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ORIGINAL PAPER

A new, simple and accurate transition curve type, for use in road and railway alignment design

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Abstract

Purpose This paper evaluates all the available transition curve types related to road and railway alignments and proposes a new, well verified, transition curve type that combines the accuracy of clothoid curve and the simplicity of cubic parabola curve.

Method A methodology similar to clothoid's curve formation is used to introduce a new transition curve type called "of clothoid Symmetrically Projected Transition Curve (SPTC). All three transition curve types are being compared to each other, for a variety of transition length value versus Radius value combinations. The cubic parabola is a simple function of the form of y=f(x). Clothoid is a transition curve in the form of x=f(l), y=f(l), having as main characteristic the linearity of curvature variation versus its length. A new transition curve will be defined in the form of y=f(x) having also as main characteristic the linearity of curvature variation versus its projection length on axis X. By using the same calculation procedure as the clothoid, the new transition curve will be fully defined. A relation similar to (1) was used as base, by defining a parameter A similar to the one used in the clothoid. The new curve will be called Symmetrically Projected Transition Curve (SPTC).

Results Some remarkable results that derived from transition curves comparison are: There are no significant differences between the 3 curves in the area of short transition lengths. For long transition lengths, cubic parabola is diverging from the other 2. The deviation of the cubic parabola from the other curves for large values of X, ratios $X/A>0.7$, as well as the affinity of the clothoid with the SPTC are obvious. The most remarkable observation than can be made in the table is the fact that ΔX always zero for the SPTC (10terms). Thus, the SPTC curve is symmetrically projected on its basic tangent. This property contributes to the simplicity of the alignment design. That is another reason to prefer the SPTC curve.

Conclusions The use of cubic parabola in combination with approximate value of diversion can lead to design problems. The new transition curve can be used instead of cubic parabola especially when long transition lengths are required. The new transition curve can also be used successfully to join 2 homo-bending arcs. However, referring to cubic parabola calculations, for a ratio $X/A \geq 0.5$ and taking in to account the approximate calculation procedure of ΔR it can lead to alignment design errors. Consequently, the usage limits for each transition curve should be well known. A new transition curve was also proposed in this work. The new curve is called Symmetrically Projected Transition Curve (SPTC). SPTC was found, in most cases, to have better performance than cubic parabola. Symmetry is an important characteristic of the SPTC and contributes to simplicity, accuracy and audit ability of the designed alignment. Finally SPTC can also be used as a transition curve between two adjacent circular arcs in the same direction.

Keywords Alignment · Railroad tracks · Road design · Transition curve · Clothoid · Cubic parabola

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1 Introduction

The selection of a suitable transition curve is of major importance towards a proper alignment design in road and railway projects. There are two very well known transition curves. The transition curve that is exclusively used in road alignments is the Clothoid [2]. The cubic parabola is used, for historical reasons only, in railway alignments. It is noted that currently there is no research underway related to each curve's accuracy and usability.

A comparison of the cubic parabola with the clothoid reveals that for small lengths of transition curves in relation to the radius of curvature corresponding to the end of the transition curve there are no significant differences between the two transition curve types. Thus, the use of the cubic parabola has no advantage over the clothoid other than its simplicity which is its dominant characteristic. However, the selection of transition curve type based on simplicity cannot be justified nowadays that computer software is widely used to design road and railway alignments. Thus, the criterion of a suitable transition curve based on the simplicity of the calculations must be reconsidered.

The use of the cubic parabola is acceptable only for small values of transition curve length. There are cases, however, where the length of the transition curve exceeds a certain limit. One such case is to obtain a "flatter" alignment. Another case is when the transition curve is used to adjust two circular arcs in the same direction. In such cases the use of the cubic parabola has serious limitations. When the length of the transition curve has to be greater than the half of the radius of curvature, the differences between cubic parabola and clothoid become significant. This is due to approximations applied in order to link cubic parabola to an arc [1].

The detailed analysis of the differences between cubic parabola and clothoid led to the introduction of a new transition curve type. This curve is called Symmetrically Projected Transition Curve (SPTC). The new curve is a simple function of the form of $y=f(x)$ like the cubic one but is not based on any approximation used in cubic parabola calculations. On the other hand it is equally accurate to the clothoid.

