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THE INITIAL VALUE PROBLEM FOR THE TRADE CYCLE IN EUCLIDIAN SPACE

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Ten Raa (1984) has shown how arithmetics ideas carry over to distributions over space and can be used to solve open, static spatial problems such as the determination of urban equilibrium. This article extends the approach to dynamic spatial economics by tracing spatial distributions through time. It is shown that the basic ideas of ordinary differential equations carry over to the present context, provided that 'functions' are spatially distributed valued. The consequent differential equations for the distributions are solved. Puu's (1982) spatial trade cycle model falls out as a special case and its associated initial value problem can now be completely solved.

1. Introduction and summary

Ten Raa (1984) has shown how arithmetics ideas carry over to distributions over space and can be used to solve open, static spatial problems such as the determination of the indirect effects of an expenditure program or the determination of urban equilibrium. This article extends the approach to dynamic spatial economics.

The issue of mathematical space selection for a dynamic spatial economy is delicate. Such an economy combines dynamic and spatial elements, say investment and consumption. Perhaps the most natural commodity space to embed those elements is $\mathscr{D}'(R \times R^2, R^n)$ which consists of n-vector distributions over time and space jointly, where n is the number of physically differentiated commodities. [For an introduction to real-valued distributions see Griffel (1981, p. 17).] However, often one traces the development of a spatial economy, considered as a whole, through time. This view is particularly useful in the study of initial value problems for spatial economies, such as the ones posed by Puu (1982) and Beckmann and Puu (1985). Then the problems can be solved as if they were textbook initial value problems,

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the only modification being that values are spatial distributions instead of reals. In this case one takes the alternative commodity space of distributions over time with values in the space of spatial distributions. The static manipulations of spatial distributions by Ten Raa (1984) will thus be extended to dynamic analysis.

An n-vector (spatial) distribution-valued distribution (over time), A, is a linear continuous functional from test functions on time (R), $\phi \in \mathscr{D}(R)$, to n-vector distributions over space (R^2), $A(\phi) \in \mathscr{D}'(R^2, R^n)$. [Test functions are defined to be infinitely differentiable and to have compact support, see Griffel (1981, p. 16).] The linearity and continuity conditions are captured elegantly by the following formal definition: $A: \mathscr{D}(R) \to \mathscr{D}'(R^2, R^n)$ is a *distribution-valued distribution* if $\phi \mapsto \langle A(\phi), \psi \rangle$ is a (real-valued) distribution for all $\psi \in \mathscr{D}(R^2)$.

Summing up, we take as the commodity space either $\mathscr{D}'(R \times R^2, R^n)$, consisting of n-vector distributions over time-space, or $\mathscr{L}[\mathscr{D}(R), \mathscr{D}'(R^2, R^n)]$, consisting of n-vector spatial distribution-valued distributions over time. It may be said that the choice is a matter of convenience. The justification of this proposition lies in a deep theorem which states that the space of distributions over time-space and the space of spatial distribution-valued distributions over time are essentially equal. [More precisely, by the Schwartz (1953–1954) kernel theorem there is a bijection between $A \in \mathscr{L}[\mathscr{D}(R), \mathscr{D}'(R^2, R^n)]$ and (its kernel) $a \in \mathscr{D}'(R \times R^2, R^n)$. a is obviously defined for separable test functions on time-space, say $\phi \otimes \psi$, where $\otimes$ is the direct tensor product, namely: $\langle a, \phi \otimes \psi \rangle = \langle A(\phi), \psi \rangle$. The deepness of the theorem rests in the extension of a to all test functions on time-space. An encyclopedic reference is Gel'fand and Vilenkin (1964).]

To illustrate the use of our commodity framework for the analysis of specific models we now briefly discuss the application to the trade cycle model of Puu (1982). Detailed analysis is relegated to the remainder of the article.

Puu studies local income, Y, and local net export, X, as functions of time, t, and location in space, denoted by Euclidean coordinates x_1 and x_2, or briefly vector x. He regards X and Y as deviations from equilibrium. Puu assumes that income adjusts in proportion to the degree savings fall short of net export:

$$\dot{Y} = \lambda(X - \sigma Y),$$

where σ is the savings quote, λ denotes adjustment speed and a dot time differentiation. He notes that it is usual to relate net exports to income 'abroad' relative to local income. Relative income 'abroad' is measured by the 'curvature' of Y, that is $\partial^2 Y/\partial x_1^2 + \partial^2 Y/\partial x_2^2$, or the Laplacean, ΔY. Assuming an import propensity μ and an adjustment process with the same delay as above, Puu obtains

$$\dot{X}=\lambda(\mu\Delta Y-X).$$

The model is reduced by elimination of X:

$$\ddot{Y}+\lambda(1+\sigma)\dot{Y}+\lambda^2\sigma Y=\lambda^2\mu\Delta Y.$$

This is Puu's equation of a dynamic spatial economy. The initial value conditions are

$$Y(x,0)=Y_0(x) \quad \text{and} \quad \dot{Y}(x,0)=Y_1(x).$$

We consider the unknown Y as a distribution over time (with spatial distribution values) and incorporate the initial value conditions in the equation by going to HY where H is the Heavyside function [zero on the negatives and one on the positives, Griffel (1981, p. 19)]. Then HY can be shown to fulfill

$$(HY)^{\cdot\cdot}+\lambda(1+\sigma)(HY)^{\cdot}+\lambda^2\sigma HY=\lambda^2\mu\Delta(HY)+[\lambda(1+\sigma)Y_0+Y_1]\delta+Y_0\dot{\delta}.$$

This is a second-order differential equation in HY.

Reconsidering HY as a distribution over time-space and letting E be the *fundamental solution* of the differential operator, defined by $DE=\delta$, we obtain by convolution with E,

$$HY=[\lambda(1+\sigma)Y_0+Y_1]*E+Y_0*\dot{E},$$

where $*$ denotes the convolution product with respect to space.

This is the formal solution to the initial value problem. For the concepts involved we refer to Gel'fand and Shilov (1964, p. 103), Schwartz (1978) or Griffel (1981). The main task that remains to be done is disclosure of E, but that will be undertaken after the model is reposed in suitable coordinates in the next section. Section 3 offers the mathematics of the incorporation of initial values in the equation and of the consequent solution. Applications, including a detailed analysis of Puu's model to which non-mathematical readers may turn straightaway, are presented in section 4.

2. Puu's model in natural coordinates

To avoid simple but unrealistic boundary conditions, such as constant equilibrium on the boundary of a quadratic or circular region, Puu (1982) considers trade cycles on the surface of a sphere. I must admit that his approach has grandeur: he takes into account local curvature effects on the structure of the equations and signifies indirect effects of disturbances resulting from propagation swings around the world. These distinctions may be punctuated by brief mention of an analogy to fluid dynamics: Puu's expenditures propagate like a ship, of which the forward movement is facilitated by the downward curvature of the ocean, and of which the fairway is disturbed by the wake of the ship which extends all the way around the world! However subtle, these effects are generally considered distractions from the basic analysis of ship movement though. It seems to me that the same holds for spatial economics. For example, the reinforcement of a Kuwaitian expenditure shock through local trade and income effects propagating via Iran, Pakistan, ..., Vietnam, the Pacific, Mexico, the Caribbean, various African and Arabian countries, and back to Kuwait, may be dismissed as a curious by-product of Puu's rather fancy sphere modelling.

