowly with time and is even close to zero.

In terms of the relation between the fractional order and the mechanical property, we know from Fig. 3 that, during the creep of soft soils, the mechanical property first remains unchanged and exhibits a comparatively large viscosity. Then, a sharp decrease of the viscosity in soft soils follows, and, finally, the material property changes slowly until the viscosity in soft soils fades away. Further analysis is needed to demonstrate the credibility of the information mentioned above.

It is known that soft soils belong to porous media, whose viscosity results from the viscous motion of pore water and the viscous response of the solid skeleton (Schapery 1975). During the initial period of creep, viscosity mainly comes from the discharge of pore water. Thus, the fractional order of the first stage of creep is greater than that of the other two stages. When pore water is excreted, the viscous response of the solid skeleton becomes the major source of viscosity. Therefore, the viscosity of soft soils shows a sharp drop, which exactly corresponds to the mutation stage of the α_k-t curve in Fig. 3. Because the solid skeleton of soft soils can only experience a limited and slow deformation, we think that the evolution of mechanical properties in the third stage, described by the variable-order fractional creep model, is also rational. In short, the comparative analysis of viscous responses and simulated results illustrates that the fractional order from the proposed fractional creep model can be employed to represent the evolution of mechanical properties.

The experimental data from Liu et al. (2008) were also fitted using the variable-order fractional creep model and the results are shown in Fig. 4. As mentioned above, the mechanical properties reflect the pore water behavior. We therefore compared the variations of the fractional order obtained by the proposed model and the pore water pressure μ. It is observed from Fig. 5 that, according to the dissipation rate of pore water pressure, the μ-t curve can be divided into three stages, which correspond to different stages of the α_k-t curve. We found that the fractional order is related to the dissipation rate of pore water pressure, and that the larger the fractional order is, the faster the change of pore water pressure. When the fractional order is very small, the pore water pressure decreases slowly. We should also point out that further research is required to quantify the relationship between the fractional order and the discharge rate of pore water.

5 Discussion

From Fig. 2(a), it is observed that the $\ln\varepsilon$-$\ln t$ curves for stresses of 25 kPa and 50 kPa can be fitted as straight lines, which implies that the corresponding experimental results can be simulated using the constant-order fractional creep model. The simulated results of α_k-t curves obtained by the constant-order fractional model and variable-order fractional model are shown in Fig. 6, respectively. It can be observed that when the creep response can be described by both the variable-order and constant-order fractional models with sufficiently guaranteed precision, the constant fractional order from the constant-order model is at an intermediate point

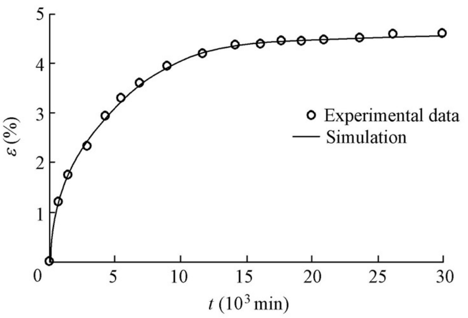


Fig. 4 ε-t curve simulated with variable-order fractional creep model and experimental data from Liu (2008) at stress of 80 kPa

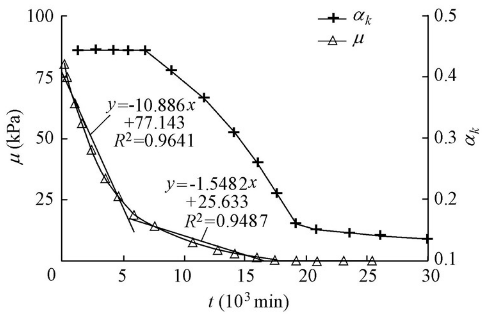


Fig. 5 Comparison between evolution of mechanical properties and dissipation rate of pore water pressure

between those orders from the variable-order model, which vary slightly during creep. This illustrates that the mechanical properties of some soft soils, which can be characterized utilizing the constant-order fractional model, are not necessarily fixed but may change slightly during creep.

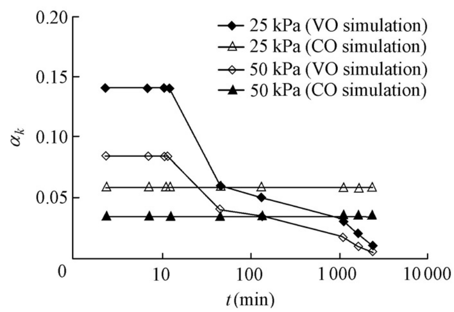


Fig. 6 α_k-t curves obtained by constant-order fractional model and variable-order fractional model

We need to point out that the focus of our study is on the characterization of the evolution of mechanical properties of soils rather than on requiring fewer parameters to fit experimental data. We believe that using a simple function to fit the order-time curve may hide the real process of the evolution of mechanical properties of materials.

6 Conclusions

In order to describe the evolution of mechanical properties of soft soils during creep, we presented a variable-order fractional creep model, in which the fractional order was expected to represent the mechanical property of soils. The proposed model was realized on the creep response of some soft soils. The simulated results show that the evolution of mechanical characteristics can be divided into three stages. A comparative analysis illustrates that the evolution of mechanical properties, shown by the simulated results, exactly corresponds to the motions of pore water and the solid skeleton. This demonstrates that the proposed variable-order fractional model can be employed to characterize the evolution of the mechanical properties of soft soils during creep. We observed that the fractional order from the proposed model is related to the dissipation rate of pore water pressure, and that the larger the order is, the faster the dissipation of pore water pressure.

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