2 Transition curves

2.1 The clothoid

The new transition curve has certain similarities with the clothoid. Consequently, the clothoid is briefly described first in order to clarify and distinguish it from the proposed curve. The clothoid's main characteristic is the linear variation in curvature versus the covered length on the curve [2] (Kasper et al. 1954). A transition curve with no linear variation in curvature has also been proposed [4]. It is known, that for every point of the clothoid the following relation which links the distance L (on the curve) from the starting point with the curvature radius R on that point is valid:

$$RL = A^2 \tag{1}$$

Thus, the product of R and L is constant for every curve characterized by the specific parameter A.

With reference to Fig. 1 and using the above relation as well as the relation:

$$dL = Rd\tau \tag{2}$$

the following relation is obtained:

$$LdL = A^2 d\tau \tag{3}$$

where, τ is the tangent angle on the particular point of the clothoid in relation to the axis x and $d\tau$ the elementary angle corresponding to length dL.

Integration of Eq. (3) yields:

$$L^2 \big/ 2 = A^2\tau + C \tag{4}$$

Given that $L=0$ for $\tau=0$ the value of C is equal to 0 and the deflection angle τ (in rad) will be:

$$\tau = \frac{L^2}{2A^2} \tag{5}$$

Based on Fig. 1 the following relations are obtained:

$$dX = dL\cos\tau, \quad dY = dL\sin\tau \tag{6}$$

After taking into account Eq. (5) dX and dY are as follows:

$$dX = dL\cos\left(\frac{L^2}{2A^2}\right), \quad dY = dL\sin\left(\frac{L^2}{2A^2}\right) \tag{7}$$

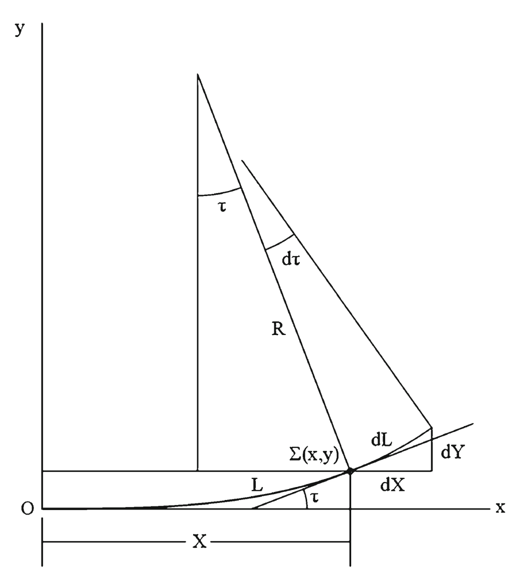

Fig. 1 Typical transition curve graph

Integration of the above equation yields the following relations used to calculate the coordinates of any point of the clothoid at a distance ℓ away from its starting point [3]

$$x = \ell - \frac{\ell^5}{40A^4} + \frac{\ell^9}{3456A^8} - \frac{\ell^{13}}{599040A^{12}} \cdots \tag{8}$$

$$y = \frac{\ell^3}{6A^2} - \frac{\ell^7}{336A^6} + \frac{\ell^{11}}{42240A^{10}} - \frac{\ell^{15}}{9676800A^{14}} \cdots \tag{9}$$

2.2 The cubic parabola

Using only the first terms of the relations (8) and (9) the following relation is obtained:

$$y = \frac{x^3}{6A^2} \tag{10}$$

which is known as the cubic parabola but is normally used as follows:

$$y = \frac{x^3}{6RX} \tag{11}$$

where R is the radius of the circle, which links to the end of the cubic parabola. L is the actual length of cubic parabola and X is its respective projection's length on axis x. The length L of the cubic parabola is considered to be equal with the projection of X on axis x. As noted above relation (11) is commonly used instead of (10). This is because in the cubic parabola parameter A has not been defined.

According to Esveld [1] the same relation with (11) can be derived starting from the general equation of the cubic parabola:

$$y = kx^3 \tag{12}$$

In order for it to link with a circle of a radius R, its curvature, its second derivation with respect to x, at a distance X from its starting point, has to be the same with that of the circle, as shown in the following relation:

$$\left.\frac{d^2y}{dx^2}\right|_{x=X} = 6kX = \frac{1}{R} \tag{13}$$

from which the value of k is derived which, if set in (12) then we get (11). This relation gives the y coordinate of any point of the cubic parabola versus its projection on axis x.