But if sphere modelling distracts, how else are unrealistic boundary conditions avoided? Well, in fluid dynamics one assumes that a ship in the ocean feels no boundary effects at all and the Euclidean coordinates in which the equations are put are assumed to extend into infinity. In other words, one simply considers the case in which there is no spatial boundary at all.

Now Puu's equations are expressed in Euclidean coordinates. Thus an analysis of the basic equations in the Euclidean plane itself would constitute a valid rejoinder to Puu's outline of spherical trade cycles. As Puu (p. 3) himself implies, such an analysis is not easy. Yet we shall face the challenge. The purpose of this article is to solve Puu's equations in the Euclidean plane for completely general initial conditions. Our approach will be powerful and bear on general spatial initial value problems that are linear in time.

Puu studies local income, Y, and local net export, X, as functions of time, t, and location in space, denoted by Euclidean coordinates x_1 and x_2, or briefly vector x. He regards X and Y as deviations from equilibrium.

Puu assumes that income adjusts in proportion to the degree savings fall short of net export:

$$\dot{Y} = \lambda(X - \sigma Y), \tag{1}$$

where σ is the savings quote, λ denotes adjustment speed and dot time differentiation.

Puu notes that it is usual to relate net exports to income 'abroad' relative to local income. Relative income 'abroad' is measured by the curvature of Y, that is $\partial^2 Y/\partial x_1^2 + \partial^2 Y/\partial x_2^2$, or the Laplacean, ΔY. Assuming an import propensity μ and an adjustment process with the same delay as above, Puu obtains

$$\dot{X} = \lambda(\mu \Delta Y - X). \tag{2}$$

The model is complete. Puu summarizes it through elimination of X:

$$\ddot{Y}+\lambda(1+\sigma)\dot{Y}+\lambda^2\sigma Y=\lambda^2\mu\Delta Y. \tag{3}$$

This is the *Puu equation* of a dynamic spatial economy.

As Puu notes, his equation is of the wave type. To solve it, however, we must know more about the structure. The first result of our paper is that Puu's equation is essentially the Klein–Gordon equation,

$$(\Box+m^2)f=0 \tag{4}$$

with $\Box=\ddot{}-\Delta$, the wave operator, and $m=\frac{1}{2}i(1-\sigma)/\sqrt{\mu}$, the mass parameter.

The Klein–Gordon equation occurs in quantum mechanics where it governs relativistic waves for particles with mass m. We have now come across it in spatial economics. The only modification is that the mass is now purely imaginary (proportional to $i=\sqrt{-1}$). Note that the modulus of the mass equals half the propensity to consume, $1-\sigma$, over the square root of the import propensity, μ.

Proposition 1. The Puu equation (3) is equivalent to the Klein–Gordon equation (4), by the change of variable defined by

$$f(x,t)=Y(x,t/\lambda\sqrt{\mu})\,\mathrm{e}^{[(1+\sigma)/2\sqrt{\mu}]t}.$$

Proof.

$$Y(x,t/\lambda\sqrt{\mu})=f(x,t)\,\mathrm{e}^{-[(1+\sigma)/2\sqrt{\mu}]t} \quad \text{or} \quad Y(x,t)=f(x,\lambda\sqrt{\mu}t)\,\mathrm{e}^{-\frac{1}{2}\lambda(1+\sigma)t}.$$

Therefore,

$$\dot{Y}(x,t)=\lambda\sqrt{\mu}\dot{f}(x,\lambda\sqrt{\mu}t)\mathrm{e}^{-\frac{1}{2}\lambda(1+\sigma)t}-\tfrac{1}{2}\lambda(1+\sigma)f(x,\lambda\sqrt{\mu}t)\,\mathrm{e}^{-\frac{1}{2}\lambda(1+\sigma)t}$$

and

$$\begin{aligned}\ddot{Y}(x,t)=\;&\lambda^2\mu\ddot{f}(x,\lambda\sqrt{\mu}t)\,\mathrm{e}^{-\frac{1}{2}\lambda(1+\sigma)t}-\tfrac{1}{2}\lambda(1+\sigma)\lambda\sqrt{\mu}\dot{f}(x,\lambda\sqrt{\mu}t)\,\mathrm{e}^{-\frac{1}{2}\lambda(1+\sigma)t}\\&-\tfrac{1}{2}\lambda(1+\sigma)\lambda\sqrt{\mu}\dot{f}(x,\lambda\sqrt{\mu}t)\,\mathrm{e}^{-\frac{1}{2}\lambda(1+\sigma)t}\\&+\tfrac{1}{4}\lambda^2(1+\sigma)^2f(x,\lambda\sqrt{\mu}t)\,\mathrm{e}^{-\frac{1}{2}\lambda(1+\sigma)t}.\end{aligned}$$

(3) becomes, upon substitution of the f-expressions for Y, $\dot{Y}$ and $\ddot{Y}$ and

division by $\lambda^2 \mu e^{-\frac{1}{2}\lambda(1+\sigma)t}$,

$$\ddot{f} - \frac{1+\sigma}{\sqrt{\mu}}\dot{f} + \frac{(1+\sigma)^2}{4\mu}f + \frac{1+\sigma}{\sqrt{\mu}}\dot{f} - \frac{(1+\sigma)^2}{2\mu}f + \frac{\sigma}{\mu}f = \Delta f \quad \text{or}$$

$$\Box f - \frac{(1-\sigma)^2}{4\mu}f = 0,$$

i.e., (4). In the same way, (4) becomes, upon substitution of f, (3). Q.E.D.

The change of variable in Proposition 1 consists of compounding interest at a rate $(1+\sigma)/2\sqrt{\mu}$ and choosing a time unit of $\lambda\sqrt{\mu}$. These economic devices merely affect time; spatial variables remain essentially the same. In particular, initial values for f and Y coincide:

$$f(x,0) = Y(x,0) = Y_0(x). \tag{5}$$

Here $Y_0(x)$ is a prescribed initial income distribution over space. The initial conditions are completed by a prescription of the trade picture at time zero:

$$X(x,0) = X_0(x).$$

By (1) we know the equivalent initial conditions for Y and $\dot{Y}$. And by the change of variable in Proposition 1 we know the initial conditions for f and $\dot{f}$, i.e., (5) and, say,

$$\dot{f}(x,0) = f_1(x). \tag{6}$$

In fact, the proof of Proposition 1 and (1) show that

$$f_1(x) = \frac{1}{\lambda\sqrt{\mu}}\dot{Y}(x,0) = \frac{1}{\sqrt{\mu}}[X_0(x) - \sigma Y_0(x)].$$

The question is how income evolves from f_0 at an initial speed f_1 under Puu's law (3). In other words, the issue is initial value problem (4), (5), (6).

3. Initial value problems

The usual approach to initial value problems for wave type equations is the method of spherical means. This method is ad hoc and, in particular, breaks down when the equation becomes of the heat type by lowering the order of reaction over time as described by Puu (1982, pp. 1–2). Economic processes can be either wave or heat like, the precise nature depending on the specification of the laws that govern the dynamics. This matter is discussed by Beckmann (1970, 1971), Beckmann and Puu (1985), and Ten Raa (1986). To circumvent the problem, we now approach spatial dynamics in a way that is robust with respect to the specification of the laws through the development of a unified analysis of initial value problems.

To fix ideas we begin considering the ordinary initial value problem for a twice locally summably differentiable function, f (or, equivalently, an absolutely continuously differentiable function, $\dot{f}$) with a differential operator

$$D=a_0+a_1\dot{}\;+a_2\ddot{}\;=a_0+a_1\frac{\mathrm{d}}{\mathrm{d}t}+a_2\frac{\mathrm{d}^2}{\mathrm{d}t^2}:$$

$$Df=0 \quad \text{on} \quad R_+, \tag{7}$$

$$f(0)=f_0, \tag{8}$$

$$\dot{f}(0)=f_1. \tag{9}$$

A crucial step of the analysis is the incorporation of the initial data, (8)–(9), in the eq. (7). This is done by going from f to Hf where H is the Heavyside function which is defined to be 1 on R_+ and 0 elsewhere [Griffel (1981, p. 19)]. Clearly, it suffices to find Hf. Its equation is given by

Proposition 2. If twice locally summably differentiable f fulfills (7), (8), (9), then Hf fulfills

$$D(Hf)=(a_1f_0+a_2f_1)\delta+a_2f_0\dot{\delta},$$

where δ is the Dirac measure [unit point mass at the origin, see Griffel (1981, p. 17)]. In fact,

$$Hf=(a_1f_0+a_2f_1)*E+a_2f_0*\dot{E},$$

where E is the fundamental solution of D [Griffel (1981, p. 42)].

Corollary. Since the fundamental solution is defined by $DE=\delta$, the first part of the proposition demonstrates that it is given by $E=Hf$ with f the solution to the case of (7), (8), (9) in which $f_0=a_1^{-1}$ and $a_2=0$ (for the first-order equation) or $f_0=0$ and $f_1=a_2^{-1}$ (for the second-order equation). The use of the proposition here is justified by noting that this solution f is twice locally summably differentiable, by the theory of ordinary differential equations, which also establishes existence and uniqueness.

Proof of Proposition 2. $D(Hf)=a_0Hf+a_1(Hf)^{\cdot}+a_2(Hf)^{\cdot\cdot}$. By the product rule, $(Hf)^{\cdot}=\delta f(0)+H\dot{f}$, as f is a fortiori locally summably differentiable and $\dot{H}=\delta$ by Griffel (1981, p. 23). Applying the rule once more, $(Hf)^{\cdot\cdot}=\dot{\delta}f(0)+\delta\dot{f}(0)+H\ddot{f}$, as $\dot{f}$ is locally summably differentiable. Substituting and rearranging, $D(Hf)=a_0Hf+a_1H\dot{f}+a_2H\ddot{f}+a_1\delta f(0)+a_2\delta\dot{f}(0)+a_2\dot{\delta}f(0)$. Since

$$a_0Hf=Ha_0f \tag{10}$$

(similarly for the second and third terms),

$$a_1\dot{f}=(a_1f)^{\cdot} \tag{11}$$

(similarly for the third term) and

$$\delta f(0)=f(0)\delta \tag{12}$$

(similarly for the last two terms), we have

$$D(Hf)=HDf+[a_1f(0)+a_2\dot{f}(0)]\delta+a_2f(0)\dot{\delta}.$$

By (7), (8), (9), $D(Hf)=(a_1f_0+a_2f_1)\delta+a_2f_0\dot{\delta}$. D can be interpreted as a distribution (namely $a_0\delta+a_1\dot{\delta}+a_2\ddot{\delta}$) that applies through the convolution product: $D*(Hf)=(a_1f_0+a_2f_1)\delta+a_2f_0\dot{\delta}$. Commuting and multiplying through with E in the sense of convolutions, $(Hf)*D*E=(a_1f_0+a_2f_1)\delta*E+a_2f_0\dot{\delta}*E$. [For the convolution product see Gel'fand and Shilov (1964, p. 103).] But the fundamental solution fulfills $D*E=\delta$ and this is the unit element in the convolution algebra, by which also

$$a_1f_0\delta*E=a_1f_0\cdot E, \tag{13}$$

and

$$a_2f_0\dot{\delta}*E=a_2f_0\delta*\dot{E} \tag{14}$$

which can be written out analogous to (13). Therefore,

$$Hf=(a_1f_0+a_2f_1)*E+a_2f_0*\dot{E}. \qquad \text{Q.E.D.}$$

Thus we have found f on R_+. The result is in perfect agreement with the textbook solution. Our formulation, however, lends itself to the treatment of spatial initial value problems.

Take the heat equation [Treves (1975)]. The basic idea is to adopt the just developed view point of ordinary differential equations by letting f n-dimensional spatially valued and a_i 'coefficients' which map spatial distributions to themselves. Specifically, $f(t)$ is a distribution on R^n, (a_0, a_1, a_2) equals $(-\Delta, 1, 0)$, and f_1 is free since the 'ordinary' differential equation is of first order. Let us try heuristically to find the solution by analogy to Proposition 2, i.e., $Hf = (a_1 f_0 + a_2 f_1) \cdot E + a_2 f_0 \cdot \dot{E}$. Consider the first term on the right-hand side. The 'coefficient' is a spatial distribution. E is also a distribution. The ordinary product is not meaningful [Griffel (1981, p. 15)], but the proof of Proposition 2 suggests we should employ the convolution product instead. Consistency of dimensions requires the convolution product to be taken with respect to (geographical) space only. Thus we obtain

$$Hf = (a_1 f_0 + a_2 f_1) * E + a_2 f_0 * \dot{E}.$$

Substituting the 'coefficients', $(a_0, a_1, a_2) = (-\Delta, 1, 0)$, the solution reduces to

$$Hf = f_0 * E.$$

Substituting the well-known fundamental heat solution for E [Treves (1975)], we find

$$(Hf)(t) = f_0 * (4\pi t)^{n/2} \exp\left(-\frac{\|x\|^2}{4t}\right) H(t).$$

This coincides precisely with the well-documented solution to the initial value problem for the heat equation [Treves (1975)]. Consequently, spatialization of Proposition 2 seems promising. It will be undertaken rightaway.