From the relations (10)–(11), the deflection angle τ for the cubic parabola can be calculated, using the relation:

$$\tan\tau = \frac{dy}{dx} = \frac{x^2}{2A^2} = \frac{x^2}{2RX} \tag{14}$$

As it is already mentioned, the length L of the cubic parabola is considered to be equal with the projection of X on axis x. This is an approximation, which is usually satisfactory. Relation (13) is based on this approximation that is valid for minor values of x in relation to radius R.

To calculate the real length L on the cubic parabola the following relation is used:

$$L = X\left[1 + \frac{1}{10}\left(\frac{X}{2R}\right)^2\right] \tag{15}$$

This relation is derived from the following procedure. From Fig. 1 we get:

$$dL = dX \big/ \cos\tau = dX\sqrt{1+\tan^2\tau} \tag{16}$$

which after the use of (14) becomes:

$$dL = \sqrt{1 + \frac{X^4}{4A^4}}dX \tag{17}$$

Analyzing it in Tailor series we get:

$$dL = \left(1 + \frac{X^4}{8A^4} - \frac{X^8}{128A^8} + ..\right)dX \tag{18}$$

Its integration results in the following relation for the final length of the cubic parabola (x=X):

$$L = X\left[1 + \frac{1}{10}\left(\frac{X}{2R}\right)^2 - \frac{1}{72}\left(\frac{X}{2R}\right)^4 + ..\right] \tag{19}$$

which, if only the first two terms are considered as important, coincides with relation (15).

2.3 The new transition curve

The cubic parabola is a simple function of the form of $y = f(x)$ and is based on the acknowledgment that its length is equal to its projection on axis X. Clothoid is a transition curve in the form of $x = f(l)$, $y = f(l)$, having as main characteristic the linearity of curvature variation versus its length. A new transition curve will be defined in the form of $y = f(x)$ having also as main characteristic the linearity of curvature variation versus its projection length on axis X. By using the same calculation procedure as the clothoid, the new transition curve will be fully defined without having to acknowledge that its length L is equal to its projection on axis X. A relation similar to (1) was used as base, by defining a parameter A similar to the one used in the clothoid. The new curve will be called Symmetrically Projected Transition Curve (SPTC). The analysis below will be based on the relation:

$$RX = R_x x = A^2 \tag{20}$$

where, R is the curvature radius at the end of the curve and X its projection on axis x. Moreover, R_x is the curvature radius at an intermediate point with respective projection x.

With reference to Fig. 2 the following relation is obtained:

$$dx = d\ell\cos\tau = R_x d\tau\cos\tau \tag{21}$$

which, using the relation (20), results in the following

$$dx = \left(A^2/x\right)\cos\tau d\tau \tag{22}$$

or

$$xdx = A^2\cos\tau d\tau \tag{23}$$

By integration of Eq. (23)

$$\int xdx = A^2\int\cos\tau d\tau + C \tag{24}$$

the following relation is obtained

$$x^2/2 = A^2\sin\tau + C \tag{25}$$

However, at the beginning of the transition curve $x = 0$ and $\tau = 0$. Thus, the angle τ (in rad) is expressed with the relation:

$$\sin\tau = \frac{x^2}{2A^2} = \frac{x}{2R_x} \tag{26}$$

which is similar to (5) and to (14) and expresses the angle τ versus the projection of an intermediate point of the curve on the axis x and of the constant A or of the curvature radius R_x on that point.

The relation (26) is also valid at the end of the transition curve and becomes:

$$\sin\tau = \frac{X^2}{2A^2} = \frac{X}{2R} \tag{27}$$

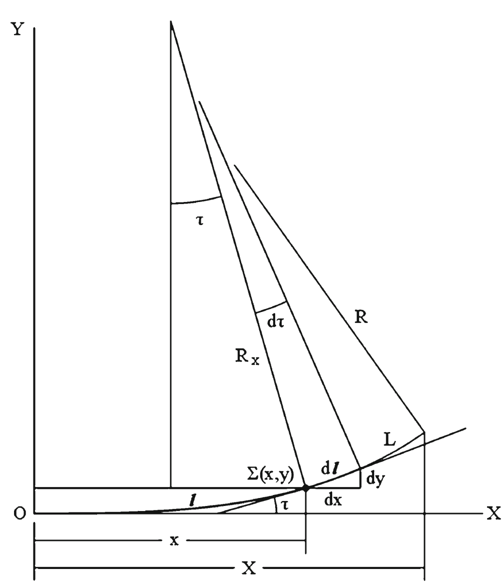

Fig. 2 Transition curve graph in detail

where X is the projection of the end of the curve and R the respective curvature radius.