Proposition 3. Let a_0, a_1 and a_2 continuously and linearly map spatial distributions to themselves, and let

$$D = a_0 + a_1 \frac{\mathrm{d}}{\mathrm{d}t} + a_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2}.$$

If spatially distributed valued function f of time is twice locally summably differentiable and fulfills (7), (8), (9) with f_0 and f_1 now spatial distributions, then Hf fulfills

$$D(Hf) = (a_1 f_0 + a_2 f_1)\delta + a_2 f_0 \dot{\delta}.$$

In fact,

$$Hf = (a_1 f_0 + a_2 f_1) * E + a_2 f_0 * \dot{E},$$

where E is the fundamental solution of D.

Remark. Hf is a function of time. By Schwartz (1978) it can be reconsidered as a generalized function or distribution on time. Then $D(Hf)$ is also a well-defined distribution on time with spatially distributed values. And so is the right-hand side of the equation for $D(Hf)$. For example, the first term, $a_1 f_0 \delta$, is the distribution on time that is concentrated in the origin and assigns (to a test function that is 1 in the origin) value $a_1 f_0$. The second equation in the statement of Proposition 3 presents Hf as a distribution on time and space jointly with real values; the convolution product is taken, as in the heat example, with respect to space only. The alternative interpretation of Hf as a distribution on time with spatially distributed values versus a real-valued distribution on time and space jointly, are consistent by the kernel theorem of Schwartz (1953–1954). This is one element of the proof which is outlined next. For full mathematical detail the reader is referred to Ten Raa (1985), in compliance with editorial policy.

Outline of proof of Proposition 3. f is a differentiable (spatially distributed valued) function, but Hf is not. To subject the latter to D, we must associate a distribution with it. The association has to be injective to derive the equation for Hf as a function of t, in the last part of the proposition. This is done in

Lemma 1. Locally summable functions with spatially distributed values can be mapped into distributions with spatially distributed values.

The construction of the associated mapping with spatially distributed values is straightforward. The only difficult part of the proof of Lemma 1 is to establish the distributional requirement that the mapping is continuous in the sense of Schwartz (1978). This is done in two steps. First, it is shown to be lower semicontinuous in test functions with support in a fixed compact set using Fatou's lemma [Rudin (1964, p. 246)]. Second, this is shown to imply continuity by means of Baire's lemma [Kolmogorov and Fomin (1970, p. 61)].

The application of D to Hf in the sense of distribution-valued distributions yields

Lemma 2. Let a_0, a_1 and a_2 be as in the statement of Proposition 3. Then, in the sense of distributions,

$$a_0 Hf = Ha_0 f,$$

$$a_1 (Hf)^{\cdot} = Ha_1 \dot{f} + a_1 f(0)\delta,$$

$$a_2 (Hf)^{\cdot\cdot} = Ha_2 \ddot{f} + a_2 \dot{f}(0)\delta + a_2 f(0)\dot{\delta}.$$

The notation does not show that Hf is taken in the sense of distribution-valued distributions and, therefore, is sloppy. The full proof takes away confusion [Ten Raa (1985)]. $\mathscr{D}'(R^2)$ is reflexive [Schwartz (1978, p. 75)]. This fact is used to show the first equality and $a_1(Hf)^{\cdot}=(a_1Hf)^{\cdot}$.

The right-hand side is rewritten using the first equality and the product rule. By linearity and continuity, the coefficients can be pulled outside the brackets [Schwartz (1978, p. 74)] to obtain the right-hand side of the second equality. Differentiating through and repeating arguments, the last equality of Lemma 2 is obtained.

Adding up the equalities of Lemma 2 and substituting (7), (8), (9), the equation for Hf is established in the sense of distribution-valued distributions. To convolute through with the fundamental solution of D, we must associate a real-valued distribution on joint time-space. The association has to be injective to derive the equation for Hf as a function of t. This is done in

Lemma 3. Distributions with spatially distributed values can be mapped into real-valued distributions on time-space. Moreover, the time derivative [Ten Raa (1985)] is mapped into the partial derivative with respect to time.

The first part of this lemma is a corollary to the kernel theorem of Schwartz (1953–1954) and the second part is established on the product space of time and space test functions which is dense in the joint test space. An encyclopedic reference is Gel'fand and Vilenkin (1964).

Applying Lemma 3 and convoluting through, the solution to the equation for Hf is established in the sense of real-valued distributions on time-space. By injectivity it holds in the sense of distribution-valued distributions and in the sense of functions of time, respectively. This completes the outline of the proof of Proposition 3.

4. Applications

Before applying Proposition 3 to the initial value problem (4), (5), (6) that describes Puu's dynamic spatial economy, we first consider a simpler example, namely the wave equation. The example is interesting. The wave equation is usually *not* solved by our unified method, but by use of spherical means. This method is ad hoc and hinders analysis of initial spatial value problems that is robust with respect to the specification of the 'coefficients' in the equation, or, at a deeper level, the laws that govern the economy. The subsequent analysis overcomes this and, in particular, serves as model for the solution of problem (4), (5), (6).

The wave equation reads $\square f=0$ with $\square=\ddot{}-\Delta$. Hence the coefficients are $(a_0,a_1,a_2)=(-\Delta,0,1)$. Proposition 3's differentiability condition on f is fulfilled according to Treves' (1975, p. 105) Fourier analysis. [In the case at hand, with zero right-hand side, f is in fact infinitely differentiable by Treves (1966, p. 456). The above weaker differentiability condition, however, remains sufficient for non-zero right-hand sides, as Ten Raa (1985, appendix) shows. In other words, the present argument is more robust with respect to driving forces.] As before, let the initial values be $f(x,0)=f_0(x)$, and $\dot{f}(x,0)=f_1(x)$. Take $x \in R^3$, the traditional three dimensional space. Proposition 3 yields

$$Hf = f_1 * E + f_0 * \dot{E},$$

where E is the fundamental solution of $\Box$, i.e., $\delta(t-\|x\|)/4\pi\|x\|$ [Treves (1975)]. Substituting, the first term of the solution becomes, employing spherical coordinates r and Ω about x,

$$f_1 * \frac{\delta(t-\|x\|)}{4\pi\|x\|} = \int\int f_1(x') \frac{\delta(t-\|x-x'\|)}{4\pi\|x-x'\|} \mathrm{d}x'$$

$$= \int\int f_1(r,\Omega) \frac{\delta(t-r)}{4\pi r} r^2 \, \mathrm{d}r \, \mathrm{d}\Omega$$

$$= \frac{t}{4\pi} \int f_1(t,\Omega) \mathrm{d}\Omega = t M_{x,t}(f_1)$$

with $M_{x,t}$ defined as the spherical average about x at radius t. The second term becomes, repeating the use of spherical coordinates,

$$f_0 * \dot{E} = \frac{\partial}{\partial t}(f_0 * E) = \frac{\partial}{\partial t}[t M_{x,t}(f_0)].$$

In sum,

$$Hf = t M_{x,t}(f_1) + \frac{\partial}{\partial t}[t M_{x,t}(f_0)].$$

This result, however quickly established, coincides precisely with the solution obtained by the ad hoc method of spherical means.