For the calculation of y for every x from the Fig. 2 there is the relation:

$$dy = \tan\tau dx = \frac{\sin\tau}{\sqrt{1-\sin^2\tau}}dx \tag{28}$$

which, based on (26) becomes:

$$dy = \frac{\frac{x^2}{2A^2}}{\sqrt{1-\frac{x^4}{4A^4}}}dx \tag{29}$$

Integration of Eq. (29) yields:

$$y = \int_0^x \frac{\frac{x^2}{2A^2}}{\sqrt{1-\frac{x^4}{4A^4}}}dx \tag{30}$$

In order to calculate the integral a Tailor series analysis is performed, beginning from the following simple relation:

$$g(x) = \frac{1}{\sqrt{1-x}} = (1-x)^{-1/2} = 1 + \frac{1}{2}\frac{x^1}{1!} + \frac{3}{4}\frac{x^2}{2!} + .. \tag{31}$$

which is written as follows:

$$g(x) = \sum_{i=0}^{n} a_i\frac{x^i}{i!} = \sum_{i=0}^{n} b_i \tag{32}$$

with coefficients a_i, b_i of the form:

$$a_i = a_{i-1}\frac{2i-1}{2},\quad a_0 = 1 \tag{33}$$

$$b_i = b_{i-1}\frac{2i-1}{2i}x,\quad b_0 = 1 \tag{34}$$

If x is replaced with x^4 the following relation is obtained:

$$g(x) = \frac{1}{\sqrt{1-x^4}} = \sum_{i=0}^{n} a_i\frac{x^{4i}}{i!} = \sum_{i=0}^{n} b_i \tag{35}$$

where coefficients a_i are the same as in Eq. (33) and coefficients b_i are given by:

$$b_i = b_{i-1}\frac{2i-1}{2i}x^4 \tag{36}$$

Subsequently, function $h(x)$ is defined:

$$h(x) = \frac{x^2}{\sqrt{1-x^4}} = \sum_{i=0}^{n} b_i \tag{37}$$

with b_i as (36) and $b_0=x^2$. Using relation (37) and by replacing x with $x/\left(A\sqrt{2}\right)$ relation (30) becomes:

$$y=\int_0^{x}\frac{\frac{x^2}{2A^2}}{\sqrt{1-\frac{x^4}{4A^4}}}dx=\int_0^{x}f(x)dx \tag{38}$$

where the equation:

$$f(x)=\sum_{i=0}^{n}b_i \tag{39}$$

with coefficients b_i of the form:

$$b_i=b_{i-1}\frac{2i-1}{2i}\frac{x^4}{4A^4},\quad b_0=\frac{x^2}{2A^2} \tag{40}$$

The integration will give an equation of the form:

$$y=\int_0^{x}\frac{\frac{x^2}{2A^2}}{\sqrt{1-\frac{x^4}{4A^4}}}dx=\sum_{i=0}^{n}\frac{c_i}{4i+3} \tag{41}$$