Return to the initial value problem of this paper, (4), (5), (6). The coefficients become $(a_0, a_1, a_2) = (m^2 - \Delta, 0, 1)$. The differentiability condition on f is fulfilled just as with the wave equation. Now $x \in R^2$. Proposition 3 yields

$$Hf = f_1 * E + f_0 * \dot{E},$$

where E is the fundamental solution of $\square + m^2$. According to Treves (1975, p. 67),

$$E = \sum_{p=0}^{\infty} (-m^2)^p \pi^{-\frac{1}{2}} \frac{(t^2 - \|x\|^2)^{p-\frac{1}{2}}}{2^{2p+1}\Gamma(p+1)\Gamma(p+\frac{1}{2})}, \quad \|x\| \leqq t \quad \text{(and 0 for other } x\text{)}.$$

By the proof of Proposition 1 it follows that

$$\begin{aligned} HY(x,t) &= f(x, \lambda\sqrt{\mu}t)\mathrm{e}^{-\frac{1}{2}\lambda(1+\sigma)t} \\ &= \mathrm{e}^{-\frac{1}{2}\lambda(1+\sigma)t}\left[\left(f_1 + \frac{f_0}{\lambda\sqrt{\mu}}\frac{\partial}{\partial t}\right)\right. \\ &\quad \left. * \sum_{p=0}^{\infty}\left[\frac{(1-\sigma)^2}{4\mu}\right]^p \pi^{-\frac{1}{2}} \frac{(\lambda^2\mu t^2 - \|x\|^2)^{p-\frac{1}{2}}}{2^{2p+1}\Gamma(p+1)\Gamma(p+\frac{1}{2})}\right]. \end{aligned}$$

Recall that (7) yields $f_1 = (X_0 - \sigma Y_0)/\sqrt{\mu}$ and (5) that $f_0 = Y_0$. Elimination of f_1 and f_0 yields

$$\begin{aligned} HY(x,t) &= \mathrm{e}^{-\frac{1}{2}\lambda(1+\sigma)t}\left\{\left[\frac{X_0 - \sigma Y_0}{\sqrt{\mu}} + \frac{Y_0}{\lambda\sqrt{\mu}}\frac{\partial}{\partial t}\right]\right. \\ &\quad \left. * \sum_{p=0}^{\infty}\left[\frac{(1-\sigma)^2}{4\mu}\right]^p \pi^{-\frac{1}{2}} \frac{(\lambda^2\mu t^2 - \|x\|^2)^{p-\frac{1}{2}}}{2^{2p+1}\Gamma(p+1)\Gamma(p+\frac{1}{2})}\right\} \end{aligned}$$

with $0 \leqq \|x\| \leqq \lambda\sqrt{\mu}t$, and zero elsewhere. By (1), using shorthand,

$$\Sigma = \sum_{p=0}^{\infty}\left[\frac{(1-\sigma)^2}{4\mu}\right]^p \pi^{-\frac{1}{2}} \frac{(\lambda^2\mu t^2 - \|x\|^2)^{p-\frac{1}{2}}}{2^{2p+1}\Gamma(p+1)\Gamma(p+\frac{1}{2})},$$

$$\begin{aligned} X(x,t) &= \left(\lambda^{-1}\frac{\partial}{\partial t} + \sigma\right)Y(x,t) \\ &= \left(\lambda^{-1}\frac{\partial}{\partial t} + \sigma\right)\left\{\mathrm{e}^{-\frac{1}{2}\lambda(1+\sigma)t}\left\{\left[\frac{X_0 - \sigma Y_0}{\sqrt{\mu}} + \frac{Y_0}{\lambda\sqrt{\mu}}\frac{\partial}{\partial t}\right] * \Sigma\right\}\right\} \\ &= \lambda^{-1}\mathrm{e}^{-\frac{1}{2}\lambda(1+\sigma)t}[-\tfrac{1}{2}\lambda(1+\sigma)]\left\{\left[\frac{X_0 - \sigma Y_0}{\sqrt{\mu}} + \frac{Y_0}{\lambda\sqrt{\mu}}\frac{\partial}{\partial t}\right] * \Sigma\right\} \\ &\quad + \mathrm{e}^{-\frac{1}{2}\lambda(1+\sigma)t}\left(\lambda^{-1}\frac{\partial}{\partial t} + \sigma\right)\left\{\left[\frac{X_0 - \sigma Y_0}{\sqrt{\mu}} + \frac{Y_0}{\lambda\sqrt{\mu}}\frac{\partial}{\partial t}\right] * \Sigma\right\} \end{aligned}$$

$$=\lambda^{-1}\mathrm{e}^{-\frac{1}{2}\lambda(1+\sigma)t}[-\tfrac{1}{2}\lambda(1+\sigma)]\left\{\left[\frac{X_0-\sigma Y_0}{\sqrt{\mu}}+\frac{Y_0}{\lambda\sqrt{\mu}}\frac{\partial}{\partial t}\right]*\Sigma\right\}$$

$$+\mathrm{e}^{-\frac{1}{2}\lambda(1+\sigma)t}\left[\left(\sigma\frac{X_0-\sigma Y_0}{\sqrt{\mu}}+\frac{(X_0-\sigma Y_0)/\sqrt{\mu}+\sigma Y_0/\sqrt{\mu}}{\lambda}\right.\right.$$

$$\left.\left.\times\frac{\partial}{\partial t}+\frac{Y_0}{\lambda^2\sqrt{\mu}}\frac{\partial^2}{\partial t^2}\right)*\Sigma\right]$$

$$=\mathrm{e}^{-\frac{1}{2}\lambda(1+\sigma)t}\left\{\left\{-\tfrac{1}{2}(1+\sigma)\frac{X_0-\sigma Y_0}{\sqrt{\mu}}+\sigma\frac{X_0-Y_0}{\sqrt{\mu}}\right.\right.$$

$$\left.\left.+\left[\tfrac{1}{2}(1+\sigma)\frac{Y_0}{\lambda\sqrt{\mu}}+\frac{X_0}{\lambda\sqrt{\mu}}\right]\frac{\partial}{\partial t}+\frac{Y_0}{\lambda^2\sqrt{\mu}}\frac{\partial^2}{\partial t^2}\right\}*\Sigma\right\}$$

$$=\mathrm{e}^{-\frac{1}{2}\lambda(1+\sigma)t}\left\{\left[(\sigma-1)\frac{X_0-\sigma Y_0}{2\sqrt{\mu}}+\frac{X_0-\frac{1}{2}(1+\sigma)Y_0}{\lambda\sqrt{\mu}}\frac{\partial}{\partial t}+\frac{Y_0}{\lambda^2\sqrt{\mu}}\frac{\partial^2}{\partial t^2}\right]\right.$$

$$\left.*\sum_{p=0}^{\infty}\left[\frac{(1-\sigma)^2}{4\mu}\right]^p\pi^{-\frac{1}{2}}\frac{(\lambda^2\mu t^2-\|x\|^2)^{p-\frac{1}{2}}}{2^{2p+1}\Gamma(p+1)\Gamma(p+\frac{1}{2})}\right\}$$

with $0\leqq\|x\|\leqq\lambda\sqrt{\mu}t$, and zero elsewhere.