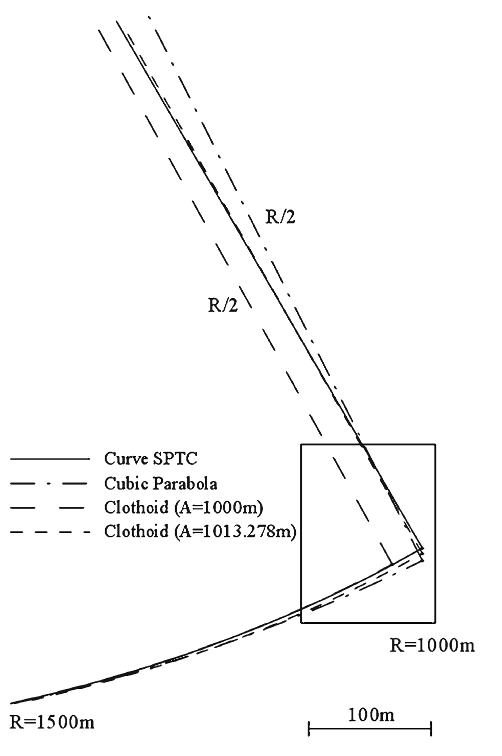

Fig. 3 Comparison of transition curves used

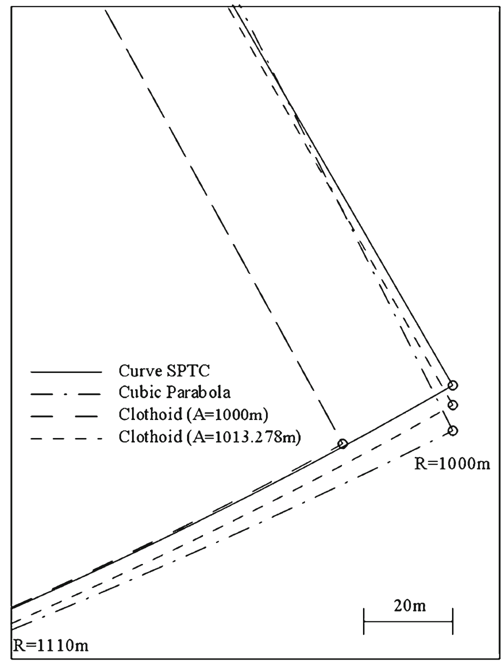

Fig. 4 Comparison of transition curves in detail

with coefficients:

$$c_i=c_{i-1}\frac{2i-1}{2i}\frac{x^4}{4A^4},\quad c_0=\frac{x^3}{2A^2} \tag{42}$$

By taking into account only the first four terms, the following relation emerges:

$$y=\frac{x^3}{2A^2}\left[\frac{1}{3}+\frac{1}{14}\left(\frac{x^2}{2A^2}\right)^2+\frac{3}{88}\left(\frac{x^2}{2A^2}\right)^4+\frac{1}{48}\left(\frac{x^2}{2A^2}\right)^6+..\right] \tag{43}$$

The above relation gives the value of y at intermediate points of the transition curve versus the projection x. If only the first term is used relation (10) is obtained. Thus, cubic parabola is a first approximation of the proposed transition curve.

For Υ at the end of the transition curve the following relation will be valid:

Table 1 Basic characteristics of transition curves used

Curve	A (m)	X (m)	L (m)
Cubic Parabola	[1000]	1000	1025
Curve SPTC	1000	1000	1028.057
Clothoid (1)	1000	975.288	1000
Clothoid (2)	1013.278	1000	1026.732

Table 2 Displacement Y (tangent offset) for various values of X (tangent distance)

Projection X(m)	Clothoid Y(m)	Cubic Parabola Y(m)	SPTC (10 terms) Y(m)	SPTC (3 terms) Y(m)	SPTC (2 terms) Y(m)
100	0.167	0.167	0.167	0.167	0.167
200	1.333	1.333	1.333	1.333	1.333
300	4.502	4.500	4.502	4.502	4.502
400	10.682	10.667	10.681	10.681	10.681
500	20.908	20.833	20.904	20.904	20.903
600	36.271	36.000	36.254	36.254	36.250
700	57.976	57.167	57.924	57.923	57.902
800	87.444	85.333	87.303	87.297	87.206
900	126.491	121.500	126.143	126.105	125.771
1000	177.678	166.667	176.858	176.661	175.595

$$Y=\frac{X^2}{2R}\left[\frac{1}{3}+\frac{1}{14}\left(\frac{X}{2R}\right)^2+\frac{3}{88}\left(\frac{X}{2R}\right)^4+\frac{1}{48}\left(\frac{X}{2R}\right)^6+\dots\right] \tag{44}$$

The length ℓ on the curve will be calculated with the use of the relation:

$$d\ell=dx/\cos\tau=dx\sqrt{1-\sin^2\tau}=\sqrt{1-\frac{x^4}{4A^4}}\,dx \tag{45}$$

An analysis similar to the previous one also results in:

$$\ell=\int_0^x\frac{dx}{\sqrt{1-\frac{x^4}{4A^4}}}=\sum_{i=0}^{n}\frac{b_i}{4i+1} \tag{46}$$

with the coefficients given by the relations:

$$b_i=b_{i-1}\frac{2i-1}{2i}\frac{x^4}{4A^4},\quad b_0=x \tag{47}$$

By employing the first four terms, relation (48) emerges which gives the value of length ℓ on the transition curve versus the projection x on intermediate points:

$$\ell=x\left[1+\frac{1}{10}\left(\frac{x^2}{2A^2}\right)^2+\frac{1}{24}\left(\frac{x^2}{2A^2}\right)^4+\frac{5}{208}\left(\frac{x^2}{2A^2}\right)^6+\cdots\right] \tag{48}$$

At the end of the transition curve length L is:

$$L=X\left[1+\frac{1}{10}\left(\frac{X}{2R}\right)^2+\frac{1}{24}\left(\frac{X}{2R}\right)^4+\frac{5}{208}\left(\frac{X}{2R}\right)^6+\cdots\right] \tag{49}$$

which coincides with relation (15) if only its first two terms are considered important.