In the solution for $Y(x,t)$, consider the elementary *example* with $\sigma=0.2$, $\mu=0.16$, $\lambda=2.5$, $X_0=\sqrt{\mu}\delta$ and $Y_0=0$. Then

$$Y(x,t)=\mathrm{e}^{-\frac{3}{2}t}\sum_{p=0}^{\infty}\pi^{-\frac{1}{2}}\frac{(t^2-\|x\|^2)^{p-\frac{1}{2}}}{2^{2p+1}\Gamma(p+1)\Gamma(p+\frac{1}{2})}$$

and

$$X(x,t)=\mathrm{e}^{-\frac{3}{2}t}\left[-\tfrac{4}{5}\tfrac{1}{2}\delta+\tfrac{2}{5}\delta\frac{\partial}{\partial t}\right]*\sum_{p=0}^{\infty}\pi^{-\frac{1}{2}}\frac{(t^2-\|x\|^2)^{p-\frac{1}{2}}}{2^{2p+1}\Gamma(p+1)\Gamma(p+\frac{1}{2})}$$

$$=\tfrac{2}{5}\mathrm{e}^{-\frac{3}{2}t}\left(\frac{\partial}{\partial t}-1\right)\sum_{p=0}^{\infty}\pi^{-\frac{1}{2}}\frac{(t^2-\|x\|^2)^{p-\frac{1}{2}}}{2^{2p+1}\Gamma(p+1)\Gamma(p+\frac{1}{2})}.$$

Note that Y is the fundamental solution, E, with purely imaginary mass and tempered by $\mathrm{e}^{-\frac{3}{2}t}$. The income distribution at each period (1 through 5) is plotted (figs. 1 through 5; note that the scales differ). They are lined up, along with intermediate plots (for each subperiod of 0.2), in fig. 6 where time runs to the right and distance sticks out in forward direction. For each distri-

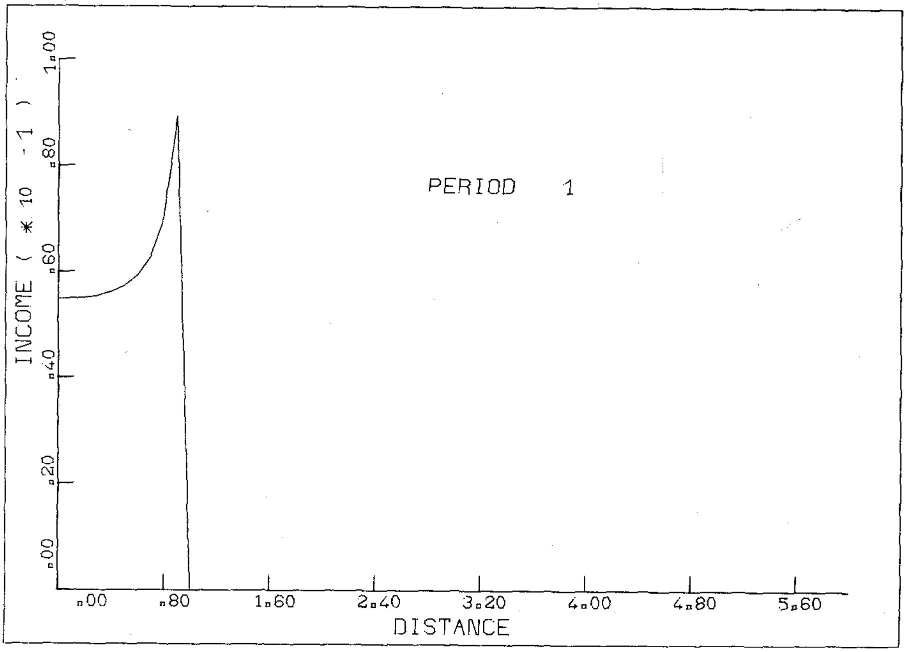

PERIOD 1
INCOME (* 10 - 1)
.00
.20
.40
.60
.80
1.00
.00
.80
1.60
2.40
3.20
4.00
4.80
5.60
DISTANCE

Fig. 1

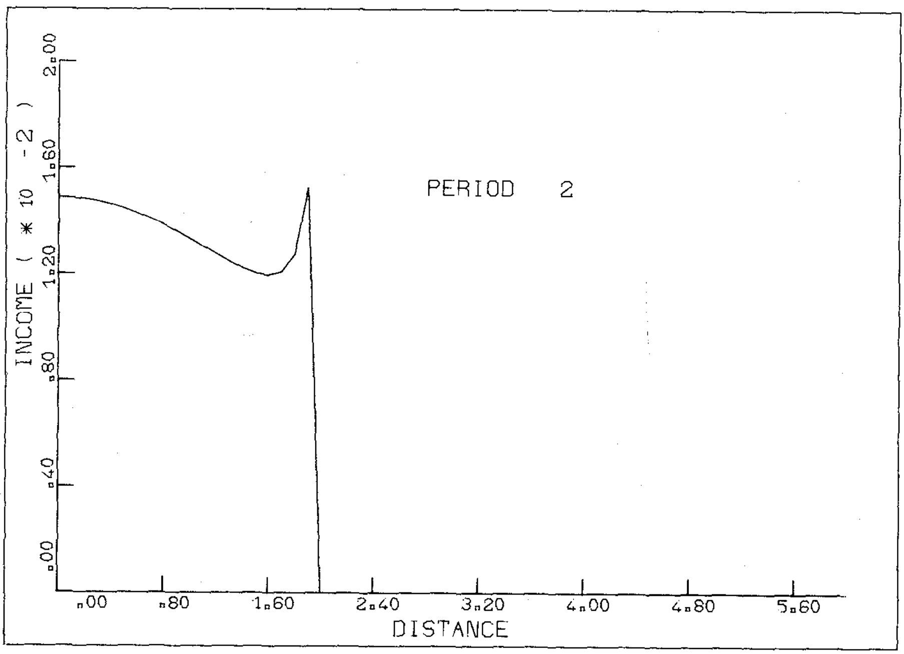

PERIOD 2
INCOME (* 10 - 2)
.00
.40
.80
1.20
1.60
2.00
.00
.80
1.60
2.40
3.20
4.00
4.80
5.60
DISTANCE

Fig. 2

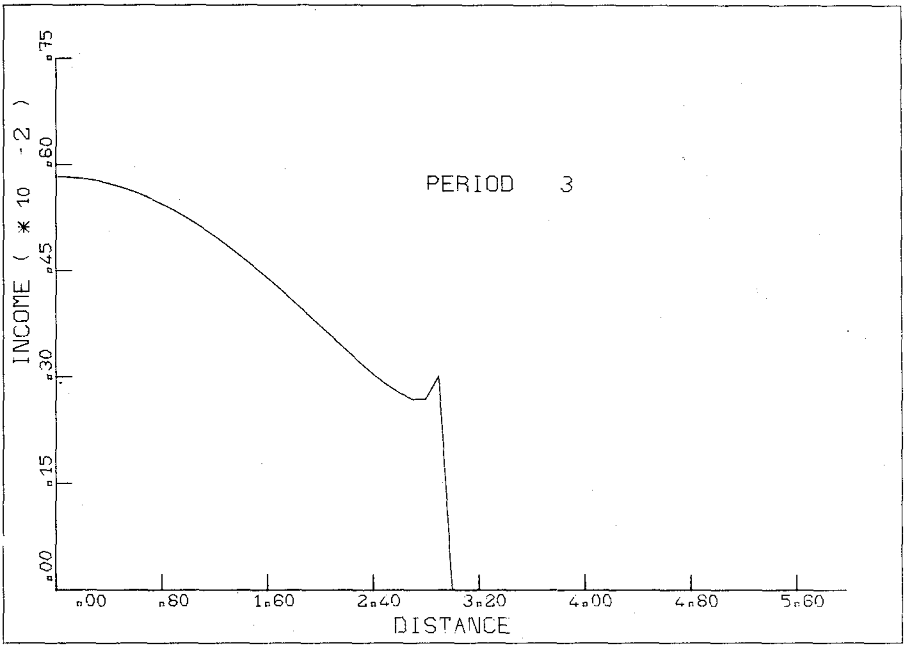

PERIOD 3
INCOME (* 10 -2)
.00
.15
.30
.45
.60
.75
.00
.80
1.60
2.40
3.20
4.00
4.80
5.60
DISTANCE

Fig. 3

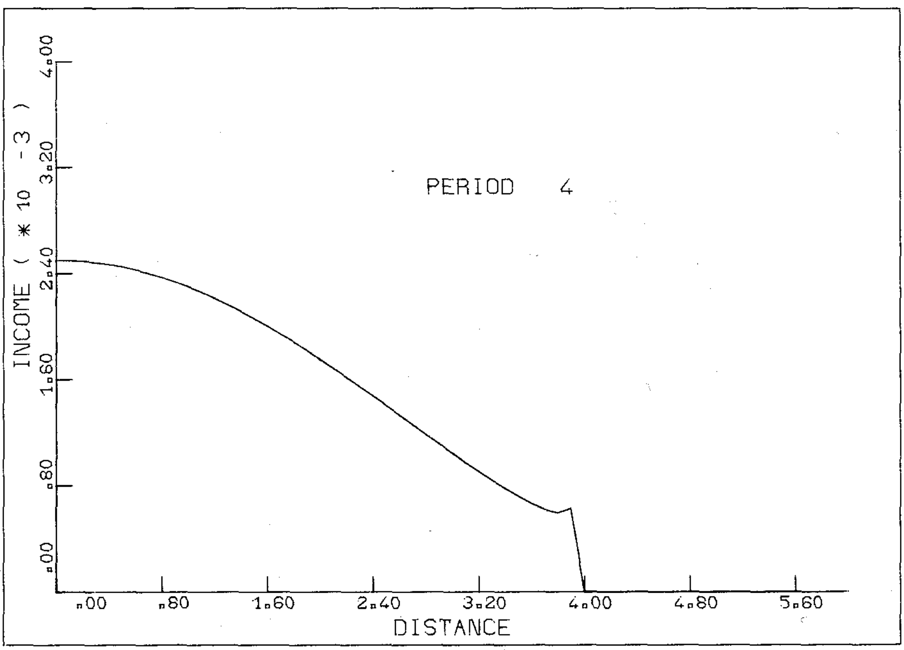
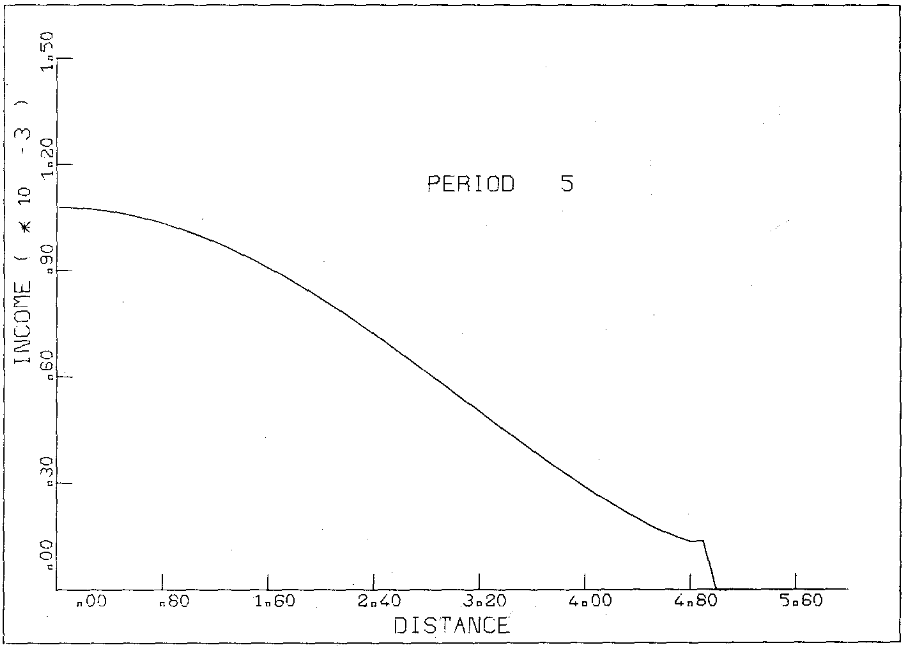

PERIOD 4
INCOME (* 10 -3)
.00
.80
1.60
2.40
3.20
4.00
.00
.80
1.60
2.40
3.20
4.00
4.80
5.60
DISTANCE

Fig. 4



PERIOD 5
INCOME (* 10 -3)
.00
.30
.60
.90
1.20
1.50
.00
.80
1.60
2.40
3.20
4.00
4.80
5.60
DISTANCE

Fig. 5

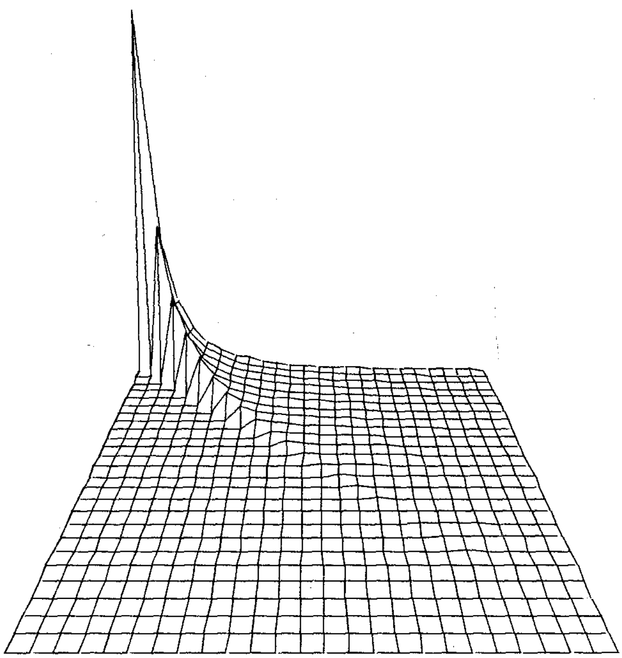


Fig. 6

bution, total income can be found easily:

$$\int_0^t Y(x,t)\,\mathrm{d}x = \mathrm{e}^{-\frac{3}{2}t} \sum_{p=0}^{\infty} \pi^{-\frac{1}{2}} \frac{\int_0^t (t^2-\|x\|^2)^{p-\frac{1}{2}}\,\mathrm{d}x}{2^{2p+1}\Gamma(p+1)\Gamma(p+\frac{1}{2})}$$

$$= \mathrm{e}^{-\frac{3}{2}t} \sum_{p=0}^{\infty} \pi^{-\frac{1}{2}} \frac{2\pi \int_0^t (t^2-r^2)^{p-\frac{1}{2}} r\,\mathrm{d}r}{2^{2p+1}\Gamma(p+1)\Gamma(p+\frac{1}{2})}$$

$$= \mathrm{e}^{-\frac{3}{2}t}\pi^{\frac{1}{2}} \sum_{p=0}^{\infty} \frac{\int_0^t (t^2-r^2)^{p-\frac{1}{2}}\,\mathrm{d}r^2}{2^{2p+1}\Gamma(p+1)\Gamma(p+\frac{1}{2})}$$

$$= \mathrm{e}^{-\frac{3}{2}t}\pi^{\frac{1}{2}} \sum_{p=0}^{\infty} \frac{-(t^2-r^2)^{p+\frac{1}{2}}\big|_0^t/(p+\frac{1}{2})}{2^{2p+1}\Gamma(p+1)\Gamma(p+\frac{1}{2})}$$

$$= \mathrm{e}^{-\frac{3}{2}t}\pi^{\frac{1}{2}} \sum_{p=0}^{\infty} \frac{t^{2p+1}}{(p+\frac{1}{2})2^{2p+1}\Gamma(p+1)\Gamma(p+\frac{1}{2})}.$$

For low periods it is small. Then it reaches a peak at an intermediate point of time and eventually it peters out again. The evolution is given in fig. 7.

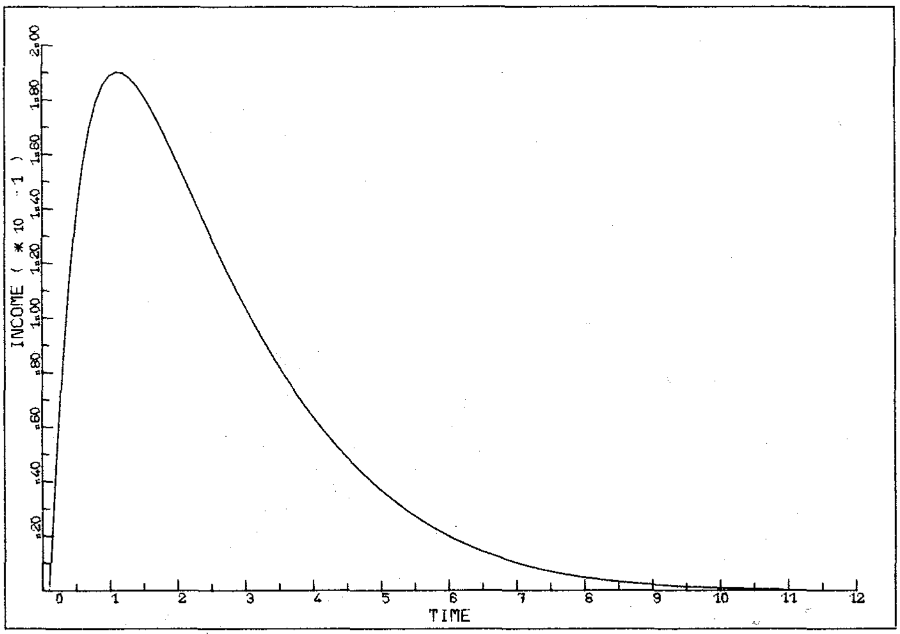


Fig. 7

Summing through time we get the grand total of income,

$$\int_0^\infty \left[\int_0^t Y(x,t)\,\mathrm{d}x \right] \mathrm{d}t = \pi^{\frac{1}{2}} \sum_{p=0}^{\infty} \frac{\int_0^\infty \mathrm{e}^{-\frac{3}{2}t} t^{2p+1}\,\mathrm{d}t}{(p+\frac{1}{2})2^{2p+1}\Gamma(p+1)\Gamma(p+\frac{1}{2})}.$$

Here

$$\int_0^\infty \mathrm{e}^{-\frac{3}{2}t} t^{2p+1}\,\mathrm{d}t = -\tfrac{2}{3}\int t^{2p+1}\mathrm{d}(\mathrm{e}^{-\frac{3}{2}t}) = \tfrac{2}{3}(2p+1)\int_0^\infty \mathrm{e}^{-\frac{3}{2}t} t^{2p}\,\mathrm{d}t$$

$$= (\tfrac{2}{3})^2(2p+1)2p \int_0^\infty \mathrm{e}^{-\frac{3}{2}t} t^{2p-1}\,\mathrm{d}t = \cdots$$

$$= (\tfrac{2}{3})^{2p+1}(2p+1)! \int_0^\infty \mathrm{e}^{-\frac{3}{2}t}\,\mathrm{d}t$$

$$= (\tfrac{2}{3})^{2p+2}(2p+1)!,$$

so that the grand total of income reduces to

$$\pi^{\frac{1}{2}} \sum_{p=0}^{\infty} \frac{(\frac{2}{3})^{2p+2}(2p+1)!}{(p+\frac{1}{2})2^{2p+1}\Gamma(p+1)\Gamma(p+\frac{1}{2})} = \tfrac{4}{3}\pi^{\frac{1}{2}} \sum_{p=0}^{\infty} \frac{(2p+1)!}{(2p+1)3^{2p+1}\Gamma(p+1)\Gamma(p+\frac{1}{2})}$$

$$= 4\pi^{\frac{1}{2}} \sum_{p=0}^{\infty} \frac{(2p)!}{3^{2p+2}\Gamma(p+1)\Gamma(p+\frac{1}{2})}.$$

Since the initial exports shock is $\sqrt{0.16} = 0.4$, the total exports multiplier for this instance of Puu's economy amounts to

$$10\pi^{\frac{1}{2}} \sum_{p=0}^{\infty} \frac{(2p)!}{3^{2p+2}\Gamma(p+1)\Gamma(p+\frac{1}{2})} = 1.685,$$

where the evaluation is numerical.

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