Table 3 Deflection angle τ for various values of X

Projection X(m)	Clothoid τ(deg)	Cubic Parabola τ(deg)	SPTC (10 terms) τ(deg)	SPTC (3 terms) τ(deg)	SPTC (2 terms) τ(deg)
100	0.28648	0.28648	0.28648	0.28648	0.28648
200	1.14601	1.14576	1.14599	1.14599	1.14599
300	2.57936	2.57657	2.57918	2.57918	2.57918
400	4.58955	4.57392	4.58857	4.58857	4.58850
500	7.18453	7.12502	7.18076	7.18075	7.18010
600	10.38118	10.20397	10.36976	10.36965	10.36572
700	14.21129	13.76630	14.18183	14.18089	14.16307
800	18.73093	17.74467	18.66292	18.65686	18.59212
900	24.03667	22.04795	23.89113	23.85986	23.66376
1000	30.29735	26.56505	30.00000	29.86525	29.35776

Table 4 Lengths L (on curve) corresponding with the values of X

Projection X(m)	Clothoid L(m)	Cubic Parabola L(m)	SPTC (10 terms) L(m)	SPTC (4 terms) L(m)	SPTC (3 terms) L(m)
100	100.000	100.000	100.000	100.000	100.000
200	200.008	200.008	200.008	200.008	200.008
300	300.061	300.061	300.061	300.061	300.061
400	400.257	400.256	400.257	400.257	400.257
500	500.787	500.781	500.786	500.786	500.786
600	601.973	601.944	601.971	601.971	601.970
700	704.321	704.202	704.311	704.310	704.307
800	808.599	808.192	808.564	808.562	808.542
900	915.990	914.762	915.879	915.867	915.771
1000	1028.386	1025.000	1028.057	1027.980	1027.604

3 Comparison of the curves

Using the relations of the previous paragraphs all the transition curves were calculated and selected segments of the transition curves ($R \leq 1500\ m$) are presented in Fig. 3, to demonstrate the differences between them. These differences are presented in detail in Fig. 4 ($R \leq 1110\ m$).

Besides the cubic parabola and the symmetrically projected transition curve, two clothoids also are illustrated. The curvature radius of the circle in which all curves end is $R = 1000\ m$. The parameter for the first three curves is $A = 1000\ m$. For the fourth curve the following parameter value was used $A = 1013.278\ m$, corresponding to the curve's length $L = 1026.732\ m$, so that its projection at the end is $X = 1000\ m$, exactly as the projection of the cubic parabola.

In Table 1 the details of all four curves are shown. It should be noted that radius values of about $R = 1000\ m$ are high for ordinary roads but are common to railway alignments.

From Fig. 4 it is shown that the symmetrically projected transition curve is very close to the clothoid with the same parameter. The length is their difference and this is due to the fact that the length of the clothoid, which is $L = 1000\ m$, is equal to the projection of the symmetrically projected transition curve. Namely with the second curve there is a variation from the radius $R = \infty$ to the radius $R = 1000\ m$ at a length greater by $28\ m$.

In the following tables some comparative characteristic quantities will be presented for five different kinds of transition curves. Quantities for the clothoid, the cubic parabola and the curve SPTC are presented with accuracy, using 10 terms of the series for clothoid and SPTC. Two approximations of the curve SPTC (with three and two terms respectively) are presented on the last two columns of each table.

All curves have parameter $A = 1000\ m$. The clothoid is calculated according to relation (9) using as value of ℓ the one corresponding to a projection equal to the first column of the Table. This is calculated according to relation (8). The clothoid was used in this way so that the resulted quantities are directly comparative to the quantities of the other curves.

In Table 2 the values of the displacement Υ (tangent offset) for various values of X (tangent distance) are shown. The deviation of the cubic parabola from the other curves for large

Table 5 Deviation ΔR for different values of X

Projection X(m)	Approximation ΔR(m)	Clothoid ΔR(m)	Cubic Parab. ΔR(m)	SPTC (10) ΔR(m)	SPTC (3) ΔR(m)	SPTC (2) ΔR(m)
100	0.04167	0.04167	0.04167	0.04167	0.04167	0.04167
200	0.33333	0.33333	0.33363	0.33335	0.33335	0.33335
300	1.12500	1.12492	1.13012	1.12524	1.12524	1.12525
400	2.66667	2.66606	2.70486	2.66850	2.66850	2.66870
500	5.20833	5.20543	5.38909	5.21710	5.21712	5.21942
600	9.00000	8.98959	9.63887	9.03157	9.03205	9.04875
700	14.29167	14.26107	16.13011	14.38539	14.39032	14.47809
800	21.33333	21.25548	25.86352	21.57540	21.61157	21.97114
900	30.37500	30.19768	40.24449	30.93898	31.14653	32.34456
1000	41.66667	41.29661	61.09386	42.88336	43.85947	47.17075

Table 6 Displacement ΔX versus projection X

Projection X(m)	Approximation ΔX(m)	Clothoid ΔX(m)	Cubic Parab. ΔX(m)	SPTC (10) ΔX(m)	SPTC (3) ΔX(m)	SPTC (2) ΔX(m)
100	0.00063	0.00021	0.00062	0.00000	−0.00005	−0.00005
200	0.02000	0.00667	0.01999	0.00000	−0.00005	−0.00004
300	0.15187	0.05062	0.15164	0.00000	−0.00005	0.00018
400	0.64000	0.21327	0.63694	0.00000	−0.00003	0.00301
500	1.95313	0.65053	1.93053	0.00000	0.00025	0.02260
600	4.86000	1.61738	4.74500	0.00000	0.00307	0.11548
700	10.50437	3.49096	10.05396	0.00000	0.02270	0.45372
800	20.48000	6.79180	19.03034	0.00000	0.12536	1.46384
900	36.90563	12.20138	32.90848	0.00000	0.55439	4.03497
1000	62.50000	20.57446	52.78640	0.00000	2.03807	9.73871

values of X, ratios $X/A > 0.5$, as well as the affinity of the clothoid with the SPTC are obvious. The last two columns give an indication of the areas where their use is satisfactory for simplified calculations without the use of a computer.

In Table 3 the values of angle τ in degrees are presented. The value of angle τ was calculated by using relation (5) for the clothoid, relation (14) for the cubic parabola and relation (26) for SPTC. The deviation of the cubic parabola from the other curves for large values of X, ratios $X/A > 0.5$, as well as the affinity of the clothoid with the SPTC are obvious.

In Table 4 the lengths L on the curve corresponding with the values of X are shown. The length L of the clothoid was calculated by using relation (8). The length L of the cubic parabola was calculated by using the commonly used approximate relation (15) instead of (19) which is more accurate. The length of SPTC was calculated by using relation (49). On the last two columns of Table 4, the approximations with four and three terms respectively are presented. The approximation with two terms is equal to the approximate length of cubic parabola. The deviation of the cubic parabola from the other curves for large values of X, ratios $X/A > 0.7$, as well as the affinity of the clothoid with the SPTC are obvious. The use of relation (19), that gives more accurate results, will yield smaller values of the length of cubic parabola and thus the deviation of cubic parabola from the other curves would be greater.

In Table 5 the values of the deviation ΔR for different values of X are shown. The values of the deviation on the second column of the Table result from the approximate relation:

$$\Delta R = X^2 \Big/ (24R) \tag{50}$$

which is the one most frequently used [1], [3] (Esveld 2001, Lamm et al. 1999). The following accurate relation was used for the other columns [3] (Lamm et al. 1999):

$$\Delta R = Y - R(1 - \cos\tau) \tag{51}$$

The divergence of the real deviation of the cubic parabola from the approximate relation (50) for large values of the ratio $X/A > 0.5$ is obvious. The alignment design with the simultaneous use of the cubic parabola and of the approximate value of the deviation can lead to serious mistakes.

The use of the cubic parabola for ratios $X/A \leq 0.3$ does not lead to mistakes but does not differentiate it from other curves as results from Tables 2, 3, 4 and 5. This means that apart from the simplicity, which characterizes it, it has nothing to offer compared to the clothoid.

In Table 6 the values of quantity ΔX versus X are presented. Quantity ΔX (Fig. 5) represents the distance of the projection of the center of the curvature at the end of the curve from the middle of the linear segment, which defines the projection of the whole curve. The two projections are viewed on axis X.

In the second column of the Table the value of ΔX is given based on the approximate relation:

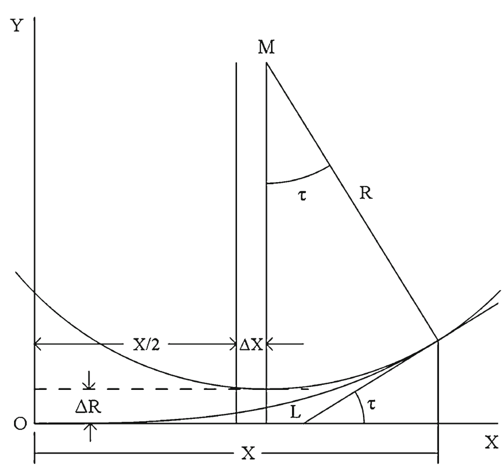

Fig. 5 Displacement ΔX of a transition curve

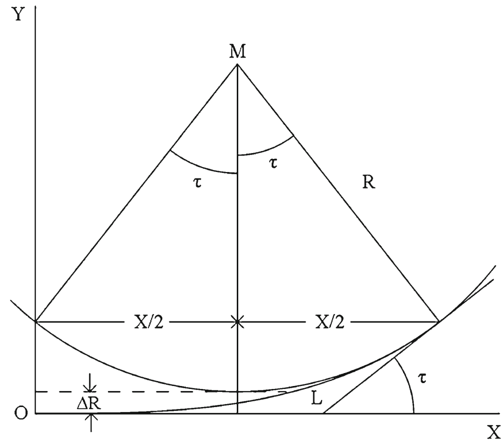

Fig. 6 Symmetrically Projected Transition Curve (SPTC)

$$\Delta X = X^3 \Big/ \left(16R^2\right) \quad (52)$$

which is only valid for the cubic parabola [1] (Esveld 2001). The exact values in the following columns are calculated based on the relation:

$$\Delta X = X \Big/ 2 - R_X \sin\tau_X \quad (53)$$

where with indicator X are symbolized the radius R and angle τ for the particular point of the transition curve that its projection is at a distance X from its starting point [1] (Esveld 2001). It is worth noting the good approximation of the ΔX of the cubic parabola from the relation (52) for values of the ratio $X/A<=0.5$, as shown by the comparison of the second with the fourth column of the Table 6.

The most remarkable observation than can be made in the table is the fact that ΔX always zero for the SPTC curve whereas in the other curves it has a value which increases as X/A increases. Thus, the SPTC curve is symmetrically projected on its basic tangent. This property can be easily explained by examining relation (27) and Fig. 6. This property contributes to the simplicity of the alignment design. That is another reason to prefer the SPTC curve. The comparison of all transition curves for different ratios of transition curve length versus arc radius, leads to remarkable conclusions. The results are presented in Tables 2, 3, 4, 5 and 6 to demonstrate the usability of each transition curve. In particular they allow the designer to know the accuracy of the calculations.

4 Conclusions

A comparison of the cubic parabola transition curve and the clothoid revealed that for a ratio $X/A \leq 0.3$ the curves have no significant differences. Thus, beyond simplicity reasons the cubic parabola has no other obvious advantage compared to the clothoid. However, referring to cubic parabola calculations, for a ratio $X/A \geq 0.5$ and taking in to account the approximate calculation procedure of ΔR it can lead to alignment design errors. Consequently, the usage limits for each transition curve should be well known. A new transition curve was also proposed in this work. The new curve is called Symmetrically Projected Transition Curve (SPTC). SPTC was found, in most cases, to have better performance than cubic parabola. By employing its approximate forms it can offer the correct solutions in cases where the use of the cubic parabola was preferred, due to simplicity reasons, to the clothoid. The SPTC will be more attractive than the clothoid to the engineers who prefer the cubic parabola as a transition curve because it is of the form $y=f(x)$. SPTC can also give the relation between tangent distance and tangent offset of the clothoid because the two curves are identical for $X/A \leq 0.5$. Symmetry is an important characteristic of the SPTC and contributes to simplicity, accuracy and audit ability of the designed alignment. Finally SPTC can also be used as a transition curve between two adjacent circular arcs in the same direction.

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